DOCUMENT RESUME

ED 188 901 SE 031 209

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TITLE	CSMP Mathematics for the Intermediate Grades, Part VI, Teacher's Guide: General Introduction, The World of Numbers, Workbooks, The Languages of Strings and Arrows, Geometry and Measurement, Probability and Statistics. Experimental Version.
INSTITUTION	Central Midwestern Regional Educational Lab., St. Ann, Mo.
SPONS AGENCY	National Inst. of Education (DHEW), Washington, D.C.
PUB DATE	80
NOTE	807p.: For related documents, see SE 031 204-210 and ED 175 634-652. Not available in hard copy due to copyright restrictions. Contains colored charts and activities which may not reproduce well.
EDRS PRICE	MF05 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	Algorithms: Diagrams: Elementary Education: *Elementary School Mathematics: Geometric Concepts: Grade 6: *Lesson Plans: Manipulative Materials: *Mathematics Curriculum: *Mathematics Instruction: Measurement: Number Concepts: Number Systems: Probability: Statistics: *Teaching Methods: Worksheets
IDENTIFIERS	*Comprehensive School Mathematics Program

ABSTRACT

This two-volume teacher's guide is to be used with a second-semester, sixth-grade mathematics program. The manner of presentation and the pedagogical ideas and tools are based on the works of Georges and Frederique Papy. They are recognized as having introduced colored arrow drawings ("papygrams") and models of our numeration system (the Papy "minicomputer"). This two-volume set contains six sections: (1) General Introduction: (2) World of Whole Numbers: (3) Languages of Strings and Arrows: (4) Geometry and Measurement: (5) Probability and Statistics: and (6) Workbooks. Each section has its own table of contents. Suggestions for planning and conducting lessons that include lesson descriptions, suggestions on the use of worksheets and manipulative materials, and general remarks relating to certain sequences of lessons are also included. (Author/MK)

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CSMP Mathematics for the Intermediate Grades

Part VI

Comprehensive School Mathematics Program
CEMREL, Inc.
Experimental Version
6-41001

Teacher's Guide

General Introduction
The World of Numbers
Workbooks

U.S. DEPARTMENT OF HEALTH,
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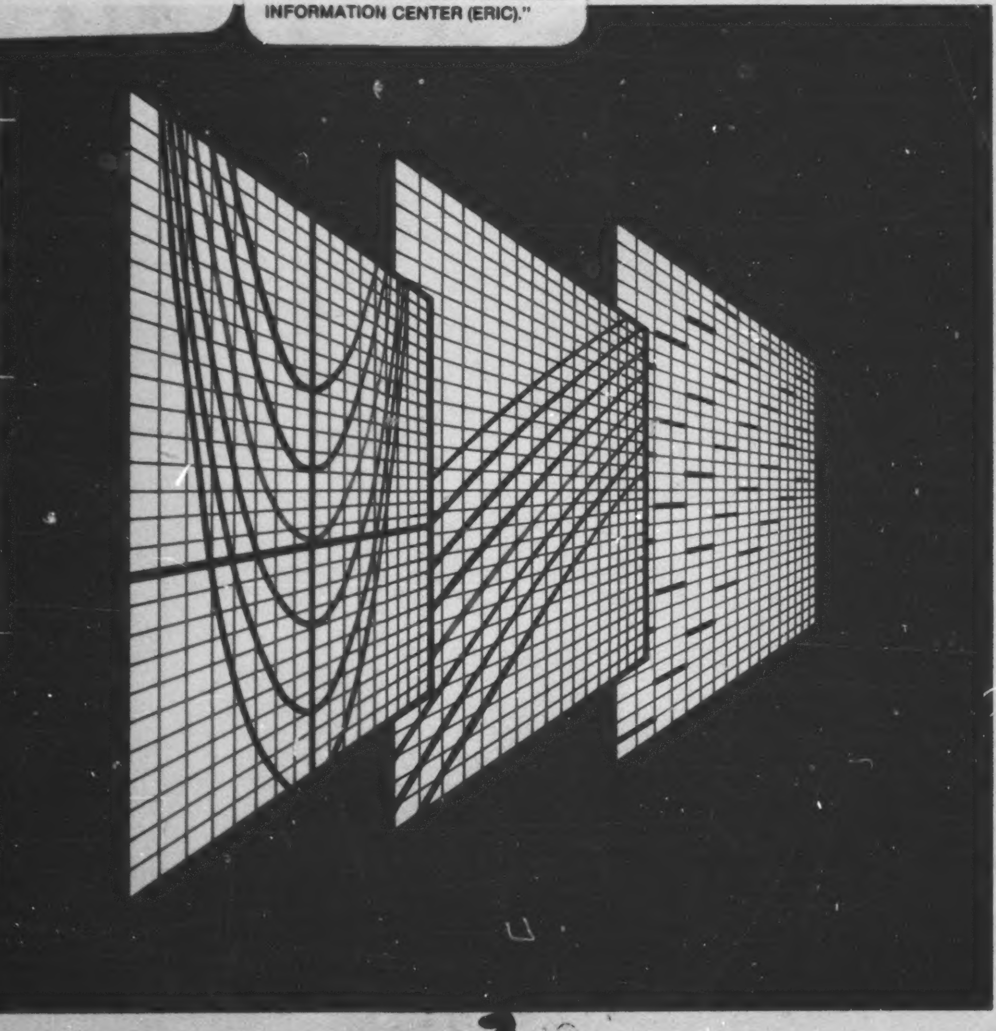

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PREFACE

CSMP is firmly committed to the proposition that the study of mathematics should be an interesting, challenging, intellectual, and aesthetic experience for all children. The teachers who teach its curriculum should be similarly committed to that proposition. The CSMP curriculum expects to truly engage children in the study of mathematics. It expects to call upon their curiosity, their intuition, their powers of reasoning, their imagination, their perseverance, and their organizational skills. We believe that the study of mathematics not only calls upon these faculties of children, but will help to develop them. But the curriculum cannot do the job by itself -- there must be teachers who share the basic intellectual values built into the curriculum.

There is an almost infinite variety of uses to which mathematics can be and has been put. Certainly, a reasonable mastery of arithmetic skills is part of the mathematical equipment of a competent, modern adult. Our job as educators is to provide children with a set of mathematical experiences far broader than the acquisition of simple arithmetic skills. We believe it is important to expose children to a wide variety of ways in which mathematics can help them to formulate and think about problems. We believe further that the arithmetic skills themselves can be much more successfully taught in the context of a rich and challenging mathematical environment. It is especially important to prepare children with a variety of mathematical activities in view of the fact that no one can possibly know what career and life-style an elementary school child will ultimately want to choose.

It is CSMP's conviction that mathematics is a unified whole and should be learned as such. The organizational basis of the CSMP curriculum is the spiral approach. Thus, the program is sequenced so that students come into contact with each area of content continuously throughout the program. This approach consciously precludes atomizing the content and mastering each bit before continuing to the next. Rather, students encounter repeatedly the ideas, building interlocking experiences of ever increasing sophistication.

CSMP believes that no single method of classroom management suffices to meet the needs of every student. Hence, the program is constructed to allow numerous opportunities for whole class participation, small group participation, and independent individual study.

The CSMP materials are constructed to maximize teacher-student and student-student interaction. Children begin immediately to investigate mathematical concepts in concrete situations appealing and meaningful to them at their level of experience.

Individualization at this level is achieved through graded worksheets and workbooks with problems at varying levels of difficulty all in one booklet. Worksheets and workbooks are integrated into the basic sequence of lessons. Due to the spiral nature of the curriculum, each student can engage in a workbook for a period of two (possibly more) weeks without losing touch with the progress of the class and, at the same time, working his or her way to the limits of his or her capacity.

Unlike most modern programs, the content is not presented as an artificial structure external to the experience of children and into which they are thrust but rather as an extension of experiences children have encountered in their development, both at the real life and the fantasy levels. Children are led through sequences of problem solving experiences presented in game-like and other story settings. Nonverbal languages give children immediate access to the mathematical ideas and methods necessary not only for solving these problems but also for continually expanding the students' understanding of the mathematical concepts themselves. These languages include:

1. The language of strings. This language mimics the basic method of organizing and collecting data related to problems at hand. That is, it deals with the fundamentally useful and important mathematical notion of sets.

2. The language of arrows. This graphical language models the fundamental

process involved in comparing and analyzing sets and operations on them. That is, it deals with the useful and important mathematical notions of relations and functions including, but not restricted to, permutations of sets, ordering relations and the numerical processes of adding to, subtracting from, multiplying by, and sharing equally among.

3. The language of the Papy Minicomputer. The Papy Minicomputer, an extremely effective abacus invented by Georges Papy, models the positional structure of our system of numeration and lends itself to very sophisticated algorithmic processes, including the standard algorithms for the basic numerical operations.

Through these languages the children are able to apply the basic mathematical ideas effectively and creatively to explore the content directly without wading through the cumbersome, sterile terminology and notation which, in most other programs, pervade the early stages of the curriculum and stand as barriers between the child and the content.

The content is learned in an atmosphere of constant connection with applications, from simple story situations in the early grades to challenging nontrivial simulations of real world problems in the upper grades. The emphasis is at all times on a two level approach to learning: understanding the content and its applications, and equally important, developing the techniques and processes of learning the content. It is the latter form of knowledge which gives the power to apply the former.

NOTE ON AUTHORSHIP

"Teacher's Guide to CSMP Mathematics for the Intermediate Grades, Part VI" is the product of the concerted efforts of a number of people over several years' time. This edition was produced during the 1977-78 academic year by the following members of the CSMP staff and with substantial contributions from the staff associates.

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The lessons in this guide were developed and tried out with one first semester seventh grade class at the Brittany Middle School, University City, Missouri and one second semester sixth grade class at St. Timothy's Elementary School, St. Louis, Missouri. The students, representing a full range of abilities, had been introduced to the CSMP curriculum at the beginning of fourth grade in a one semester program using "CSMP Mathematics for the Intermediate Grades, Entry Module I". The entry program prepared the students to begin the first part of the CSMP Intermediate Grades Program for veteran students during the second semester of the fourth grade.

We are extremely grateful to Mr. Jack Fitzgerald, Mathematics department head at Brittany School, for his cooperation in arranging for CSMP's teaching of the two classes for development purposes. We also wish to express appreciation to Ms. Jacque Hummel, the regular teacher of the class at St. Timothy's School, for her daily cooperation and constructive criticisms. Our greatest motivation for this work was the enthusiasm and steady progress of the students.

GENERAL INTRODUCTION

Comprehensive School Mathematics Program

CSMP Mathematics for the Intermediate Grades, Part VI

General Introduction

CEMREL, Inc.

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1. The CSMP Viewpoint

The CSMP curriculum is based on the belief that mathematics belongs to that part of the school curriculum which deals with ideas rather than that part which stresses techniques. This is not to say that there are no techniques involved in the learning of mathematics. There are, just as there are in the learning of English. But these techniques, like spelling and grammar in the learning of English, are not ends in themselves; they are important in so far as they help children to understand and use some area of mathematics.

What are the objects of mathematical study? For us, they include numbers of various kinds; operations on and relationships between numbers; geometrical figures and their properties; relations and functions (e.g., the 2 x function, the "less than" relation, etc.) and operations on them.

Among all objects of mathematical study, from the simplest to the most complex, there is one common property: if we separate these objects of study from the real life situations that they model, they are abstract; that is, they are things of the mind. For this reason, mathematics is essentially an intellectual activity. For example, the number 2 represents the phenomenon of twoness, a quality which is present in an endless variety of contexts in life. We can picture the number 2 by two dots or by a stick two centimeters long, and these representations are certainly useful in suggesting to our minds what may or may not be true about the number 2. But ultimately the number 2 is accessible to use only through our minds. In the same sense, a geometrical line is neither a pencil line nor a stretched string nor a ray of light, but is an abstraction which may be pictured in many ways.

This is surely an adult point of view. But it is one that should be understood and taken into consideration by those who develop mathematical materials for children and those who teach the materials to children.

Since mathematical objects are ideas, one of the ways we explore the properties of and the relationships among such objects is to think about them. It is for this reason that growth in the ability to reason plays such an important role in the study of mathematics. We strongly argue, however, that in mathematics the development of the art of reasoning goes hand in hand with the growth of imagination, ingenuity, and intuition. At the elementary level, mathematical objects, their properties, and their interrelationships must be thought about and discussed in various suggestive and interesting situations.

Seen in this perspective, elementary arithmetic then takes the form of "adventures in the world of numbers". Individual numbers such as 0, 1, 2, 100, $\frac{1}{5}$, $^-7$, $\frac{^-4}{9}$, 3.72, $^-1.006$, and so on can assume definite personalities of their own in this world. The teaching of arithmetic shifts its emphasis from an obstacle race for mastery of the standard algorithms to the stimulation of exploring the world of numbers. Skills such as balancing a checkbook have their place in real life but are of absolutely no interest to a fifth or sixth grader. It is unrealistic to view such activities as goals of an elementary school mathematics education. Far more genuinely rewarding are the stretching of the powers of imagination and the challenging of the mind. Facility grows naturally, along with knowledge and understanding. Necessary skills become by-products of such an educational experience.

Such a curriculum requires a special kind of pedagogy. The pedagogy employed should include teaching strategies which encourage creativity and allow freedom to explore but still foster sound intellectual habits. Attitudes toward student achievement that allow for intellectual and emotional growth and respect the individuality of each student are essential to the successful attainment of the goals of this type of curriculum.

The pedagogy which CSMP develops in this spirit can be described as a "pedagogy of situations".[†] The introductions to the various strands of the CSMP Intermediate Grades Program and those to the chapters within the strands refer to many examples which illustrate aspects of this pedagogy. All these examples are taken from the lessons themselves, and a prime source of

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[†] This phrase is due to the mathematician and psychologist, Caleb Gattegno.

"pedagogical situations" is the lessons of the program. At the risk of oversimplification, a pedagogy of situations may be described as one which is based on the belief that learning occurs in reaction to the experience of confronting a situation (real, simulated, or imagined) that is rich in consequences, is worthy of confrontation, and has genuine intellectual content.

Children can react to a situation that intrigues them. They may grapple with problems of symbolization and the relation of a symbol to the thing it represents. Arguments about the consequences of the given information may be close to those which are the stuff of mathematics. Although adults may protest that some of the situations in the lessons are not "real", they are real to children, who can often involve themselves in and feel at home with "pretend" situations more readily than with many "real" situations. On the basis of several years' experience in a wide variety of classes, we can say with confidence that a pedagogy of situations:

- involves children at a personal level (they are interested in the situations presented, and sometimes want to dramatize or devise related situations);
- presents an intellectual challenge that is accessible to a broad range of abilities (even the slowest students understand the challenge and can learn something from their experience in confronting a situation; some may even be able to realize the full learning potential of that challenge);
- provides opportunities for creativity (some children are motivated by well chosen situations to generalize from their experiences or to make conjectures to be tested in similar situations);
- supports experiences that have mathematical content.

The CSMP curriculum fulfills what we consider to be the fundamental role of mathematics in the elementary school by presenting a large number of varied, yet interrelated situations that provide the experiences out of which mathematics grows. Two questions immediately arise. How do students learn necessary "number facts" via a pedagogy of situations? And how can we "grade the achievement" of a child as he or she reacts to a situation?

Believe it or not, the "number facts" are among the things which students learn almost inadvertently under a pedagogy of situations. Numbers become personalities; their interrelations are the subject of many stories in which numbers talk to each other and react to each other. In confronting some of these situations, students make various calculations out of necessity. Calculation which serves some higher, more motivating purpose presents nowhere near as many difficulties as does calculation for its own sake. Further, more specific remarks concerning calculation occur scattered through the latter part of this introduction and in the introductions to chapters of the World of Numbers strand.

As for "grading of achievement", CSMP takes the position that learning often stops when the purpose of a lesson or sequence of lessons is the mere mastery of skills. Every child should learn something from each situation; otherwise, it is not a fruitful situation and should be revised or eliminated. But to measure what or how much has been learned in any one situation is probably not possible. Learning is not a linear process. It occurs in quantum jumps that take place at times when you least expect them. It is possible to monitor the progress of students, and you have your own ways of doing this, but who is to say at what precise time everyone in a class is supposed to have gained a particular skill or concept from his or her experiences? We comment further on this matter in the next section of this General Introduction.

A more substantive question is: How can we present situations rich in mathematical content before children have the verbal equipment and the computational skills to deal with them? It is presupposed by many people that children must be able to read and to calculate before they can tackle any mathematical situation other than trivial ones. Yet if one analyzes the types of thinking and the few basic concepts underlying elementary school mathematics, one finds that there is nothing esoteric or even difficult about these thinking modes or these concepts. What has been difficult for students in the past is the language in which these modes of thinking and these concepts are traditionally couched. For some reason, in our natural languages these modes of thinking are difficult to communicate precisely. That is why mathematicians devise languages of their own, languages that minimize the ambiguities of natural languages. Part of the problem in the past has been that elementary school children have found it difficult to deal with such tricky uses of language at a time when they do not yet have a firm command of their natural language.

On the other hand, since the underlying ideas and modes of thinking are inherently simple, there is no reason why very young children should not have the pleasure of such thinking and of exploring mathematical ideas. All that is needed are languages with which to express such ideas and to communicate the thinking that is generated by the ideas. These languages must be very simple, essentially nonverbal, and precise.

CSMP uses three such languages that fit the criteria and that have been highly successful in allowing students to do the kind of thinking that we described above. We call them the language of strings, the language of arrows, and the language of the Papy Minicomputer.

The first two are picture languages: brightly colored strings and dots enable precise descriptions of and the accurate recording of thinking about classifications of objects; colored arrows between pairs of dots stimulate thinking about relations between objects. For detailed descriptions and discussions of these two languages you are referred to the introduction to the Languages of Strings and Arrows strand.

The third language is based on a simple abacus called the "Papy Minicomputer". Using this language, one may represent all decimal numbers (positive or negative) in such a way as to encourage creative thinking about the nature and the properties of those numbers. With the support provided by the Minicomputer, the children, even at an early age, have the opportunity to develop excellent mental arithmetic skills and to gain sophisticated insights into the world of numbers. The ability to perform mechanical computation is one of the incidental by-products of such an approach to work with numbers.

2. Program Description

a) Description of Materials

The CSMP Intermediate Grades curriculum is designed as a program for students in grades four, five, and six. There are six parts to the curriculum, each corresponding to one semester's work. This is Part VI of the six.

In addition, there are two Entry Modules, one each for grades four and five. As the name implies, each Entry Module is designed to facilitate transition into the CSMP Elementary School Curriculum.

The materials of Part VI of CSMP Mathematics for the Intermediate Grades require some explanation. This collection of two volumes (six books) which are directed specifically to you, the teacher, is known as the "Teacher's Guide to CSMP Mathematics for the Intermediate Grades, Part VI". (By way of abbreviation, in what follows we shall refer to this collection simply as the "Teacher's Guide.") With the exception of this book, "General Introduction," each book of the Teacher's Guide addresses itself to a single area of the curriculum, and each of these areas is known as a "strand." The four content strands are:

- the World of Numbers - (N);
- the Languages of Strings and Arrows - (L);
- Geometry and Measurement - (G);
- Probability and Statistics - (P).

The six books comprising the "Teacher's Guide" are bound into two volumes, with a separation page inserted between the various books of a given volume.

This volume contains both this General Introduction as well as the World of Numbers strand. The other volume contains the Languages of Strings and Arrows strand, the Geometry and Measurement strand, the Probability and Statistics strand, and the Workbook strand. Each of these six books has its own table of contents and is paginated individually.

Each strand has the following arrangement of material.

- A table of contents. This enables you to locate whatever you need in that particular strand.
- A suggested chronological order of the lessons in that strand. This provides an easy, quick reference uncluttered by extraneous information. It shows you in which order the lessons described in the strand will be taught if you follow the suggested schedule.
- An introduction to the strand. This explains why certain material finds its place in that strand, what the overall goals of study are, and how this material fits in with that which is described in other strands.
- Suggestions for the planning and conducting of lessons in that strand. These include lesson descriptions, suggestions on the use of worksheets and manipulative materials, general remarks relating to certain sequences of lessons, etc.

The suggested lessons are grouped in chapters, all the lessons described in the same chapter having some particular feature(s) in common. Because of this commonality of purpose, it is possible to precede each chapter by an introduction which indicates where the lessons in that chapter are going and, on occasion, gives a few more details concerning the mathematical underpinnings of the lessons described in the chapter than are contained in the lesson descriptions themselves.

Many of the suggested lessons require student worksheets for individual work.

Classroom sets of these worksheets are printed and packaged separately from this guide. However, an answer key for each of the worksheets is included in the guide. Worksheets are numbered consecutively within each strand and their answer keys can be found at the end of the lessons for which the worksheets are needed. For example, an answer key for Worksheet L05 is at the end of Section 2.6 of the Languages of Strings and Arrows strand since it is suggested that this worksheet be used with the lesson described in that section.

In order to accommodate the various ability levels in a given class, the worksheets generally are written at several levels of difficulty denoted by stars. "*" represents an easy worksheet, "**" represents an average level worksheet, and "***" and "****" represent difficult level worksheets. All students should begin with the "*" level worksheets for a given lesson or sequence of lessons and proceed as far as possible through the other levels. In classes containing the usual range of abilities, most of the students should succeed with the * worksheets; the ** worksheets should be appropriate for about two-thirds of the class; and approximately one-third of the students will find the *** and **** worksheets challenging and enjoyable to do. No one should ever be required to attempt all the worksheets provided for a given lesson. Each student should progress as far as his or her abilities allow.

There are various other items which are (or could be) used in the program. Among them are the following: various manipulative materials such as translators, rulers, compasses, grid board, and so on (these are described in more detail in the lessons in which they are used); several workbooks, storybooks, and story-workbooks (these are described in more detail in Sections 2 (b) and (e) of this General Introduction, and in the introduction to the Workbook strand); a two-volume book entitled "Math Play Therapy", which includes a collection of suggested small group activities that could be instrumental in stimulating an open and positive attitude toward mathematics on the part of those of your students who find the subject difficult. See the introductions to both volumes of "Math Play Therapy" for suggestions as to its use in the program.

b) Role and Use of Workbooks

It is suggested that approximately one day each week be scheduled for students to do individual work in designated workbooks. The workbooks are designed to review ideas encountered in the content strands and to apply these ideas to new situations posed as stories. Workbooks are written with problems of varying levels of difficulty in order to accommodate differences in students' abilities and interests.

More detailed descriptions of the workbooks, their purposes, and answer keys are given in the Workbook strand. There you will also find suggestions for keeping records of student progress.

c) The Spiral Development

The learning process occurs through many interrelated experiences. But no single experience, particularly a 40 to 50 minute mathematics lesson, is an end in itself. The approach in the CSMP curriculum keeps this in mind. It is neither intended nor expected that every student will meet the full challenge of a particular situation in any given lesson. Furthermore, we find that it is most effective to vary the situations and strands from day to day rather than to continue with one type of situation or piece of content until so-called mastery has occurred. Thus, we suggest that the lessons be scheduled so that a different topic is studied each day of the week. In a typical week, for example, your class might work with numerical games one day, probability the next day, workbooks the third day, numerical relations the fourth day, and geometry the fifth day.

The spiral development, in which a student experiences each of several ideas a little at a time and then proceeds through increasing levels of sophistication as the situations become more challenging, reflects the fact that learning is not a linear process. Not only do different children learn at different times and at different rates, but even on an individual level learning can occur in quantum leaps that take place at times when you least expect them. The spiral development gives each student a new chance to catch on to an idea at each turn of the spiral.

Of course, acceptance of the CSMP pedagogy requires a certain act of faith on your part. If you are accustomed to staying with a given topic until everybody (or nearly everybody) in the class knows it "cold", then you may be initially very reluctant to leave a lesson at the end of the time suggested, believing that several students "haven't got it yet". By keeping the spiral idea in mind, you should become more comfortable with these time schedules. Although it may appear to you that little learning is taking place in a sequence of lessons, you may be surprised that suddenly even your slowest students are understanding the ideas when they encounter them later in the spiral. Whereas with a mastery approach, presented in a purely individualized mode, slow students often spend most of the school year trying to "master" low level skills (with generally poor results), the spiral approach gives these students the opportunity to benefit from the richness of the situations presented throughout the curriculum. In this manner students are allowed to be involved in interesting and enjoyable situations which are modeled by important mathematical notions. They are not deprived of a variety of experiences just because they haven't mastered every bit and piece along the way. Therefore, in the long run, their interest in, understanding of, and even facility with mathematics has a much better chance to develop.

d) Evaluation

On the basis of the preceding discussion, it should come as no surprise, as stated earlier, that CSMP takes the position that learning often stops when the purpose of a lesson or sequence of lessons is the mere mastery of skills. Furthermore, we do not believe it is desirable or reasonable to attempt to assess student mastery of skills from day to day or even from week to week. As we said before, learning has a habit of occurring when you least expect it, so how does one determine when everyone in a class should have mastered a specific skill or attained understanding of a particular concept as a result of his or her experiences? We know of no method for measuring what or how much has been learned from any one situation except in the most trivial cases. On the other hand, teachers, parents, and even the students themselves can benefit from some continuous monitoring process that records involvement in activities and assesses progress through the program. Indeed, it is possible to monitor your students' progress without contradicting the statements made above; all of you have your own ways of doing this. Such a process is suggested in the introduction to the Workbook strand. Of course, you should also continue to use any monitoring devices with which you feel comfortable.

As far as traditional computational skills are concerned, the standardized test data collected during the extended pilot test stage of the development of this curriculum indicate that CSMP classes generally do better than average on the items which test "basic" skills.

About 75% of the lessons for the CSMP Intermediate Grades curriculum involve numerical concepts. The situations involve not only positive rational numbers (as is generally the case at this level), but introduce the students to many other citizens of the world of numbers at as early a stage as possible. You should not infer from this that your students are expected to have mastered the arithmetic of the whole numbers, of the integers, of the decimal numbers, and of the rational numbers before they are ready for Part VI of the CSMP Intermediate Grades curriculum. On the contrary, it means that the students will become more and more friendly and at ease with the various types of numbers and will develop their own methods for calculating with them long before some standard mechanical algorithms are introduced.

It is essential that the creative freedom of the student not be impaired by a premature emphasis on the standard algorithms for adding, subtracting, multiplying, and dividing. We believe that many of the present computational difficulties experienced by elementary and junior high school students were caused by their spending an inordinate amount of time trying to master the algorithms too early in their mathematical education. This shuts the students off from other experiences necessary for their mathematical growth. Students need plenty of time to explore the world of numbers. Mastering an algorithm should be the last step of this exploration, a sort of natural synthesis of previous experience. Furthermore, as electronic hand-calculators become available to nearly everyone, algorithmic mastery as a prerequisite to other numerical experiences becomes less important.

We strongly urge you to follow the suggested schedule of lessons, teaching the algorithms as they come, but not giving them more emphasis than they deserve. Skill in mental arithmetic and estimation, however, is extremely important in our opinion and should be promoted at every opportunity.

While CSMP disagrees with the philosophy underlying standardized testing, we realize that students as well as teachers are under extreme pressure to perform in certain ways with regard to such tests. If your students or their parents express concern about the apparent lack of emphasis on mastery of standard algorithms (such as long division by three or four digit numbers), they should be reassured that CSMP students have a definite advantage when it comes to applying mathematics to novel situations, thinking strategically, organizing data or information, and understanding methods of mathematics. Because CSMP feels it is wrong for such tests to dictate emphasis on the mastery of standard algorithms and that drill for algorithmic mastery for its own sake defeats CSMP's purposes, you will find that even after algorithms have been introduced into the CSMP curriculum the role they play will be as tools for doing other interesting things. Therefore, if you feel pressured by standardized tests or by other measures to put more emphasis on drill activities, then we suggest you use any supplementary materials that have proved successful in the past to provide this extra practice. We leave scheduling choices for these kinds of activities to your judgment.

e) Role and Use of Storybooks and Story-Workbooks

Among the instructional materials for the CSMP Elementary School curriculum is a collection of mathematical storybooks, STORIES BY FREDERIQUE. This collection consists of twenty-four stories for young people, presented in colorful newsprint booklets. They are intended to appeal to youngsters in the 5-14 age bracket. The stories are completely independent of each other and need little or no external explanation. They may be used for individual or collective reading, at home or at school.

Each of the stories is portrayed by the mathematical languages of dots, strings, and arrows which reinforce the ideas of the story. Three of the stories (TWO BY TWO, THE WEIRD STORY OF 24, and SUMMER SCHOOL IN THE OLD DAYS) also use the language of the Minicomputer. The amazing versatility of these nonverbal languages is fully exploited.

Young readers become personally involved in these stories; there are questions to answer, pictures to relate to the stories, and new adventures to invent for the characters in the stories. The reader discovers that emotions and aesthetic senses can be involved in the world of mathematics just as in the world of people.

Twelve of the stories are intended for very young children (5 to 8 years of age). Here the focus is on the anecdotal aspect of the story. One of the goals of these twelve books is to familiarize children with several nonverbal mathematical languages and to leave them with the impression that these languages are as natural and useful as the usual language.

Nine of the remaining stories concern numbers and their relationships with the world. These particular stories are about a boy and some of his friends who just happen to be numbers. The personification of numbers introduces the "games" of mathematics into one of the most important domains of childhood -- the world of fantasy. Free from the usual constraints so often placed upon them, the numbers invent marvelous games and dance to the tunes of vividly colored arrows. These games are never frivolous. There are rules to be followed and as a result, orderly patterns emerge as the numbers play or dance.

Of the remaining three books, two are detective stories introducing some interesting mathematical problems, and the other one emphasizes the various and different aspects of a simple straight line.

The titles of the books in the collection and the intended age range for each story are:

Ages 5 to 8

THE PLAYFUL NUMBERS
81 ROSES
I AM A VERY HAPPY BOY
ONE OUT OF SEVEN
THE HAPPY PUPPET
TWO BY TWO
THE LITTLE DREAMER
THE LITTLE DONKEY
THE MAGIC BOX
THE BABY IS BORN
THE WEIRD STORY OF 24
THE OLD SHOEMAKER

Ages 8 to 12

WHERE'S MY NOSE?
SINGING FRIENDS
DANCING FRIENDS
THE LIVING LINES
I AM NOT MY NAME
NABU WINS AN AWARD
THE SQUARE TRAP
SUMMER SCHOOL IN THE OLD DAYS

Ages 10 to 14

THE HIDDEN TREASURE
A VERY STRANGE NEIGHBORHOOD
ELECTION DAY IN THE NUMBER WORLD
A VALENTINE MYSTERY

One of the storybooks, A VERY STRANGE NEIGHBORHOOD, is the basis for a lesson in the Workbook strand. The classroom set of materials for Part VI contains thirty copies of this storybook. You may wish to order copies of other storybooks for a classroom library. Although these other storybooks are in the third grade program or in previous parts of the Intermediate Grades Program, your students may enjoy reading them from time to time.

Besides the collection, STORIES BY FREDERIQUE, another collection, THE CSMP LIBRARY, has been developed as part of the instructional materials for the CSMP Elementary School curriculum. This is a series of story-workbooks which combines the spirit of the storybooks and the workbooks into a single integrated experience for the children, engaging them in reading about and doing mathematics in stimulating, challenging story situations. The current list of story-workbooks in the collection is:

Ages 8 to 11

SUMMER SCHOOL - 0'S DISCOVERY
ROLLERSKATING 37
TO PICTURE
A SHORT STORY ABOUT 2 FRIENDS
A SHORT STORY ABOUT 3 FRIENDS
THE ISLAND OF TAM-TAM
THE TALKATIVE NUMBERS

Ages 9 to 14

SUMMER CAMP
HALLOWEEN PUZZLES
1,000'S DREAM
A STRANGE COUNTRY
NOT TOO CLOSE
WHO ARE WE?
CLINTON STREET
A BOOK ABOUT ME
SEVEN SECRET NUMBERS
SHONDA'S NEWSSTAND

f) Role and Use of Hand-Calculators

The electronic hand-calculator, if used creatively, can have a remarkably beneficial effect on the students' interest in and understanding of arithmetic, and its use can be a contributing factor in the improvement of their capabilities in that area. The reasons for this are many and varied, but among them are certainly the following:

- By removing the burden of mechanical computation from the children's shoulders, the calculator frees them to concentrate on the thinking that is a necessary part of the solution to any meaningful problem.

- The instant feedback provided by the calculator enables the children to indulge in making and testing conjectures, an area into which they would not otherwise venture because the quantity of computation involved would be too forbidding.

This pedagogical tool is particularly useful in the introduction of new numerical notions such as negative integers, decimal numbers, multiples of a given whole number, and so on, because through suitably chosen, rich situations these concepts can be made to arise in a natural way.

A classroom set of battery powered hand-calculators is recommended for Part VI. Many different kinds of hand-calculators are available from CSMP as a distributor for National Semi-Conductor and Texas Instruments. Any of the products of these two companies may be purchased at discounted prices.

If it is not possible to supply all students with hand-calculators, encourage the students to bring calculators from home. The hand-calculators that are available from CSMP have algebraic logic and function keys which are permanently in a constant mode. You should check to be sure that calculators that you supply or that are brought in by students also have these characteristics.

- Algebraic logic. This means that the calculator responds to instructions given in the order in which they are usually written. To check that this is the case, press $\boxed{2}$ $\boxed{+}$ $\boxed{3}$ $\boxed{=}$. If you have "5" on the display, then all is well.

- Constant mode. Some calculators have a switch so that the calculator can be set either in a constant mode or in a chain mode. If your calculator lacks a switch, it is quite likely that the calculator has an automatic constant. For calculators with an automatic constant, the first number entered during multiplication is usually the constant, while the second number entered during the other three operations (addition, subtraction, division) is usually the constant.

  To check your calculator for the constant mode operation of $\boxed{+}$ is a simple procedure. Press $\boxed{0}$ $\boxed{+}$ $\boxed{2}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ and so on. If the number on the display is increasing by two at each depression of $\boxed{=}$, then $\boxed{+}$ is operating in a constant mode. Check the constant mode operation of $\boxed{-}$ in a similar fashion.

To check the constant mode operation of $\boxed{x}$ (where the first number entered will be the constant), press $\boxed{2}\ \boxed{x}\ \boxed{1}\ \boxed{=}\ \boxed{=}\ \boxed{=}$ and so on. If the number on the display doubles at each depression of $\boxed{=}$, then $\boxed{x}$ is operating in a constant mode.

While it would be possible to use calculators without these characteristics in the lessons, the adjustments that would be required in the instructions make the calculators unsuitable.

The hand-calculator relations $\boxed{+}\ \boxed{n}\ \boxed{=}$ . . . and $\boxed{+}\ \boxed{=}$ . . . are explored quite extensively in several lessons. Even if your calculators have algebraic logic and the constant mode, it may be necessary for you to check to see if your calculators operate in the way that is described in the lessons.

If you find that your calculators do not operate in this manner, then try letting

$$\boxed{+}\ \boxed{n}\ \boxed{=}\ \ldots \qquad \text{mean}$$

$$\boxed{+}\ \boxed{n}\ \boxed{=}\ \boxed{+}\ \boxed{n}\ \boxed{=}\ \boxed{+}\ \boxed{n}\ \boxed{=}\ \ldots$$

and

$$\boxed{+}\ \boxed{=}\ \ldots \qquad \text{mean}$$

$$\boxed{n}\ \boxed{+}\ \boxed{0}\ \boxed{=}\ \boxed{+}\ \boxed{n}\ \boxed{=}\ \boxed{+}\ \boxed{n}\ \boxed{=}\ \ldots$$

where $\boxed{n}$ is some starting number.

Furthermore, with some calculators there is a correspondence between a multiple of a number and the number of times $\boxed{=}$ is pressed in the $\boxed{+}$ $\boxed{=}$ . . . relation. For example,

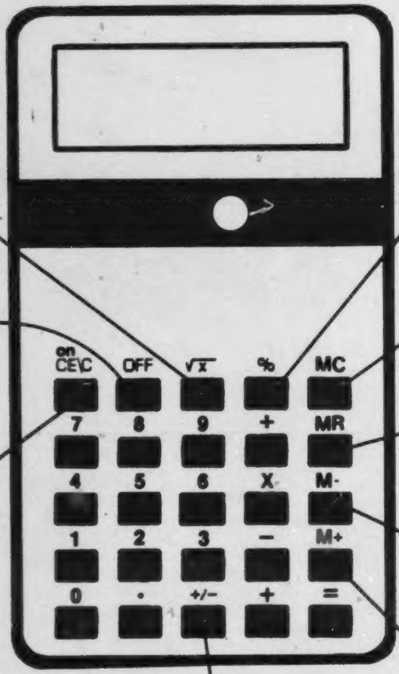
> Starting with "4" on the display and pressing $\boxed{+}$ $\boxed{=}$ yields 4 (= 1 x 4). Pressing $\boxed{=}$ again yields 8 (= 2 x 4); pressing $\boxed{=}$ again yields 12 (= 3 x 4); pressing $\boxed{=}$ again yields 16 (= 4 x 4), pressing $\boxed{=}$ again yields 20 (= 5 x 4); and so on. Notice that when $\boxed{=}$ has been pressed five times, the fifth positive multiple of 4 (i.e., 20) is displayed.

Frequently, calculators that do not behave in the manner described above will differ by one less (i.e., the fifth positive multiple of 4 appears after pressing $\boxed{=}$ four times).

The calculators used in the descriptions of the Lessons of Chapter 1, HAND-CALCULATOR PUZZLES, are this latter kind. Therefore, there may be some discrepancy between solutions your students find and the ones presented in those lesson descriptions. However, the differences should be minor and you should be able to adjust the lesson description to fit the type of calculators your students are using.

In Part VI you and your class will have several opportunities to explore the use of "special keys" that appear on many hand-calculators (see Chapters 1 and 7 of the N strand). The illustration on the following page gives a brief description of some "special" keys that might be on your hand-calculators.

Prior to the presentation of the hand-calculator lessons, you should familiarize yourself with these keys.



Square Root Key — finds a number which, when multiplied by itself, will equal the displayed number

Off Key — clears display, memory, and calculations in progress

On/Clear Entry/ Clear Key — clears an erroneous entry on the display; does not clear memory

Percent Key — allows direct calculation of percent

Memory Clear Key — clears memory; does not clear display

Memory Recall Key — displays contents of memory

Memory Subtract Key — subtracts displayed number from memory

Memory Add Key — adds displayed number to memory

Change Sign Key — changes sign of number displayed; allows entry of negative numbers

3. Guidelines for Organizing the Program

The design and content of the lessons in Part VI affords you the utmost flexibility in teaching the program successfully. It is a truism to say that your students this year are very different, both individually and as a class, from the students you taught last year, and indeed from other students in any other CSMP class. And your own teaching style and interests are probably very different from those of your colleagues. It is difficult for those of us who develop curricula to guess correctly as to which topics will be very easy for your students and which lessons will require more instructional time. You are the only one who will be able to judge that as you begin using the materials. This program is not meant to be a static sequence of exercises to be followed rigidly; the schedule and design of lessons in this guide should be regarded as suggestions; they simply reflect our own experience teaching the materials. You and your students will make the program dynamic as together you explore the mathematical ideas and follow your own interests and inclinations.

a) Use of Schedules

If this is the first time you are teaching a CSMP class and you would like specific suggestions for organizing the lessons, we offer some guidelines and a suggested schedule of lessons. Please remember that our suggested sequence is but one of many that could have been assembled. You may wish to follow it at the beginning of the school year and then deviate from it as the needs of your students dictate.

The master schedule included at the end of this section reflects our suggestions for a hypothetical sixth grade class of average ability, and it illustrates some important ideas about organizing the program. You will notice that several different strands are scheduled each week and that even within a strand the topics are interspersed. This was done to carry out the idea of a spiral curriculum. If you organize your own schedule, it is best to follow a similar pattern rather than to spend two or three full weeks, for example, "covering" one particular topic such as composition of relations or decimal numbers. In a usual five day week, two days should be devoted to the World of Numbers strand, one day to the Workbook strand, and the other two days should be shared by the P, G and L strands.

In the schedule, a block represents one full period of approximately 45-50 minutes duration. It may very well be, however, that you and your class will spend more than one period on some lessons. Exercise your own professional judgment in deciding when to stop one particular lesson and when to take it up again. Just be sure that you do not insist that every single child understands every single aspect of a lesson before you go on. It is better to stop a lesson at a point when student interest is high and a problem remains unsolved than to teach so methodically that everyone in the class understands everything, but is completely bored.

Inside each block, the lesson of the day is designated in several ways. First, there is listed a number which indicates its suggested chronological order among the lessons of a strand. Hence, "5N" indicates that the lesson in question is suggested as the fifth lesson in the N (World of Numbers) strand. This identifying number also appears in the upper outside corner of each page in the description of that lesson, thus enabling a lesson to be located simply by flipping through the pages of the strand book. Second, you are given the section number and title of the lesson. Third, you are told on which page of the strand book the description of the lesson in question begins.

Please bear in mind that the model schedule was designed for a hypothetical class with the usual range of abilities. In order to be successful with the program, you should adjust the schedule to the ability, maturity, and interests of your class. For example, you may wish to extend a lesson or a workbook for an additional period beyond those scheduled.

b) Use of Adjustment Days

From time to time an adjustment day appears in the schedule. Since we have intended that Part VI would be the mathematics curriculum for one semester of the Intermediate Grades Program and since a usual school semester is eighteen weeks, the schedule indicates eighteen weeks of lessons. However, it is also usual for there to be days when no mathematics lesson is taught; e.g. parent-teacher conference days, field trip days, snow days, and so on. The adjustment days should account for such non-class days, and should also give you opportunities to do many kinds of things. You could use an adjustment day to review a lesson you feel needs review, or to complete an unfinished lesson, or to extend some interesting ideas from completed lessons. You could also use an adjustment day to give your students time to become more familiar with the manipulative materials in the program, such as the translators or the compass, or with the storybooks. Your students may especially enjoy playing games such as Minicomputer Golf, Minicomputer Tug-of-War, Guess My Rule, the String Game, or the Table Game and an adjustment day could give them another opportunity to play a game. You may wish to use an adjustment day to do some mental arithmetic exercises, or to teach a topic in mathematics you think would be especially interesting or useful to your class.

SUGGESTED SCHEDULE OF LESSONS

Master Schedule for the CSMP Intermediate Grades, Part VI

Week	World of Numbers	Languages of Strings and Arrows Probability & Statistics	Workbooks	World of Numbers	Geometry and Measurement Probability & Statistics
1	N1 1.1 Hand-Calculator Puzzles #1 N p. 2	L1 1.1 The Table Game #1 L p. 3	N2 2.1 One-Digit Distance #1 N p. 16	N3 3.1 Who is Zig? N p. 31	ADJUSTMENT DAY
2	N4 4.1 Binary Writing #1 N p. 79	L2 1.2 The Table Game #2 L p. 18	W1 Collage of Problems #1 (Lesson 1) W p. 2	N5 5.1 Fractions #1 N p. 140	G1 1.1 Introduction to Angles G p. 7
3	N6 6.1 Combinatorics #1 N p. 190	L3 1.3 The Table Game #3 L p. 20	W2 Collage of Problems #1 (Lesson 2) W p. 4	N7 7.1 Percent #1 N p. 221	G2 1.2 Constructions with Acute Angles #1 G p. 19
4	N8 6.2 Combinatorics #2 N p. 199	L4 1.4 The Table Game #4 L p. 23	W3 A Very Strange Neighborhood W p. 13	N9 3.2 Exponents and Prime Factorization #1 N p. 39	G3 1.3 Constructions with Acute Angles #2 G p. 25
5	N10 4.2 Binary Writing #2 N p. 89	P1 1.1 Four Number Cubes P p. 2	W4 Collage of Problems #2 (Lesson 1) W p. 24	N11 8.1 Decimals #1 N p. 259	ADJUSTMENT DAY
6	N12 9.1 Cartesian Graphs N p. 295	L5 2.1 The Relations Game #1 L p. 37	W5 Collage of Problems #2 (Lesson 2) W p. 25	N13 7.2 Percent #2 N p. 227	G4 1.4 Polygon Constructions #1 G p. 30
7	N14 4.3 Binary Writing #3 N p. 96	L6 2.2 The Relations Game #2 L p. 50	N15 2.2 One-Digit Distance #2 N p. 23	N16 5.2 Composition N p. 148	G5 1.5 Polygon Constructions #2 G p. 35
8	N17 6.3 Combinatorics #3 N p. 210	ADJUSTMENT DAY	W6 Collage of Problems #3 (Lesson 1) W p. 36	N18 5.3 Fractions #2 N p. 158	G6 1.6 Polygon Constructions #3 G p. 40
9	N19 1.2 Hand-Calculator Puzzles #2 N p. 10	L7 2.3 The Relations Game #3 L p. 62	W7 Collage of Problems #3 (Lesson 2) W p. 37	N20 8.2 Division with Decimals #1 N p. 268	G7 1.7 Polygon Constructions #4 G p. 46

10	**N 21** 9.2 Rounding Operation #1 N p. 308	**L 8** 2.4 The Relations Game #4 L p. 73	**W 8** Collage of Problems #4 (Lesson 1) W p. 48	**N 22** 3.3 Exponents and Prime Factorization #2 N p. 47	**P 2** 2.1 Population Growth Rate #1 P p. 33
11	**N 23** 4.4 Review of Positional Systems N p. 107	ADJUSTMENT DAY	**W 9** Collage of Problems #4 (Lesson 2) W p. 48	**N 24** 8.3 Decimals #2 N p. 274	**G 8** 1.8 Triangle Constructions #1 G p. 52
12	**N 25** 7.3 Percent #3 N p. 238	**L 9** 2.5 The Relations Game #5 L p. 78	ADJUSTMENT DAY	**N 26** 5.4 Unit Fractions N p. 166	**G 9** 1.9 Triangle Constructions #2 G p. 57
13	**N 27** 4.5 Base $\hat{2}$ Abacus #1 N p. 119	**L 10** 2.6 The Relations Game #6 L p. 91	**W 10** Collage of Problems #5 (Lesson 1) W p. 58	**N 28** 8.4 Division with Decimals #2 N p. 281	**G 10** 1.10 Triangle Constructions #3 G p. 62
14	**N 29** 4.6 Base $\hat{2}$ Abacus #2 N p. 130	**L 11** 3.1 The String Game #1 L p. 96	**W 11** Collage of Problems #5 (Lesson 2) W p. 58	**N 30** 3.4 Counting Positive Divisors #1 N p. 54	**P 3** 2.2 Population Growth Rate #2 P p. 44
15	**N 31** 7.4 Percent #4 N p. 251	**P 4** 1.2 Probability Problems P p. 15	ADJUSTMENT DAY	**N 32** 3.5 Counting Positive Divisors #2 N p. 65	**P 5** 2.3 Population Pyramid P p. 52
16	**N 33** 9.3 Rounding Operation #2 N p. 315	**L 12** 3.2 The String Game #2 L p. 103	**W 12** Collage of Problems #6 (Lesson 1) W p. 68	**N 34** 5.5 Composition and Addition of Fractions N p. 172	**P 6** 2.4 Comparative Population Graphs P p. 61
17	**N 35** 9.4 Linear Programming #1 N p. 324	ADJUSTMENT DAY	**W 13** Collage of Problems #6 (Lesson 2) W p. 68	**N 36** 8.5 Minicomputer Golf with Decimals N p. 285	**G 11** 1.11 Introduction to the Protractor #1 G p. 68
18	**N 37** 9.5 Linear Programming #2 N p. 334	**L 13** 3.3 The String Game #3 L p. 108	ADJUSTMENT DAY	**N 38** 5.6 Guess My Rule N p. 183	**G 12** 1.12 Introduction to the Protractor #2 G p. 76

4. Commonly Asked Questions about the CSMP Curriculum - and Suggested Answers

As you teach the CSMP curriculum (or even as you prepare to teach it) there will surely be some basic questions that come to your mind. They are the kind of questions that not only you as a teacher, but also parents and supervisors will ask about any curriculum that appears to be different from a usual program.

The following questions and answers are listed as an aid to you in understanding our curriculum and in responding to those who request information about it.

1. Q: FOR WHAT SORT OF STUDENT IS THE CSMP ELEMENTARY CURRICULUM DESIGNED?

A: It is designed for all children in the schools. Each learning situation is designed to involve all students at their own levels in such a way that even the slowest can succeed in dealing with some aspects of the situation that challenge them. In fact, it has been found that such a "pedagogy of situations" (see Question 2) is particularly effective with the so-called "slow" learners, and recent research suggests it might be effective also with educable mentally handicapped children.

2. Q: BRIEFLY, HOW DOES CSMP DIFFER FROM A TRADITIONAL MATHEMATICS CURRICULUM?

A: The core content of CSMP is not revolutionary. The program aims to inculcate notions and facts which have always been considered important in mathematics. The major differences are in method. Traditional teaching takes pride in disciplining the student to be submitted to the usual algorithms. CSMP students are involved as well in many other real mathematical activities, with understanding at their level.

For that purpose CSMP uses new teaching aids, the nonverbal languages of strings and arrows and of the Papy Minicomputer, which are themselves both pure and applied mathematics. These mathematical objects (graphs, arrows, etc.) are used as aids rather than as ends in themselves.

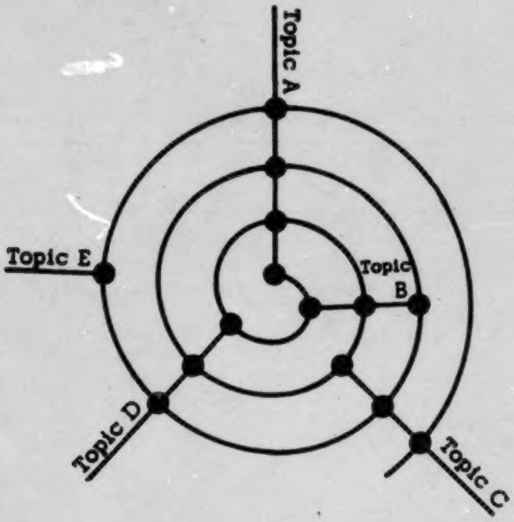
A pedagogy of situations is used to permit children to participate with creative freedom in the learning process. The students are faced with situations and are led by their reactions to important ideas in mathematics. Thereby, students learn to mathematize situations, a process which is of fundamental importance whenever mathematics is applied.

CSMP also makes use of a spiral approach (see Question 3) rather than the mastery approach found in most mathematics curricula.

3. Q: WHAT IS THE SPIRAL APPROACH? HOW DOES IT WORK?

A: The learning process occurs through many interrelated experiences. But no single experience, particularly a 40 to 50 minute mathematics lesson, is an end in itself. The CSMP curriculum was developed with this in mind. It is neither intended nor expected that each child will meet the full challenge of a particular situation in any given lesson. Furthermore, we have found that it is most effective to vary the situations and topics from day to day rather than to continue with one type of situation or piece of content until so-called mastery has occurred. Children at this age do not seem to enjoy staying on the same topic for a very long time but prefer to be involved in a variety of situations. Thus, the lessons are scheduled so that different topics will be studied during each week and similar situations appear again and again at different times and levels in a spiral development.

If a spiral curriculum seems strange at first, consider the following illustration: Imagine an academic subject that could be divided into five subtopics. Then imagine a student's progress being represented by a spiral, starting at the center and traveling outwards in the direction of increasing degrees of maturity and sophistication. In the diagram, an intersection point between the spiral and a topic line indicates a lesson on that topic.



The student is first introduced to Topic A, then to B, and to D, followed by a slightly more sophisticated lesson on A, and so on. Topics C and E are not introduced until a little later in the spiral, while Topic B may be of a more terminal nature and is considered completed after a few lessons.

This spiral development, in which a child experiences each of several ideas a little at a time and then proceeds through increasing levels of sophistication as the situations become more challenging, is wholly consistent with CSMP's pedagogy of situations.

Acceptance of the CSMP pedagogy requires a certain act of faith on your part. If you have been accustomed to staying with a given lesson until everybody (or nearly everybody) in the class knows it "cold," then you may be initially very reluctant to leave a lesson at the end of the time suggested, believing that several children "haven't got it yet." By keeping the spiral idea in mind, you may become more comfortable with these time schedules. Although it may appear to you that no learning is taking place in a sequence of lessons, you may be surprised that suddenly even your slower students are understanding the ideas when they encounter them later in the spiral. Whereas with a mastery approach, presented in a purely individualized mode, slow students often linger most of the school year trying to "master" low level skills (with generally poor results); the spiral approach gives these students the opportunity to benefit from the richness of the situations presented throughout the curriculum. In this manner, children are allowed to become involved in interesting and enjoyable situations which are modeled by important mathematical notions. They are not deprived of a variety of experiences just because they haven't mastered every bit and piece along the way.

4. Q: ISN'T THIS THE "NEW MATH"?

A: There is a current popular belief that all school mathematics today is either "new math" or "old math." Neither of these labels is well defined. "New math" usually denotes the programs that heavily emphasize the teaching of sets; introduce technical mathematical language for concepts such as "commutativity," "associativity," "distributivity," and so on; and that teach "concepts" before "skills". The "old math" refers to programs of the "good old days": memorization of the multiplication tables and complete emphasis on paper and pencil drill in calculations.

The CSMP curriculum is definitely not the "new math" nor the "old math" by any definition. It uses little technical language, yet achieves the understandings sought by the "new math". It uses no repetitive drills, yet elicits the ability to calculate stressed by the "old math." But these achievements are just by-products of a more pervasive goal: to help children use their curiosity, their intuition, their powers of reasoning to formulate and think about interesting problems and situations.

5. Q: IS THE STUDENT REALLY STUDYING MATHEMATICS IN THE CSMP CURRICULUM?

A: Yes. The subject matter of mathematics involves much more than calculation with numbers, and the methods of mathematics include more than the paper and pencil algorithms. CSMP believes that the number systems are very important and that it is essential to introduce students to the world of numbers. However, the intelligibility of that world is not simply a matter of computation. The CSMP curriculum is filled with experiences that directly and indirectly relate to the wide world of numbers, that support important internal relations in the world of numbers, and that encourage logical reasoning. The deep understanding of mathematics is not always verbal but is very often pictorial and CSMP tries to respect that aspect.

6. Q: WHY USE THE LANGUAGES OF STRINGS, ARROWS, AND THE PAPY MINICOMPUTER?

A: It is surprising that the language of strings was not used earlier and more extensively in the teaching of mathematics. Mathematicians have used strings for a very long time and they support notions so fundamental and natural that they date back to Aristotle. The scope of the string diagrams is broader than mathematics, the string diagram being a marvelous instrument for logical thinking. String diagrams permit the understanding of a situation at a glance.

Arrow diagrams are probably the most important teaching aid in CSMP. They are used everywhere in our daily lives: signals, advertisements, directions, and so on. Why not use them in teaching mathematics?

Arrows are a pictorial way to represent relations. Relations have always been important in mathematics, but in the past have seemed to be something rather abstract, elusive, and mysterious. With arrows they become vivid, concrete, and easy to understand.

The Minicomputer allows very young children to become familiar with numbers and operations on them (especially large numbers and the more difficult calculations) before they are able to work with them on paper. Later, and more importantly, the Minicomputer is used as a vehicle for posing situations involving mental arithmetic and for examining the workings of the arithmetic operations. It provides the experience necessary so that rather than memorizing an algorithm and applying it to situations, the situations give rise to the algorithm.

7. Q: IF THE STUDENTS USE THE MINICOMPUTER, WILL THEY BE ABLE TO CALCULATE WITHOUT IT?

A: This question suggests that the only purpose of the Minicomputer in the CSMP curriculum is to perform calculations. To be sure, the Minicomputer does enable very young children to deal with "large" numbers and to perform relatively difficult calculations long before they can write the paper and pencil algorithms for the calculations. But very soon these experiences with the Minicomputer suggest how the paper and pencil algorithms come about, and from that moment the Minicomputer is no longer regarded as a calculating tool. Instead, for the remainder of the curriculum, the Minicomputer assumes its primary roles: as a source of very interesting problems and games, as an effective vehicle for stimulating mental arithmetic, as a teaching device for introducing more advanced topics such as negative numbers and decimal numbers, and, in general, as an alternate representation of numbers.

8. Q: WHAT IS THE CSMP POSITION ON HAND-CALCULATORS AND WHAT ROLE DO THEY PLAY IN THE PROGRAM?

A: Hand-calculators should be a part of any mathematics curriculum. As more and more children have hand-calculators, these tools will become incorporated in the elementary and secondary curricula in a natural way with uses very similar to those of the Minicomputer. In some cases, they may be used as a tool for mechanical computation where the difficulty of the calculation might turn an otherwise interesting problem into an exercise in drudgery. In other cases, they may be used as the context for posing situations leading to the discovery of important mathematical properties or providing opportunities for interesting exercises in mental arithmetic.

Hand-calculators are of particular benefit to "slow" learners. Many slow learners have mechanical difficulties rather than problems in thinking about a situation. The hand-calculator alleviates this problem.

Some lessons in the CSMP curriculum require only two calculators, while others suggest that each student have one. Sometimes lessons of this latter sort can be modified so that students may share hand-calculators or may simply think about how a hand-calculator works. For some lessons, however, every student should be provided with a hand-calculator. In either situation, students with access to hand-calculators should always be encouraged to bring them to class.

9. Q: WHY IS THERE A LACK OF DRILL AND LITTLE EMPHASIS ON "SKILLS"?

A: CSMP believes that training for rote skill should never precede understanding. The avowed purpose of drill in arithmetic is to acquire the "skills," that is, the skills of calculation. Certain numerical skills are essential, but, unfortunately for school children, the usual drill techniques for acquiring them are dull, stultifying, and sometimes even counterproductive. The CSMP curriculum appears to place little emphasis on "skills" because it does not resort to drill. But as a matter of fact, the essential skills of both mental and written calculation are learned almost inadvertently in response to the many situations where the ability to calculate is needed in order to respond to the situation: to win the number game, to identify the secret number of the detective story, or to build a number road. Calculation is always considered as a tool for doing interesting things, rather than as a chore done for its own sake. See also the comments on Evaluation, Section 2 (d) of the General Introduction.

10. Q: HOW DO STUDENTS LEARN TO USE MATHEMATICS IN PRACTICAL SITUATIONS?

A: For students of elementary school age, a situation that intrigues them and catches their imagination or their curiosity will certainly motivate them to learn whatever is necessary in order to deal with the situation. But situations that adults consider "practical" very often are outside the realm of interest of school children and do not provide motivation for learning. CSMP has found that students will indeed learn to use mathematics in situations they consider worthy of their attention. As they approach adulthood they soon will consider "practical" situations to be most worthy of their attention, but in the meanwhile, they are more interested in games, fantasy, puzzles, stories, and so on.

11. Q: WHY ARE WE TEACHING PROBABILITY?

A: Next to arithmetic, probability has the most direct application to life in such areas as economics, meteorology, data processing, insurance, most of the natural sciences, and most of the social sciences. Probability differs from many areas of mathematics in that it is applied directly. One does not need to combine knowledge of probability with any deep theoretical knowledge in order to apply it to "real world" problems. One encounters statistics nearly every day of one's life, and the better developed one's intuitive sense of probabilistic situations is, the less likely one is to be "duped" by statistics.

12. Q: HOW ARE TELLING TIME, MONEY, AND THE METRIC SYSTEM TREATED?

A: Telling time is not included in the CSMP curriculum. It is our belief that telling time is more naturally a part of a language program than of a mathematics program, although it is frequently included in the mathematics curriculum. Since there are many excellent materials available to help you teach telling time, we urge you to teach this topic using those methods and materials you particularly like or that have proved successful for you in the past.

The topic of money is first introduced in the CSMP curriculum in lessons involving combinations of coins. It is treated again in the Upper Primary Grades and the Intermediate Grades as a support for the teaching of decimal numbers. Although money is not treated as a separate topic, these situations provide considerable experience with our monetary system. For students who need more exposure to this topic, teachers should feel free to develop supplementary materials in order to provide extra practice.

Metric units are used whenever there is an occasion to measure in standard units. Activities involving the metric system are begun in the First Grade Program and continue throughout the Geometry and Measurement strand of the Upper Primary Grades Program and the Intermediate Grades Program. We recommend that you do not ask the students to convert from the English to the metric system or vice versa (inches to centimeters, yards to meters, Celsius to Fahrenheit, grams to ounces, and so on). Since the United States has not yet "gone metric," your students will still need to be familiar with the English system. We suggest that you use whatever materials and methods you have found successful in the past for teaching this topic, but we repeat our recommendation that you not ask the students to convert from one system to the other.

13. Q: WHAT IS THE ROLE OF STORYBOOKS AND STORY-WORKBOOKS IN THE PROGRAM?

A: The storybooks in the collection STORIES BY FREDERIQUE include twenty-four stories for children, presented in colorful newsprint booklets. They are intended to appeal to children in the 5-14 age bracket. The stories are completely independent of each other and need little or no external explanation. They may be used for individual or collective reading, at home or at school. The story-workbooks, as the name implies, are a cross between storybook and workbook. They present more problems for the student to solve than do the storybooks, but within a story setting. The story-workbooks comprise the collection called THE CSMP LIBRARY. Each of the stories is portrayed by the mathematical languages of dots, arrows, and strings which reinforce the ideas of the stories. A young reader can become personally involved in the stories; there are questions to answer, pictures to relate to the stories, and new adventures to invent for the characters in the stories. The reader discovers that emotions and aesthetic senses can be involved in the world of mathematics just as in the world of people.

Among the goals of the various storybooks and story-workbooks are the following: to familiarize the students with nonverbal mathematical languages and to leave them with the impression that these languages are as natural and useful as the usual language; to introduce some mathematical games; to present detective stories which introduce some interesting mathematical problems; to provide reinforcement for all students and especially for the slower learners.

14. Q: WHAT PROVISION IS MADE FOR INDIVIDUALIZATION IN THE CSMP CURRICULUM?

A: In many of the lessons there are worksheets to be done individually by the students. Sometimes the worksheets are done at the beginning of a lesson, followed by a full group discussion of them. But ordinarily a topic is first developed in a full group setting, followed by individual student work on related worksheets. If "individualization" means a procedure for allowing students to work alone at their individual rates and levels, then the worksheets (and also the workbooks, discussed below) do provide for (partial) individualization. A series of worksheets for a given lesson offers a whole spectrum of levels from very simple to quite challenging. Students all start with the first worksheet (one-star) and proceed at their own pace and levels as far as they are capable. Ths spiral approach accommodates this kind of individual work in the same manner as the full group work; that is, no worksheet is a prerequisite for the content of any succeeding lesson.

Twelve lessons during the semester are devoted to the six IG-VI workbooks. Each workbook is also designed to move through a spectrum of levels of difficulty. We believe that the worksheets and workbooks realize the best sense of "individualization."

15. Q: WHAT PROVISIONS ARE MADE FOR THE VARIETY OF ABILITIES IN A USUAL CLASS?

A: Every lesson includes something for all students; there are opportunities for the slowest students to participate in a meaningful way, and there are challenges for the brightest students. This means that you should not exclude any students from a full group lesson. At the same time, you should not expect every student to understand everything during the course of the lesson. Furthermore, with the spiral development many situations are open-ended and allow students to experience them at levels as high as they are capable.

Evaluation has shown that students traditionally labeled "slow" are doing better in CSMP classes than in non-CSMP classes. We believe this is due to the spiral development, to full group work on mathematical activities, and to the presence in the CSMP curriculum of certain nonnumerical activities in which many supposedly "slow" students can excel.

The slow learners should participate in math with the other students in the class, but they might also benefit from additional small group work. The additional time should not be simply to rehash or drill on the situations presented in class, but rather to provide additional experiences for these students. See the book "Math Play Therapy" Volumes I and II for suggestions of small group work with slow learners.

Additionally, CSMP provides for a variety of ability levels in the student materials. Worksheets and workbooks offer problems in a range from very easy to quite challenging. Indeed, some of the "higher starred" worksheets and pages toward the end of the workbooks may be accessible only to your very best students.

In an extreme case, you might seriously consider the possibility of arranging for exceptionally bright students to take their math with a higher grade level CSMP class.

16. Q: IS IT NECESSARY TO FOLLOW THE SUGGESTED SCHEDULE EXACTLY?

A: This program is not meant to be a static sequence of exercises to be followed rigidly. The lessons in this guide are suggestions and the time allotments are merely approximations based on our own experience teaching the materials. If this will be the first time you are teaching the program and you would like specific suggestions for organizing the lessons, we offer some guidelines and a suggested schedule of lessons. You may wish to follow it at the beginning of the school year and then to deviate from it as the needs of your students dictate.

It is probably best to follow a one-day, one-strand pattern when you organize your own schedule, rather than to spend two or three full weeks "covering" one particular topic. Interchange the strands and the days of the week as you please. In a usual five day week, two days should be devoted to the World of Numbers strand, one day to the Workbook strand, and the other two days should be shared by the P, G, and L strands.

Exercise your own professional judgment in deciding when to stop a particular lesson. It is better to stop a lesson when student interest is high than to teach so methodically that everyone in the class understands everything, but is completely bored. In order to be successful with this curriculum, adjust the schedule to the ability, maturity, and interests of your class.

17. Q: HOW DO I HANDLE STUDENTS WHO HAVE AN EXTENDED ABSENCE OR STUDENTS (NEW TO CSMP) WHO TRANSFER INTO MY CLASS DURING THE YEAR?

A: The spiral approach allows a student to miss one or two weeks of school without much "catch up" work needed. When the student returns, either you or classmates can help with any material that is deemed necessary.

If several months of classes have been missed, you might need to treat the student as you would a transfer student new to CSMP. Give the student a little extra attention during a lesson especially when the questions are review for most of the class.

In most cases, the differences between new and veteran students tend to fade rather quickly. One exception might be in lessons involving the Minicomputer. Veteran students who are unusually adept with the Minicomputer very often are good tutors for the transfer students in the use of this abacus. In addition, the storybook TWO BY TWO is suggested as an aid for new students who are unfamiliar with the Minicomputer. Some of the storybooks and story-workbooks written for younger children can be used to introduce new students to the languages of strings and arrows.

It is probably best not to send worksheets and/or workbooks home to a child. Without participating in the collective lessons that introduce the worksheets and workbooks, it would be very difficult for the student to proceed.

18. Q: HOW DO I HANDLE WHOLE CLASSES WHICH ARE NEW TO CSMP?

A: CSMP does not assume that kindergarten and first grade students have any prior experience with CSMP. The Kindergarten and First Grade Programs will need no adaptation for classes which are new to CSMP.

Second grade classes which are new to CSMP will need to be introduced to the CSMP languages. For this purpose, a suggested adaptation of certain second grade lessons is included in the CSMP Mathematics for the Upper Primary Grades (UPG) Part I teacher's guide.

Third grade classes which are new to CSMP will need to become fluent in the CSMP languages before beginning many of the lessons in UPG Part III. A special schedule has been prepared for these classes which consists of prerequisite lessons and workbooks from UPG Part I and II followed by a modified UPG Part III and IV program.

Fourth grade classes new to CSMP spend one semester with CSMP Mathematics for the Intermediate Grades (IG), Entry Module I, before beginning the regular fourth grade program. Similarly, fifth grade classes new to CSMP spend approximately one semester with an Entry Module II before beginning a modified fifth grade program.

19. Q: WHAT SHOULD I DO WITH MY CLASS IF THEY DID NOT FINISH THE PREVIOUS YEAR'S SCHEDULE?

A: No special measures are required. Each part of the CSMP curriculum begins with introductory lessons. In addition, the spiral approach permits individual participation at the student's own level of understanding, even if that level is not quite as high as it might have been if the previous year's schedule had been completed.

20. Q: HOW CAN I ASSESS STUDENT PROGRESS AND ASSIGN GRADES?

A: CSMP does not believe that it is either desirable or reasonable for teachers to attempt to evaluate student mastery of skills from day to day or even from week to week. On the other hand, teacher, parents, and even the children themselves benefit from some monitoring procedure that records involvement in activities and assesses progress through the program.

Most teachers have their own monitoring devices with which they are comfortable. It is recommended that such devices be continued.

21. Q: HOW DO CSMP STUDENTS DO ON STANDARDIZED TESTS?

A: While CSMP disagrees with the philosophy underlying standardized testing, test results are available to us from several sources. The results indicate that CSMP classes generally do as well or better on tests that do non-CSMP classes. It should be pointed out that standardized testing and traditional programs are correlated and that the CSMP student has been exposed to a great deal of mathematics that is not taught in more traditional programs or considered in the construction of standardized tests. There are also many unmeasurable or unmeasured areas such as the enjoyment of mathematics, strategic thinking, organizational ability, and depth of understanding of the methods and uses of mathematics where we feel CSMP students have a definite advantage. See also the comments on Evaluation in Section 2 (d) of the General Introduction.

22. Q: WHAT PROBLEMS WOULD CSMP STUDENTS HAVE IF THEY RETURN TO A MORE TRADITIONAL PROGRAM IN THE NEXT SCHOOL YEAR?

A: There should not be problems for CSMP students who return to a traditional program. Test results indicate that CSMP students are at or above the level of non-CSMP students in material covered in traditional courses. In addition, CSMP students have encountered many situations that are not introduced in traditional programs nor tested for on standardized tests. (See Question 21) In particular, their reasoning ability should be much sharper than their non-CSMP counterparts.

There are, of course, differences in timing as to when different programs introduce the standard algorithms and often CSMP's introduction will occur later than in some of the other programs. The reason for this is our belief that it is essential that the creative freedom of students not be impaired by a premature emphasis on the standard algorithms for adding, subtracting, multiplying, and dividing. We believe that many of the present computational difficulties experienced by elementary and junior high school students were caused by their spending an inordinate amount of time trying to master the algorithms too early in their mathematical education. This shuts the students off from other experiences necessary for their mathematical growth. Students need plenty of time to explore the world of numbers. Mastering an algorithm should be the last step of this exploration, a sort of synthesis of previous experience. Furthermore, as electronic hand-calculators become available to nearly everyone, algorithmic mastery as a prerequisite to other numerical experiences should become less important.

CSMP students can usually solve a problem before having learned the standard algorithm. If they enter a program that expects an algorithm they have not learned, they are still more than adequately prepared for the algorithm and should have no trouble quickly learning to use it.

CSMP introduces the usual addition algorithm in the second grade, one of the two standard subtraction algorithms in the third grade, the multiplication algorithm in the fourth grade, and the division algorithm in the fifth grade.

23. Q: WHAT ARE SOME WAYS TO ORGANIZE MATERIALS AND MANAGE MY CLASS?

A: It is very important to organize your CSMP materials before school begins. Decide where you will store the materials once they are organized. Filing drawers, shelves, boxes, or closets are all satisfactory as long as you can easily get to the materials. Take them all out of their shipping boxes and spread them on several tables and/or desk tops in your classroom so they can be organized into the order you will be using them.

Workbooks: The teacher's guide is needed to find the order in which the workbooks are used. Locate the Workbook strand column in the Master Suggested Schedule of Lessons and collect all the workbooks in the order in which they are listed. After organizing the workbooks in order, place them in an upright position in your file drawer, box, shelf, etc., so that the first workbook to be used is at the front of the stack.

Worksheets: There is a separate worksheet booklet for each student and you should decide whether you will store these booklets and distribute them when a particular lesson calls for a worksheet, or whether you will simply let the students keep their own worksheet booklets in their desks.

Each worksheet can be removed from the booklet or left in depending on how you wish to manage worksheets. However, once a worksheet is removed from the booklet it cannot be replaced, so make sure students are careful about tearing off the correct sheets.

Manipulatives: Storage of each manipulative in a separate container, properly marked, is one way to make everything accessible. Another way is to provide a sturdy box (for example, a shoe box) for each student and to put individual manipulative materials in the box, properly labeled with the student's name.

Additional Materials: Be sure that you have plenty of high quality colored chalk, large sheets of unlined white paper, and that your students have colored pencils, pens, or crayons. Plan well ahead to be sure you have any needed materials that are not supplied by CSMP.

Keep all teacher materials where they are easily accessible. You may find the guides easier to use if they are put in looseleaf binders; the books are perforated so that pages can be removed easily. The demonstration Minicomputer boards and a number line should always be on display in the room.

24. Q: SOME OF THE CSMP MATERIAL IS NEW TO ME. HOW DO I LEARN ENOUGH TO TEACH A CLASS?

A: Every CSMP teacher should have been trained by his or her coordinator. Kindergarten and first grade teachers receive a minimum of eighteen hours training, teachers of the Upper Primary Grades Program receive up to thirty-two hours training, and teachers of the Intermediate Grades Program receive up to fifty-six hours training. During the training sessions the CSMP content is taught to new teachers and suggestions are given for organization of their classes.

If the suggested training schedule has been followed, many lessons taken from the CSMP curriculum, or very similar lessons, have been presented. The training schedule also makes use of the spiral approach so that training sessions aim not only to teach CSMP mathematics but to familiarize teachers with the spirit and pedagogy of the CSMP curriculum.

25. Q: WHAT SHOULD BE DONE WHEN THERE IS A SUBSTITUTE TEACHER?

A: Some school systems have included people who frequently substitute in their CSMP training program. When this is not the case, it might be very difficult for the untrained substitute to teach from CSMP materials. One solution is to arrange for another CSMP teacher to teach math to your class while the substitute teacher teaches another subject to that teacher's class. If that cannot be arranged and you will only miss one or two days, you may find certain CSMP material, uncompleted workbooks for example, that the substitute can manage. Otherwise, use those days to cover material not included in the CSMP curriculum, such as telling time. If your absence will be an extended one, arrangements should be made with your coordinator to train a substitute, rearrange schedules to put a CSMP teacher in your class, or otherwise provide for your students.

26. Q: THE PROGRAM MAKES USE OF THE STUDENT'S CREATIVE FREEDOM. HOW MUCH ROOM IS THERE FOR THE TEACHER'S INGENUITY?

A: CSMP not only allows for your ingenuity but depends on it. While the lessons may appear to be very rigid by presenting dialogue for teacher and student, it should be remembered that these are suggested questions, answers, and comments based on our experiences with development classes. Even if you were to follow your part of the "script," your students are likely to offer responses which differ from those we have listed. It is the students' responses that will determine the content and direction of your next question or comment.

In a traditional program your creativity is called upon the first time you develop a particular lesson, and to a lesser extent, as you revise lessons in succeeding years. There is little room for spur-of-the-moment creativity once a lesson begins. The CSMP curriculum, by creating a free and open situation emphasizing the student's creative freedom, continually presents you with unanticipated student responses and calls on your ingenuity at all times.

27. Q: HOW CAN I EXPLAIN CSMP TO PARENTS?

A: The best way would be for your coordinator to organize a workshop for interested parents. The parents can then see the languages of the program in use and have some of the same experiences their children have. Workshops can be of almost any length but anything short of two hours does not allow for much familiarity with the material. In such a workshop, we suggest that the language of the Minicomputer be explained with particular emphasis on its use in presenting situations where mental arithmetic is called upon and in developing algorithms (for example, physically carrying a checker from the ones' to the tens' board in the "8 + 2 = 10" play, paralleling "carrying" in the paper and pencil algorithm). A lesson using the storybook WHERE'S MY NOSE? provides a good example of the power of the language of strings. A nonnumerical example, such as the brother-sister graph, and a numerical example, such as building a road from 5 to 55 using +3 and 5x arrows, will provide an introduction to the language of arrows. The lessons that are the favorites of the children would be excellent for a presentation to parents as well.

Also you might extend an invitation to parents to visit your class during the math period. Many times such an invitation leads eventually to a closer parent-teacher relationship, with the parent offering voluntary services as an aide.

You will find a letter to parents at the end of this section which may be reproduced and sent home with the students.

TO THE PARENTS OF CSMP STUDENTS

Your child is participating in the Comprehensive School Mathematics Program's (CSMP) elementary school curriculum, developed under a grant from the National Institute of Education. While the program cannot be fully explained in a brief note, we hope that this letter will give you some feeling for the substance of your child's mathematics education.

The content goals of the CSMP K-6 curriculum include a development of the standard number systems and their operations (including an early introduction to the integers, rational numbers, and decimal numbers) interwoven with a development of geometry and measurement. On an equal footing with the above, the CSMP curriculum includes a strong component of probability, statistics, and combinatorics which extends throughout grades 2-6. Powerful pedagogical devices are used, such as several nonverbal languages. These make the unifying ideas of the program accessible to very young children.

These languages include:

1) The Language of Strings This "language" mimics the basic methods of collecting and classifying data related to the problems at hand. That is, it deals with the fundamentally useful and important mathematical notion of set.

2) The Language of Arrows This graphical "language" models the process of comparing and analyzing sets and operations on them. That is, it deals with the mathematical notions of relations and functions including, but not restricted to, ordering relations and the numerical processes of adding to, subtracting from, multiplying by, and sharing equally among.

3) The Language of the Papy Minicomputer The Papy Minicomputer, an extremely effective paper abacus invented by Georges Papy, models the positional structure of our system of numeration and lends itself to a multitude of numerical algorithms, including the standard algorithms for the basic numerical operations.

Among other things, these languages allow for the development of the essential operations on numbers in a natural way based on previous experiences with the languages. They also provide a context in which situations arise that call for the repeated use of arithmetic operations without resorting to drill exercises, which most children find boring.

Another tool, not unique to but used by CSMP, is the hand-calculator. The hand-calculator allows the child to study problems that call for calculation beyond the child's ability or in which the calculation is so extensive as to cause the child to lose interest. The calculator is also used in a way similar to the use of the languages mentioned above. Properly used, it can help lead to the development of important mathematical ideas and present situations for the interesting use of mental or "paper and pencil" arithmetic.

The content is learned in an atmosphere of constant applications utilizing a "pedagogy of situations."[†] These vary from simple story situations in the early primary grades to challenging applications and nontrivial simulations of real world problems in the upper grades.

One of the tenets of CSMP's development philosophy is that mathematics is a unified whole and should be learned as such. As a result, along with CSMP's concern with the variation in ability levels in the typical heterogeneous class, the curriculum is structured in a spiral format throughout the K-6 program. The content is completely sequenced in spiral form so that each student is brought into contact with each area of content continuously throughout the program. This consciously avoids atomizing the content and mastering each bit before continuing to the next. Rather, each student spirals through repeated exposures to the content, building interlocking experiences of increasing sophistication.

[†] Inspired by the work of the mathematician and psychologist, Caleb Gattegno.

Another tenet of CSMP's development philosophy, based on years of experimentation in the classroom, is that no single method of classroom management suffices to meet the needs of every student. Hence, the program is constructed to allow numerous opportunities for whole class participation, small group participation, and individual independent experiences. Individualization throughout the K-6 program is achieved through the independent work on workbooks and worksheets of varying levels of difficulty. Due to the spiral nature of the curriculum, each student can engage in a workbook without losing touch with the progress of the class while at the same time working to the limits of his or her ability.

A feature unique to the CSMP program is a collection of storybooks written by Frédérique Papy (STORIES BY FREDERIQUE) and a collection of story-workbooks (THE CSMP LIBRARY). These probe the affective domain of learning and bring children into contact with areas of mathematics until now unexplored by them.

CSMP does not have its own system of student assessment and progress reporting. While standardized testing indicates that CSMP students develop the "basic skills" as early or earlier than do non-CSMP students, we hope you will not be anxious about your child's development of these "skills". Each child is an individual and should not be judged against norms but on what he or she can and does accomplish. Many extremely important areas -- development of self-confidence, ability to think and reason, etc. -- are very difficult to measure and grade. If you have any questions regarding the progress of your child, his or her teacher is in the best position to answer those questions.

THE CSMP STAFF

THE
WORLD
OF
NUMBERS

Comprehensive School Mathematics Program

CSMP Mathematics for the Intermediate Grades, Part VI

The World of Numbers

CEMREL, Inc.

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THE WORLD OF NUMBERS

INTRODUCTION

By this time, your students have had a rich variety of experiences with various kinds of numbers and with operations and relations on them. These include encountering both positive and negative integers, decimal numbers, some rational numbers, numerical functions (e.g., $2\times$; $+3$; $+10$; -5; $\frac{3}{4}\times$), order relations (e.g. $<$ and $>$), and the notions of multiples and positive divisors of a given number. They have been introduced to "paper and pencil" algorithms for addition, subtraction, multiplication, and division of whole numbers and decimal numbers, and they have had extensive experiences using systematic methods for multiplication, division, and addition of rational numbers (fractions) in preparation for algorithms.

In the present strand, these previous numerical experiences will be revisited, extended, and deepened. New situations will be presented, including the introduction of several fascinating extensions and modifications of numerical games: Guess My Rule (Sections 5.4 and 5.6); Hand-Calculator Puzzles (Chapter 1); Binary Codes (Sections 4.1, 4.2, and 4.3); and Detective Stories (Section 2.2 and Section 3.1); and new numerical situations: One-Digit Distance (Chapter 2); a Base $\hat{2}$ Abacus (Sections 4.5 and 4.6) and Cartesian Graphs (Chapter 9). Old and new concepts will be explored in the context of these situations.

The first paragraph above, describing some of your students' previous numerical experiences, simply catalogues some of the realms of the world of numbers your students have visited. It does not indicate the travel routes nor the vehicles used to travel them.

The CSMP curriculum stresses the unity and continuity of the growth of mathematical ideas and concepts. This is reflected in numerous ways in this strand, as in the others. For example, CSMP employs a spiral approach, not pausing for mastery of each lesson but rather allowing students to encounter the elements of each content strand in different situations, interspersed with other strands throughout the year. It is important to recognize this approach consciously. If you strive for mastery of each single lesson, you will find yourself involved in a great deal of redundancy as the year progresses.

Further, CSMP presents the content in a situational framework. That is, a "pedagogy of situations" is employed which engages students in rich problem solving activities. These offer the opportunity both to develop necessary numerical skills and to gain deeper understanding of mathematical ideas and concepts in the world of numbers. At the same time, the situations presented encourage students to develop patterns of logical thinking and strategies for attacking problems.

Perhaps the most important embodiments of the CSMP approach are the three nonverbal languages used throughout its elementary program: the Papy Minicomputer, the language of arrows, and the language of strings. These are the vehicles which make it possible to develop the situations in which the content is presented to students. Further, as students grow in the CSMP curriculum, these languages become basic investigatory tools available to explore the content.

It is hard to overstate the value of developing tools which aid in attacking problems as well as in representing situations; tools which are not confined to one area of mathematical content or to one level of the development of content. In the usual elementary curriculum, students must continually learn new languages to express each new piece of mathematics that is studied. Armed with the universally applicable languages of the CSMP curriculum, the students grow more and more familiar with the syntax of these languages and are freed to explore new content directly in terms of these same languages rather than being forced to think of each new mathematical idea as tied to a certain new language. This is definitely not to say that the CSMP students do not learn the usual descriptive language of mathematics. However, in the CSMP approach the usual descriptive language is not the vehicle for learning, but only a means for describing the ideas.

For specific examples of the situations and the ways CSMP uses nonverbal languages in this strand, we refer you to the short introductory remarks preceding each chapter and, in particular, to the lessons themselves.

A WORD ABOUT STANDARD ALGORITHMS

We mentioned above that CSMP seeks to develop the necessary skills in the numerical domain as well as deeper understanding of the underlying mathematical ideas. We are fully in agreement with the thesis that along with the growth of understanding in the world of numbers, there must be a concomitant growth of familiarity and facility with numbers and the operations and relations on them. But facility must not be confused with understanding. They are partners in the growth of mathematical maturity, and it is extremely important that neither be sacrificed for the other. Rather, a balanced growth in each must be maintained.

In the usual elementary curriculum, paper and pencil algorithms for the four basic operations provide a prime opportunity for the abuse of the position stated above. It is undeniably important that a student eventually learn mechanical algorithms for the basic operations. However, if these algorithms are presented prematurely they stunt the student's ability to develop alternative algorithms, to do mental arithmetic, to estimate, and so on.

For example, rather than employing the usual division algorithm to determine the standard name for $1{,}791 \div 9$, it may be easier and more efficient to note that $200 \times 9 = 1{,}800$, so that $199 \times 9 = 1{,}791$ and $1{,}791 \div 9 = 199$. Indeed, built into this way of approaching the problem is a relatively good estimate $(1{,}800 \div 9)$ of the number involved.

If we insist on a mechanistic response to problems such as the one above (i.e. insist on the use of paper and pencil algorithms), we may not only encourage inefficiency in many situations but also may inhibit the development of flexibility necessary for problem solving. On the other hand, if we present to students a rich array of situations in which they interact with numbers and their operations and let the standard algorithms grow naturally as just one facet of the students' experience, we obtain two important results. The students will gain the necessary facility with standard algorithmic procedures and at the same time retain the openness required to respond creatively to new situations in the world of numbers.

NOTE ON THE USE OF HAND-CALCULATORS[†]

The hand-calculator provides another powerful tool for the investigation of problems and situations in the world of numbers. This tool is intimately related to the Papy Minicomputer since a hand-calculator's internal structure is an electronic simulation of the Minicomputer. Although this analogy need not be acknowledged by you or your students, it does serve to indicate that the hand-calculator is indeed a natural tool in the CSMP curriculum.

Hand-calculators are recommended materials for Part VI of the Intermediate Grades Program. They are not included in your complete classroom set; however, they may be purchased from CSMP as separate items. There are extensive opportunities for the use of hand-calculators in the lessons of Part VI, especially in Chapters 1 and 7 of this strand. In fact, some lessons require each student to have access to a hand-calculator and would be practically impossible to teach otherwise. The workbooks present, by way of review, situations involving a hand-calculator and can be understood only from previous hands-on experience with calculators. In addition, the supplementary material for the slow learner, "Math Play Therapy", relies heavily on the use of hand-calculators.

We are firmly convinced through our own experience with children that the hand-calculator is a potent pedagogical tool when used properly. The time is not far off when its presence in the classroom will be as commonplace as the pencil. For the present, CSMP will continue to investigate the use of hand-calculators in the classroom, particularly with slow learners, and will develop situations designed for their use.

[†] See General Introduction, Section 2(f) for a more detailed discussion on hand-calculators.

1

HAND-CALCULATOR PUZZLES

In this chapter, the lessons exploit the hand-calculator to present situations involving mental arithmetic. Basically, students explore various ways to display a given number on a "special" hand-calculator, that is, a hand-calculator with only four number keys ($\boxed{5}$, $\boxed{6}$, $\boxed{8}$, and $\boxed{9}$) and any other key except $\boxed{\cdot}$ or only the keys $\boxed{2}$, $\boxed{8}$, $\boxed{-}$, $\boxed{\div}$, $\boxed{=}$, $\boxed{M+}$, $\boxed{M-}$, $\boxed{MRC}$. An additional challenge is to find a solution pressing as few keys as possible.

CAPSULE LESSON SUMMARIES

Section 1.1:	Put various numbers on the display of a hand-calculator using only the keys $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, $\boxed{9}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$. Find solutions which require pressing keys at most ten times. Explore ways to use this restricted calculator to calculate 10 × a number and to divide a number by 20. Build arrow roads between two given numbers where the arrows must be for +, −, ×, ÷ some number whose digits are 5, 6, 8, or 9. Try to find shortest such arrow roads.
Section 1.2:	Put various numbers on the display of a hand-calculator using only the keys $\boxed{2}$, $\boxed{8}$, $\boxed{-}$, $\boxed{\div}$, $\boxed{=}$, $\boxed{M+}$, $\boxed{M-}$, $\boxed{MRC}$. Find solutions which require pressing keys at most ten times. Build arrow roads between two given numbers where the arrows must be for +, −, ×, ÷ some number whose digits are 2 or 8. Try to find shortest such roads.

N1

1.1 Hand-Calculator Puzzles #1

CAPSULE LESSON SUMMARY

Put various numbers on the display of a hand-calculator using only the keys $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, $\boxed{9}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$. Find solutions which require pressing keys at most ten times. Explore ways to use this restricted calculator to calculate 10 × a number and to divide a number by 20. Build arrow roads between two given numbers where the arrows must be for +, -, ×, ÷ some number whose digits are 5, 6, 8, or 9. Try to find shortest such arrow roads.

MATERIALS

Teacher: Hand-calculator; colored chalk
Student : Hand-calculator; paper; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Write this information on the board.

$$\boxed{5}\quad\boxed{6}\quad\boxed{8}\quad\boxed{9}\quad\xcancel{\boxed{\cdot}}$$

T: The only number keys you may press are $\boxed{5}$, $\boxed{6}$, $\boxed{8}$ and $\boxed{9}$. You may use any other keys except $\boxed{\cdot}$. For example, you may add, subtract, multiply, or divide by any number whose digits are 5, 6, 8, or 9. Try to put 35 on the display of your calculator.

Give the class a few minutes to explore this situation. Observe individual efforts and encourage students who need help getting started.

There are many solutions to this problem. Since we cannot predict which solutions your students will find, we present some of the solutions discovered by students in our development classes.

S: Press $\boxed{6}\ \boxed{\times}\ \boxed{8}\ \boxed{-}\ \boxed{8}\ \boxed{-}\ \boxed{5}\ \boxed{=}$.

Write the student's solution on the board.

$$35:\ \boxed{6}\ \boxed{\times}\ \boxed{8}\ \boxed{-}\ \boxed{8}\ \boxed{-}\ \boxed{5}\ \boxed{=}$$

T: How many times did you press a key?

S: Eight.

T: If I were to charge you one cent each time you press a key, your solution would cost eight cents. Are there other solutions, maybe cheaper ones?

Accept several other solutions before continuing with other numbers. Sometimes ask the class to check a solution by pressing the suggested sequence of keys on the hand-calculator; however, it is not necessary to verify every solution. Some other solutions are presented below.†

$$35:\ \boxed{6}\ \boxed{\times}\ \boxed{8}\ \boxed{-}\ \boxed{8}\ \boxed{-}\ \boxed{5}\ \boxed{=}\qquad 8¢$$

$$\boxed{6}\ \boxed{\times}\ \boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{=}\qquad 6¢$$

$$\boxed{5}\ \boxed{\times}\ \boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{=}\ \boxed{=}\qquad 7¢$$

$$\boxed{5}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\qquad 8¢$$

---

† Throughout this chapter we assume that hand-calculators with an automatic constant feature are used. For a more detailed discussion of hand-calculators, see the General Introduction Section 2(f) on Role and Use of Hand-Calculators.

N1

Continue this activity with other numbers such as 202, 0.5, and 4.5. Several solutions for each number are provided here.

202: $\boxed{5}\ \boxed{\times}\ \boxed{5}\ \boxed{6}\ \boxed{-}\ \boxed{6}\ \boxed{9}\ \boxed{-}\ \boxed{9}\ \boxed{=}$ 10¢

$\boxed{6}\ \boxed{\times}\ \boxed{=}\ \boxed{=}\ \boxed{-}\ \boxed{8}\ \boxed{-}\ \boxed{6}\ \boxed{=}$ 9¢

$\boxed{9}\ \boxed{8}\ \boxed{+}\ \boxed{=}\ \boxed{+}\ \boxed{6}\ \boxed{=}$ 7¢

0.5: $\boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{\div}\ \boxed{8}\ \boxed{=}$ 6¢

$\boxed{9}\ \boxed{\sqrt{\ }}\ \boxed{\div}\ \boxed{6}\ \boxed{=}$ 5¢

$\boxed{5}\ \boxed{+}\ \boxed{=}\ \boxed{M_+}\ \boxed{5}\ \boxed{\div}\ \boxed{\text{MRC}}\ \boxed{=}$ 8¢

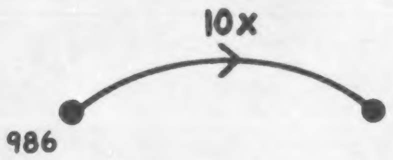
4.5: $\boxed{9}\ \boxed{\sqrt{\ }}\ \boxed{\div}\ \boxed{6}\ \boxed{\times}\ \boxed{9}\ \boxed{=}$ 7¢

$\boxed{9}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{\div}\ \boxed{6}\ \boxed{=}$ 7¢

$\boxed{5}\ \boxed{+}\ \boxed{=}\ \boxed{M_+}\ \boxed{5}\ \boxed{\times}\ \boxed{9}\ \boxed{\div}\ \boxed{\text{MRC}}\ \boxed{=}$ 10¢

Exercise 2

Draw this arrow picture on the board.



T: On the hand-calculator how can we multiply a number by 10 if the only number keys we can use are $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, and $\boxed{9}$?

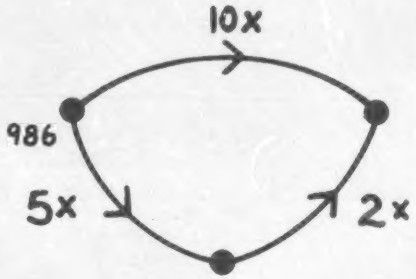
Let the students work independently for a few minutes. Perhaps someone will suggest a detour.

S: Since neither $\boxed{1}$ nor $\boxed{0}$ can be pressed, we must take a detour.

T: What do you suggest we do? Do you know a detour for 10×? Is there a composition of arrows which is the same as 10×?

S: 5× followed by 2×.

Add two arrows to your picture.



S: But we cannot use 2.

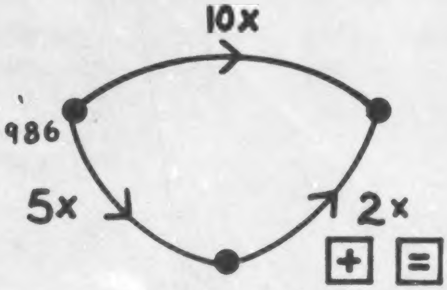
T: I think it's a good idea, though.

T (tracing the 5× arrow and then the 2× arrow): We can multiply by 5 easily. How can we double a number using our restricted calculators?

S: Press $\boxed{+}$ $\boxed{=}$.

N1

Add this information to your picture.

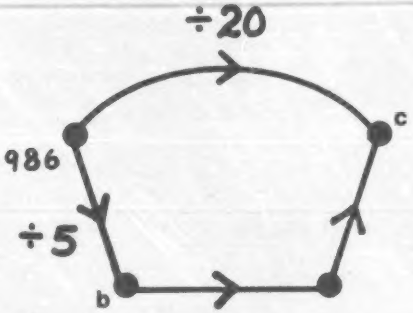


The students should verify that the above method works by putting 986 on the display, pressing $\boxed{\times}\boxed{5}\boxed{+}\boxed{=}$, and observing that the number on the display (9860) is 10 × 986.

Repeat this exercise for ÷20. For ÷20, you might suggest to the class that they look for a detour with three arrows.

T: How can we divide a number by 20 (or find $\frac{1}{20}$ of a number) with our restricted calculators? I have found a solution using a detour with three arrows.

Give the students a few minutes to work independently. Some students may need some help getting started. In that case, add three arrows to your picture as shown below.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (tracing an imagined arrow starting from b and ending at c): ÷5 followed by what is the same as ÷20 (or $\frac{1}{20}$x)?

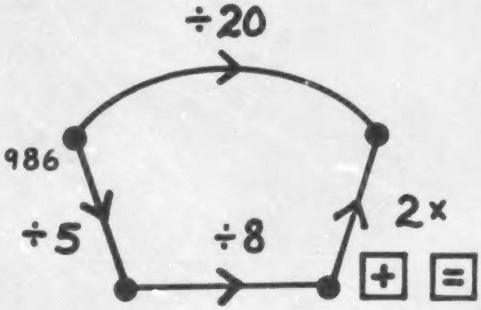
S: ÷4.

T: Good. How can we divide by 4 with our restricted calculators?

S: Divide by 8 and then multiply by 2.

If someone does not suggest this composition of functions, do so yourself.

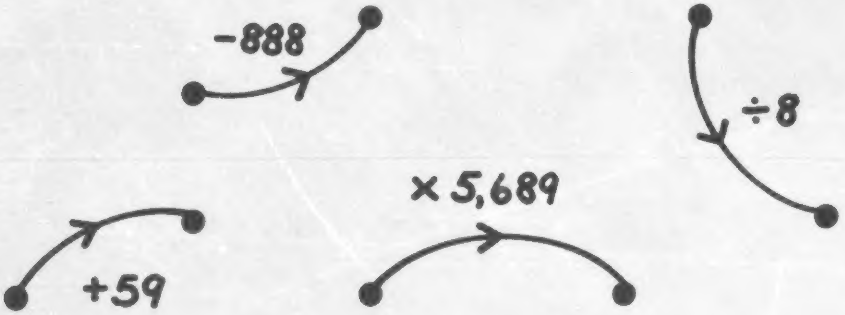
Label the arrows accordingly.



The students should verify that putting 986 on the calculator and pressing $\boxed{\div}\,\boxed{2}\,\boxed{0}$ yields the same answer as putting on 986 and pressing $\boxed{\div}\,\boxed{5}\,\boxed{\div}\,\boxed{8}\,\boxed{+}\,\boxed{=}$.

Exercise 3

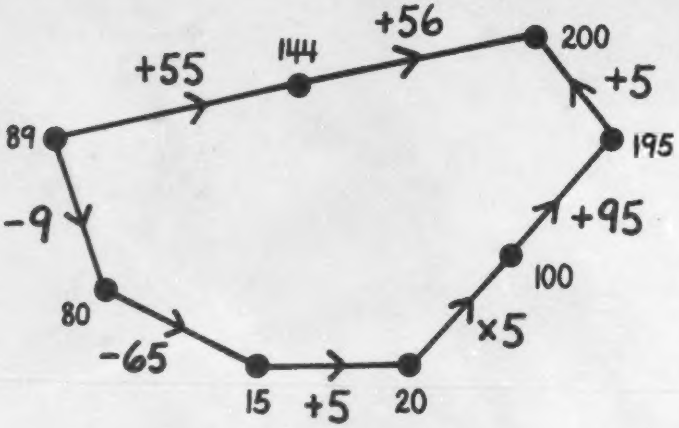
Write this information on the board.



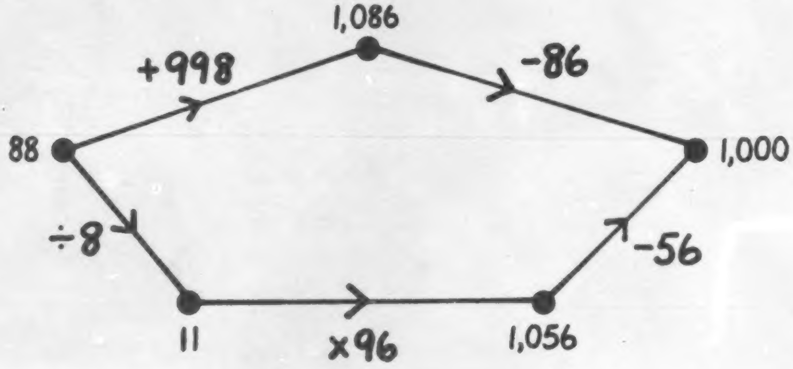
N1

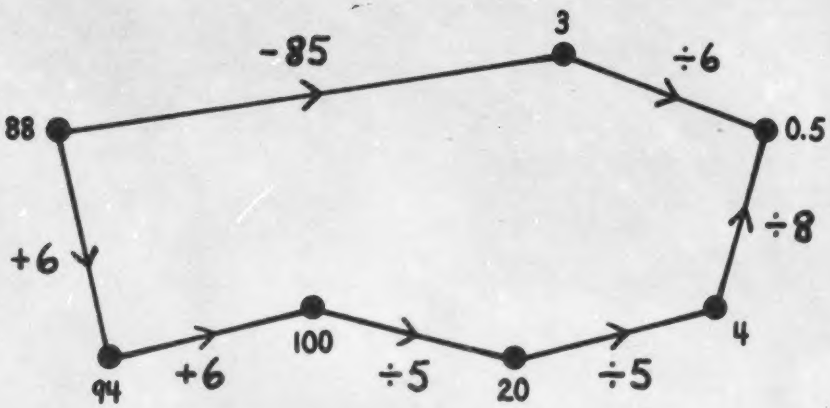
T: In our next problem arrows can be for add, subtract, multiply, or divide by some number whose digits are 5, 6, 8 or 9. A few examples of what arrows could be for are shown on the board. Build an arrow road from 89 to 200 using only these kinds of arrows. Try to find a shortest one.

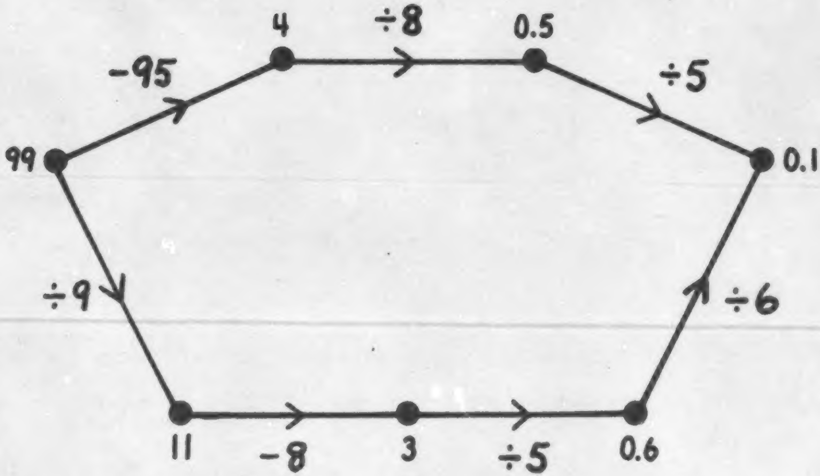
Let the students work independently on this problem. Encourage each student to find at least one solution whether or not it is a shortest road. When most students have found at least one solution, invite some students to draw their roads on the board. There are many possible solutions, two of which are given below.



Continue this activity by asking students to draw other roads, for example, from 88 to 1,000, from 88 to 0.5, or from 99 to 0.1. Some solutions are given below and on the next page.




3
-85
÷6
88
0.5
+6
÷8
100
4
+6
÷5
÷5
94
20


4
÷8
0.5
-95
÷5
99
0.1
÷9
÷6
11
-8
3
÷5
0.6

N 19

1.2 Hand-Calculator Puzzles #2

CAPSULE LESSON SUMMARY

Put various numbers on the display of a hand-calculator using only the keys $\boxed{2}$, $\boxed{8}$, $\boxed{-}$, $\boxed{\div}$, $\boxed{=}$, $\boxed{M+}$, $\boxed{M-}$, $\boxed{MRC}$. Find solutions which require pressing keys at most ten times. Build arrow roads between two given numbers where the arrows must be for +, −, x, ÷ some number whose digits are 2 or 8. Try to find shortest such roads.

MATERIALS

Teacher: Hand-calculator; colored chalk
Student : Hand-calculator; paper; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Write this information on the board.

$$\boxed{2}\quad\boxed{8}\quad\boxed{-}\quad\boxed{\div}\quad\boxed{=}\quad\boxed{M+}\quad\boxed{M-}\quad\boxed{MRC}$$

T: The only number keys you may press are $\boxed{2}$ and $\boxed{8}$. Otherwise, you may use only $\boxed{-}$, $\boxed{\div}$, $\boxed{=}$ and the memory keys. Each time you press a key it costs one cent.

Put 100 on the display of your calculator and try to find the cheapest solution.

Give the class a few minutes to explore this situation. Observe individual efforts and encourage students who have difficulty getting started.

Some of the many solutions to this problem are presented in the following dialogue.

S: Press $\boxed{2}\,\boxed{8}\,\boxed{8}\,\boxed{-}\,\boxed{8}\,\boxed{8}\,\boxed{\div}\,\boxed{2}\,\boxed{=}$. That costs 9 cents.

Write the students' solution and the cost on the board.

$$100:\ \boxed{2}\,\boxed{8}\,\boxed{8}\,\boxed{-}\,\boxed{8}\,\boxed{8}\,\boxed{\div}\,\boxed{2}\,\boxed{=}\qquad 9¢$$

Accept several other solutions before continuing with other numbers. Occasionally ask the class to check a solution by pressing the suggested sequence of keys on the hand-calculator; however, it is not necessary to verify every solution.

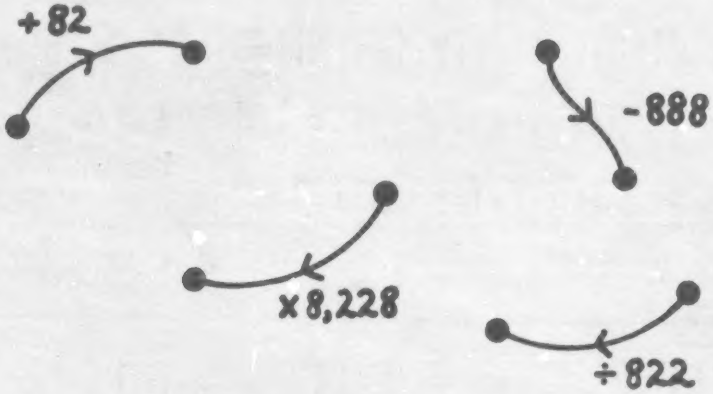
100:	$\boxed{2}\boxed{8}\boxed{8}\boxed{-}\boxed{8}\boxed{8}\boxed{\div}\boxed{2}\boxed{=}$	9¢
	$\boxed{2}\boxed{8}\boxed{8}\boxed{\div}\boxed{2}\boxed{-}\boxed{2}\boxed{8}\boxed{-}\boxed{8}\boxed{=}\boxed{=}$	12¢
	$\boxed{8}\boxed{2}\boxed{\text{M+}}\boxed{8}\boxed{\times}\boxed{2}\boxed{+}\boxed{2}\boxed{\text{M+}}\boxed{\text{MRC}}$	11¢

Continue this activity with other numbers such as 75, 7, 3, 0.1. Several solutions for each number are given here.

75 :	$\boxed{8}\boxed{8}\boxed{8}\boxed{\div}\boxed{8}\boxed{-}\boxed{2}\boxed{8}\boxed{-}\boxed{8}\boxed{=}$	11¢
	$\boxed{2}\boxed{2}\boxed{2}\boxed{\div}\boxed{2}\boxed{-}\boxed{2}\boxed{8}\boxed{-}\boxed{8}\boxed{=}$	11¢
	$\boxed{8}\boxed{2}\boxed{2}\boxed{-}\boxed{2}\boxed{2}\boxed{2}\boxed{\div}\boxed{8}\boxed{=}$	10¢
7 :	$\boxed{2}\boxed{8}\boxed{\div}\boxed{2}\boxed{=}\boxed{=}$	6¢
	$\boxed{8}\boxed{8}\boxed{\div}\boxed{8}\boxed{-}\boxed{2}\boxed{=}\boxed{=}$	8¢
	$\boxed{2}\boxed{\div}\boxed{2}\boxed{\text{M-}}\boxed{8}\boxed{\text{M+}}\boxed{\text{MRC}}$	7¢
	$\boxed{8}\boxed{\text{M+}}\boxed{\div}\boxed{8}\boxed{=}\boxed{\text{M-}}\boxed{\text{MRC}}$	7¢
3 :	$\boxed{2}\boxed{2}\boxed{\div}\boxed{2}\boxed{-}\boxed{8}\boxed{=}$	7¢
	$\boxed{8}\boxed{-}\boxed{2}\boxed{\div}\boxed{2}\boxed{=}$	6¢
	$\boxed{2}\boxed{\text{M+}}\boxed{\div}\boxed{\text{MRC}}\boxed{=}\boxed{\text{M+}}\boxed{\text{MRC}}$	7¢
0.1:	$\boxed{8}\boxed{\text{M+}}\boxed{2}\boxed{\text{M+}}\boxed{\text{MRC}}\boxed{\div}\boxed{\text{MRC}}\boxed{=}\boxed{=}$	9¢
	$\boxed{8}\boxed{\text{M+}}\boxed{\div}\boxed{2}\boxed{=}\boxed{\text{M+}}\boxed{2}\boxed{\text{M-}}\boxed{\text{MRC}}\boxed{\div}\boxed{\text{MRC}}\boxed{=}\boxed{=}$	13¢
	$\boxed{8}\boxed{\text{M+}}\boxed{\text{M+}}\boxed{2}\boxed{\text{M+}}\boxed{\text{M+}}\boxed{2}\boxed{\div}\boxed{\text{MRC}}\boxed{=}$	10¢

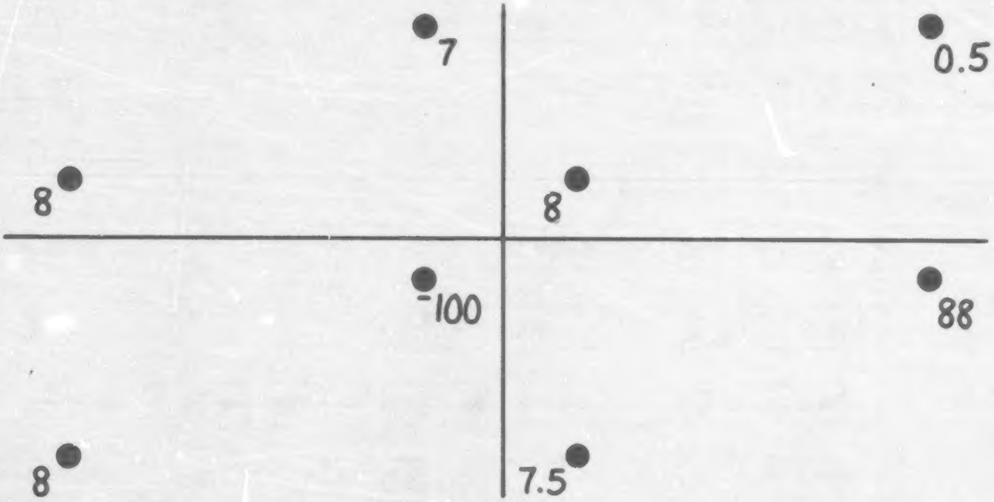
Exercise 2

Draw this arrow picture on the board.

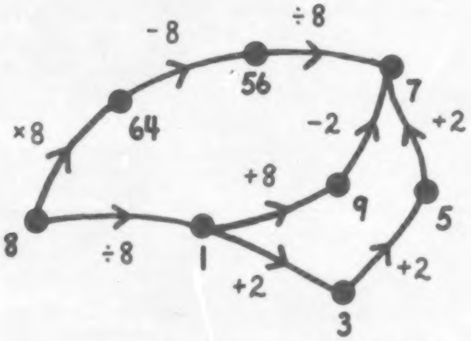


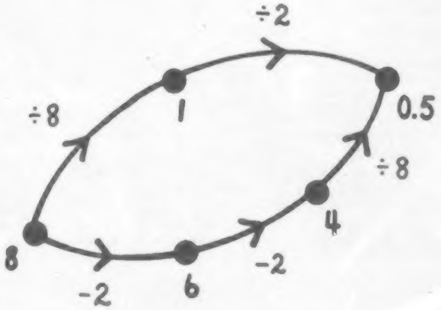
T: In our next problem, arrows can be for add, subtract, multiply, or divide by some number whose digits are 2 or 8. A few examples of what arrows could be for are shown on the board. Build an arrow road from 8 to 7 using only these kinds of arrows. Try to find a shortest road.

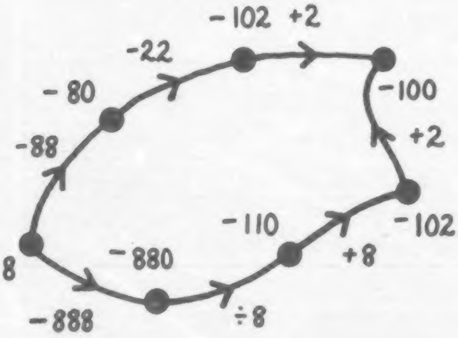
Let the students work independently on this problem. While they are working, write some additional problems on the board.

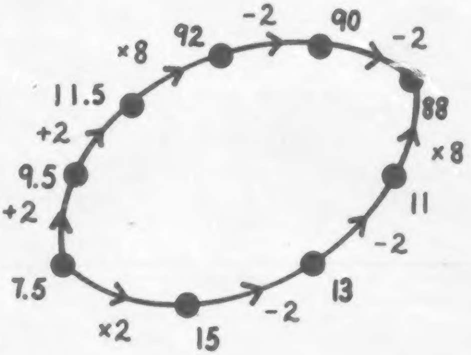


Encourage students to work on as many of the problems as they have time for. Occasionally stop and invite students to draw some of the roads at the board. Some of the many solutions are given below.


-8
÷8
×8
64
56
7
-2
+2
+8
9
5
8
÷8
1
+2
+2
3


÷2
÷8
1
0.5
÷8
4
8
-2
6
-2
-2


-102
+2
-22
-80
-100
-88
+2
-110
-102
8
-880
+8
-888
÷8


-2
90
×8
92
-2
11.5
88
+2
×8
9.5
+2
11
-2
7.5
13
×2
15
-2

2

ONE-DIGIT DISTANCE

This chapter introduces the concept of the "one-digit distance" between integers in an investigation of arrow roads where the arrows must be for +, -, ×, or ÷ some whole number from 1 to 9 (a one-digit number). "Distance" refers to the number of arrows in the shortest possible roads. After some experiences in determining shortest "one-digit" arrow roads between given numbers, the students are asked to find numbers a specified distance from a given number. Othe problems involve the one-digit distance concept in detective stories. Thus, students of the CSMP Intermediate Grades curriculum deal with still another non-geometric notion of distance.

CAPSULE LESSON SUMMARIES

Section 2.1:	Build roads using arrows for +, -, ×, or ÷ some whole number from 1 to 9. Define the one-digit distance between two integers to be the number of arrows in a shortest such road between them. Determine some of the whole numbers whose one-digit distance from 100 is 2.
Section 2.2:	Build roads using arrows for +, -, ×, or ÷ some whole number from 1 to 9. Solve a detective story in which clues involve prime numbers and the one-digit distance between numbers.

N2

2.1 One-Digit Distance #1

CAPSULE LESSON SUMMARY

Build roads using arrows for +, -, ×, or ÷ some whole number from 1 to 9. Define the one-digit distance between two integers to be the number of arrows in a shortest such road between them. Determine some of the whole numbers whose one-digit distance from 100 is 2.

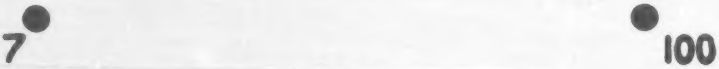
MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils

DESCRIPTION OF LESSON

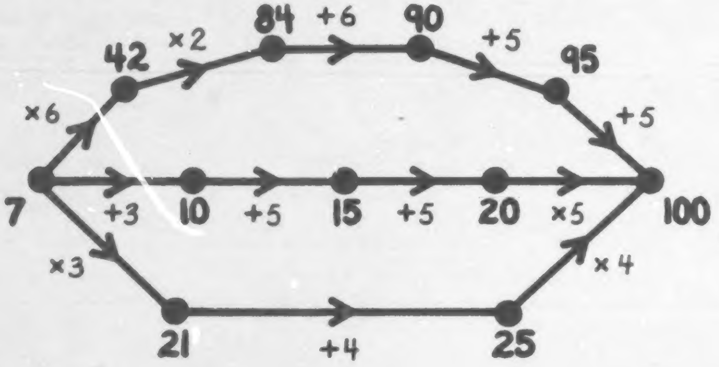
Exercise 1

Draw this picture on the board.



T: Build a road from 7 to 100 only using arrows for +, -, ×, or ÷ some whole number from 1 to 9 (one-digit positive integer).

Be sure the students understand the directions. If necessary, ask for some examples of functions that the arrows could be for: +5, -3, ×9, ÷2, and so on. Let the class work independently for a few minutes. Encourage every student to find some solution; faster students could be asked to find a shortest road. Invite several students to put their solutions on the board. Try to include a road with three arrows.



T: We have a five-arrow road, a four-arrow road, and a three-arrow road from 7 to 100. Do you think it's possible to build a road with fewer than three arrows from 7 to 100? (No)

After several attempts to build a road with only two arrows, the class should suspect that a three-arrow road is the shortest. Accept any reasonable explanations students offer to support this fact.

T: With the information we have on the board, is it easy to find an arrow road from 100 to 7? What is the length of a shortest one?

S: Just use return arrows.

S: A shortest arrow road from 100 to 7 has length 3.

T: Good. A shortest arrow road from 7 to 100 has length 3 and, using return arrows, we see that a shortest road from 100 to 7 also has length 3. Therefore, we say that the one-digit distance between 7 and 100 is 3. We write it this way.

Write this information on the board.

$$d_1(7, 100) = 3$$

Write these problems on the board.

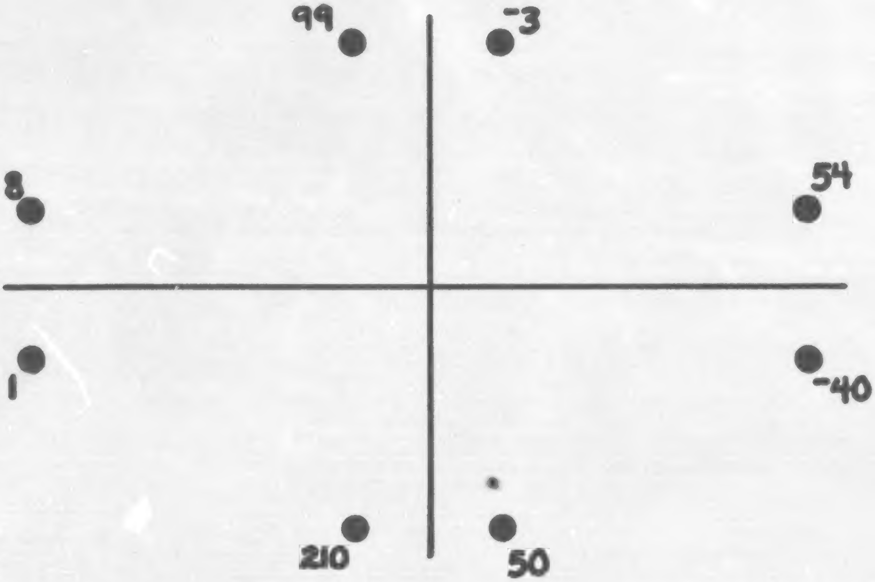
$$d_1(8, 99) = \qquad d_1(54, {}^{-}3) =$$

$$d_1(1, 210) = \qquad d_1({}^{-}40, 50) =$$

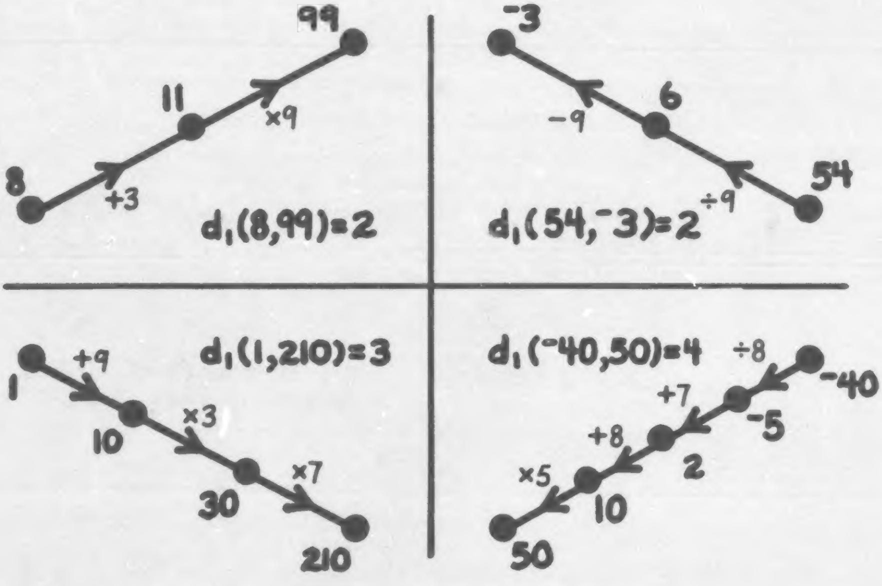
T: For each of these pairs of numbers, find the one-digit distance between them by drawing a shortest road only using arrows for +, −, ×, ÷ some one-digit positive integer.

N2

As the students work independently, draw the appropriate pair of dots for each problem on the board.


99
-3
8
54
1
-40
210
50

After most students have drawn at least two roads, invite students with shortest roads to put their solutions on the board. The one-digit distance and a shortest road for each pair of numbers are presented below. Many other roads are possible.


99
11
×9
8
+3
$d_1(8,99)=2$
-3
-9
6
54
÷9
$d_1(54,^-3)=2$
1
+9
10
×3
30
×7
210
$d_1(1,210)=3$
$d_1(^-40,50)=4$
÷8
-40
+7
-5
+8
2
×5
10
50

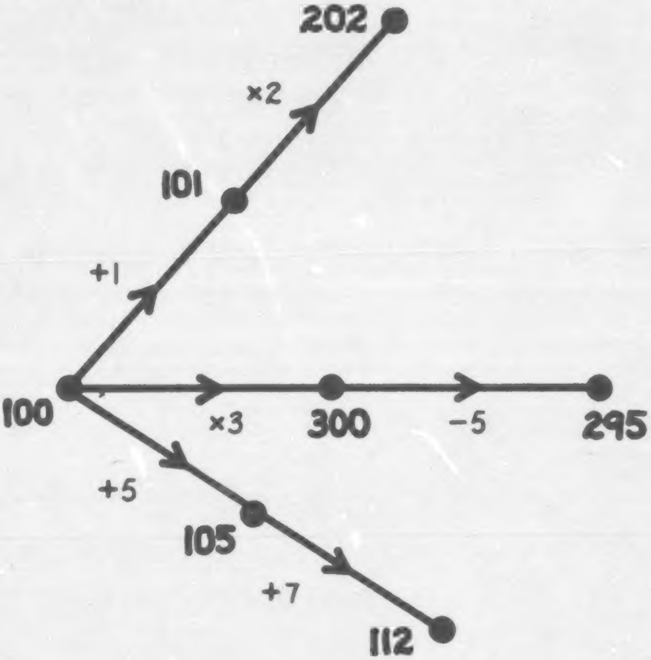
Exercise 2

Write this information on the board.

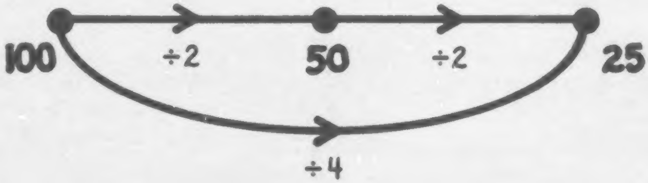
$$d_1(100,\square)=2$$

T: Find some whole numbers whose one-digit distance from 100 is 2.

Let the students work independently for a few minutes. Put several examples that students suggest on the board.


202
×2
101
+1
100
×3
300
−5
295
+5
105
+7
112

Be alert for common mistakes such as ending numbers of roads that are not shortest. For example,


100
÷2
50
÷2
25
÷4

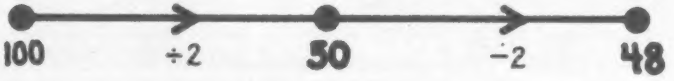
$$d_1(100,25)=1$$

N2

T: Is 48 a one-digit distance of 2 from 100?

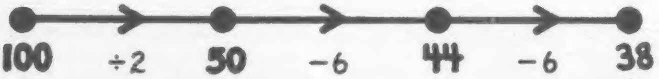
Let the students have a few minutes to draw the road.

S: Yes, the one-digit distance from 100 to 48 is 2.



T: Good. Is 38 a distance of 2 from 100?

S: No, the one-digit distance from 100 to 38 is 3.



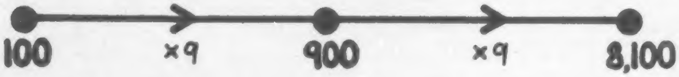
T: What about 25?

S: 25 is a one-digit distance of 1 from 100, because 100 ÷ 4 = 25.

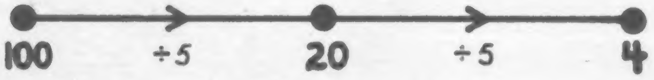
T: Which is the largest whole number that is a distance of 2 from 100?

Invite a student to draw the appropriate road on the board.

S: 8,100 is the largest number.



Ask the class to find the smallest whole number at a one-digit distance of 2 from 100. (Answer: 4)

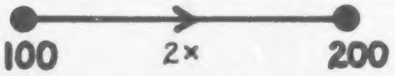


T: Let's try to find all the whole numbers between 200 and 300 that could be in the box. First, we choose a number a distance of 1 from 100. How could we start?

Draw a dot and label it "100".

S: 2 × 100 = 200, so 200 is a distance of 1 from 100.

Add this information to your drawing.



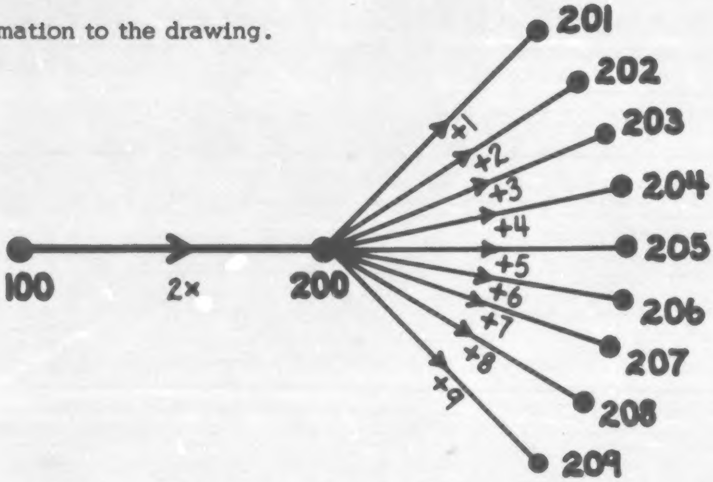
T: Now can you find some numbers that are a distance of 2 from 100 and are between 200 and 300?

S: 200 + 1 = 201, so 201 is a distance of 2 from 100.

S: You could add 2 or 3 or 4 or 5 or 6 or 7 or 8 or 9 and get a number between 200 and 300 which is a distance of 2 from 100.

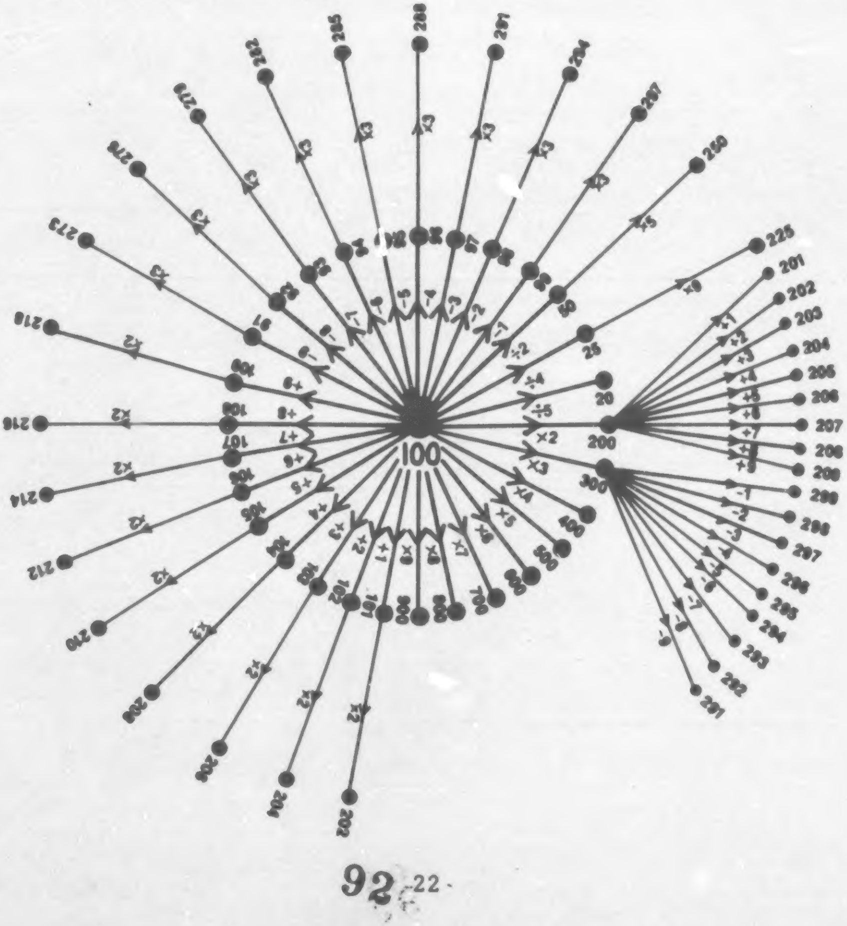
N2

Add this information to the drawing.



100
2×
200
+1
+2
+3
+4
+5
+6
+7
+8
+9
201
202
203
204
205
206
207
208
209

When it is obvious that most students understand the problem, let them work independently for a while. On the board list the numbers as they are found. A complete solution is given below; however, you should not expect the students to find the complete solution.



2.2 One-Digit Distance #2

CAPSULE LESSON SUMMARY

Build roads using arrows for +, -, ×, or ÷ some whole number from 1 to 9. Solve a detective story in which clues involve prime numbers and the one-digit distance between numbers.

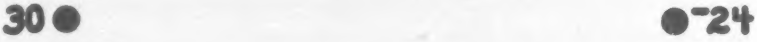
MATERIALS

Teacher: Colored chalk
Student : Paper; colored pencils

DESCRIPTION OF LESSON

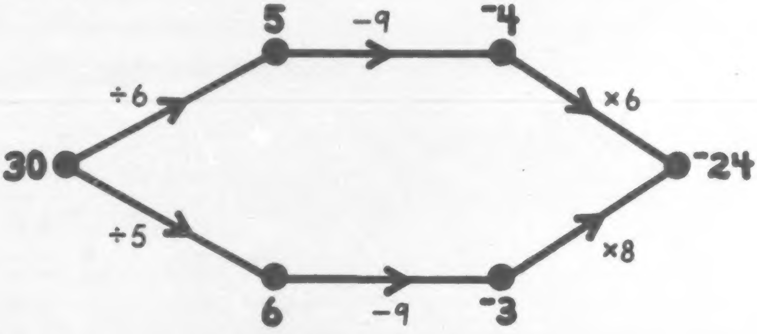
Exercise 1

Draw this picture of the board.



T: Build a road from 30 to $^{-}24$ only using arrows for +, -, ×, or ÷ some whole number from 1 to 9 (one-digit positive integer).

Let the class work independently for a few minutes. After finding at least one appropriate road, encourage students to find a shortest road. Invite several students to put their solutions on the board.



N 15

Write this information beneath your arrow road.

$$d_1(30, {}^{-}24) =$$

T: What is the one-digit distance between 30 and $^{-}24$?

S: 3, because a shortest road from 30 to $^{-}24$ has three arrows and a shortest road from $^{-}24$ to 30 has three arrows.

Complete the number story.

$$d_1(30, {}^{-}24) = 3$$

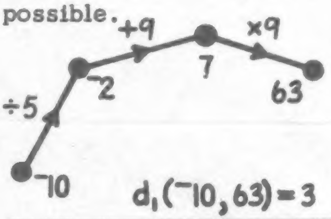
Write these problems on the board.

* $d_1({}^{-}10, 63) =$ ** $d_1(5, {}^{-}55) =$ *** $d_1(4, 888) =$

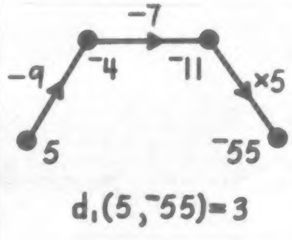
$d_1(17, 175) =$ $d_1({}^{-}37, 41) =$

T: For each of these problems, find the one-digit distance by drawing a shortest road between the two numbers.

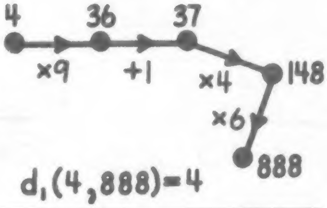
After most students have solved at least two problems, check a few of the distances by inviting students to draw roads on the board. The one-digit distance and a shortest road for each problem are given below. Many other roads are possible.



$d_1({}^{-}10, 63) = 3$

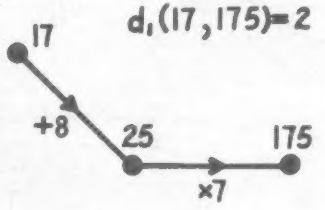


$d_1(5, {}^{-}55) = 3$

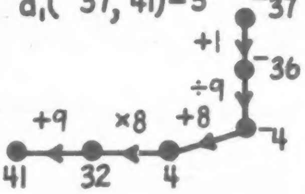


$d_1(4, 888) = 4$

$d_1(17, 175) = 2$



$d_1({}^{-}37, 41) = 5$



Exercise 2

Present this detective story about a secret number named "Kip".

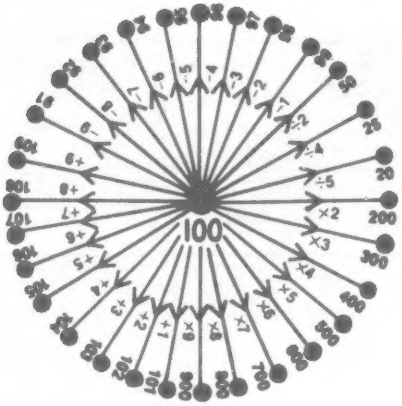
First clue

Write this information on the board.

$$d_1(100, Kip) = 2$$

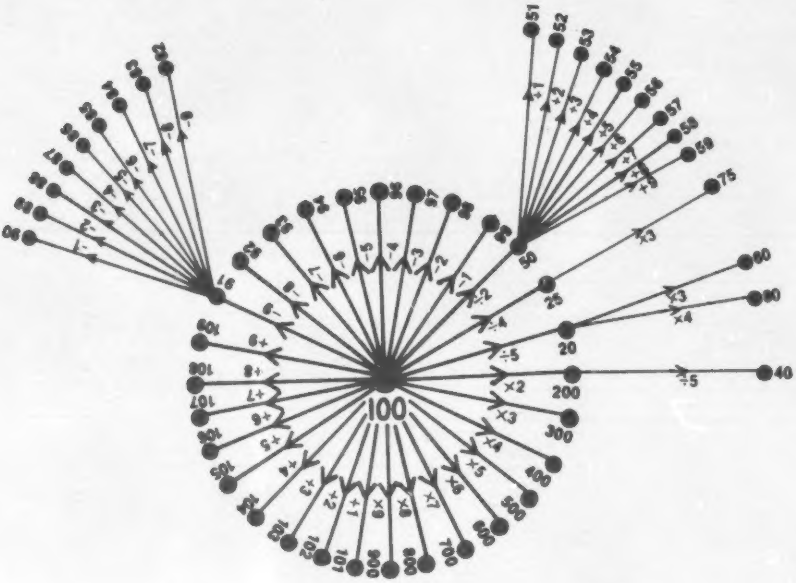
T: Kip is a whole number between 50 and 100. The one-digit distance between Kip and 100 is 2. Which numbers could Kip be?

Let the students work independently on this problem for a few minutes. Assist students who have difficulty getting started. Discuss this clue collectively and help the students develop a systematic method of finding these numbers by first thinking about all the numbers at a distance of 1 from 100.


100
-1
-2
-3
-4
-5
-6
-7
-8
-9
÷2
50
÷4
25
÷5
20
×2
200
×3
300
×4
400
×5
500
×6
600
×7
700
×8
800
×9
900
+1
101
+2
102
+3
103
+4
104
+5
105
+6
106
+7
107
+8
108
+9
109

N 15

Once the numbers at a one-digit distance of 1 from 100 have been found, then whole numbers at a distance of 2 from 100 (between 50 and 100) can be more readily determined.


100
91
90
89
88
87
86
85
84
83
82
-1
-2
-3
-4
-5
-6
-7
-8
-9
92
93
94
95
96
97
98
99
101
102
103
104
105
106
107
108
109
+1
+2
+3
+4
+5
+6
+7
+8
+9
×2
×3
×4
×5
×6
×7
×8
×9
200
300
400
500
600
700
800
900
÷2
÷4
÷5
50
25
20
51
52
53
54
55
56
57
58
59
75
60
80
40

The students should conclude that Kip could be one of the numbers at the end of a two-arrow road starting at 100.

Second clue

T: The second clue is that Kip is a prime number. Which numbers could Kip be?

Let the students eliminate from your list any number that is not prime. This activity should go rather quickly. Be sure to stop and discuss any number that is disputed. The students should conclude that Kip could be 53, 59, 83, or 89.

Third clue

Write this information on the board.

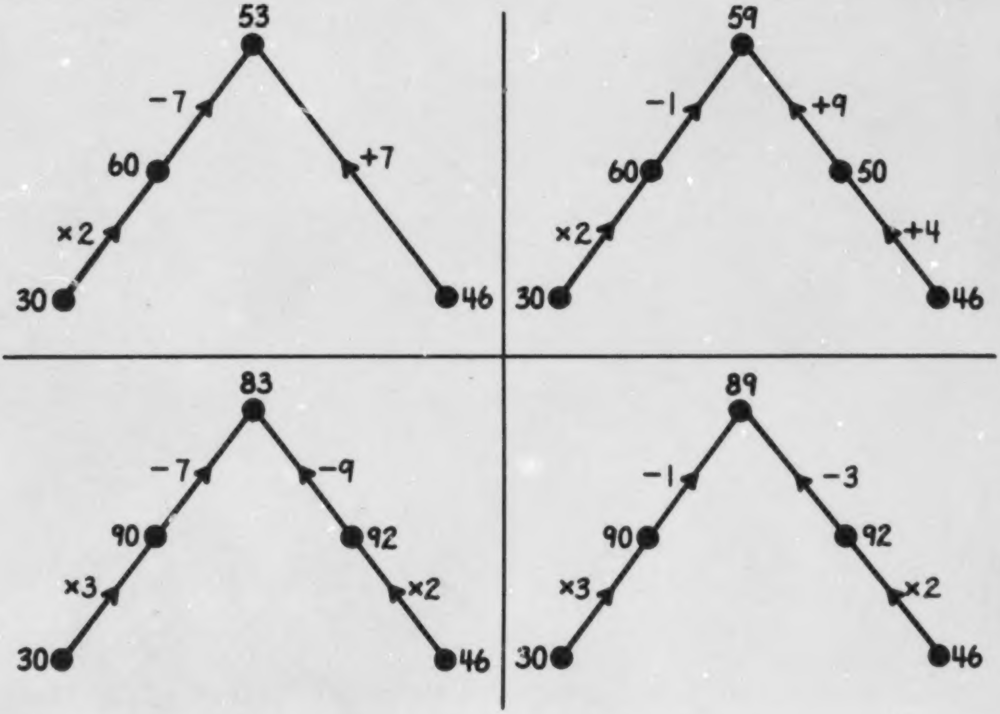
$$d_1(\text{Kip}, 30) \neq d_1(\text{Kip}, 46)$$

T: What new information does this clue give us about Kip?

S: The one-digit distance between Kip and 30 is different than the one-digit distance between Kip and 46.

T: Good. Who is Kip?

Let the students work independently on this problem for several minutes. When most students have finished, invite several of them to draw their roads on the board.


53
-7
60
×2
30
+7
46
59
-1
+9
60
50
×2
+4
30
46
83
-7
-9
90
92
×3
×2
30
46
89
-1
-3
90
92
×3
×2
30
46

The class should conclude that Kip is 53.

3

MULTIPLES AND DIVISORS

This chapter reviews and extends many previous experiences with multiples and divisors; for example, numbers on the Minicomputer with weighted checkers, the modulo n relations, and exponential notation. Several lessons concentrate on the prime factorization of a whole number and then make use of the prime factorization to introduce a technique for counting the positive divisors of a whole number.

Detective stories, arrow diagrams, and string pictures provide opportunities to explore these familar ideas and to discover new properties of multiples, divisors, and the various relations and operations involving them. A probability problem employs the prime factorization of a number in its solution.

CAPSULE LESSON SUMMARIES

Section 3.1:	From a list of numbers, select those that can be put on the Minicomputer with exactly one of the checkers ②, ③, . . ., ⑨. Put some numbers on the Minicomputer using a ⑦-checker and a ⑨-checker or using a ⑤-checker and an ⑧-checker. Solve a Detective Story in which the clues involve the prime numbers between 50 and 100, the modulo 3 relation, moving one checker on the Minicomputer, and the one-digit distance relation. (Worksheets N01, N02, N03)
Section 3.2:	Review exponential notation. Write multiples and divisors of a given number as products of prime numbers. Write various numbers as products of prime numbers making use of exponential notation.

Section 3.3:	Review prime numbers and express some numbers as products of prime numbers. Given the prime factorization of a number, express some of its multiples and divisors as products of prime numbers making use of exponential notation. Draw an arrow picture for the relation "is a multiple of" and include all possible arrows. (Worksheets N04, N05, N06)
Section 3.4:	Express some numbers as products of prime numbers. Use string pictures as an aid for counting the number of positive divisors of $3^2 \times 7^3$. (Worksheets N07, N08, N09)
Section 3.5:	Express a number as a product of prime numbers and count its positive divisors. Use string pictures to count the number of positive divisors of $r \cdot p^2 \cdot q^4$ where r, p, and q are distinct prime numbers. Find the probability that a positive divisor of $2^3 \times 3 \times 5^3$ is a multiple of 2.

3.1 Who is Zig?

CAPSULE LESSON SUMMARY

From a list of numbers, select those that can be put on the Minicomputer with exactly one of the checkers ②, ③, . . . , ⑨. Put some numbers on the Minicomputer using a ⑦-checker and a ⑨-checker or using a ⑤-checker and an ⑧-checker. Solve a detective story in which the clues involve the prime numbers between 50 and 100, the modulo 3 relation, moving one checker on the Minicomputer, and the one-digit distance relation.

MATERIALS

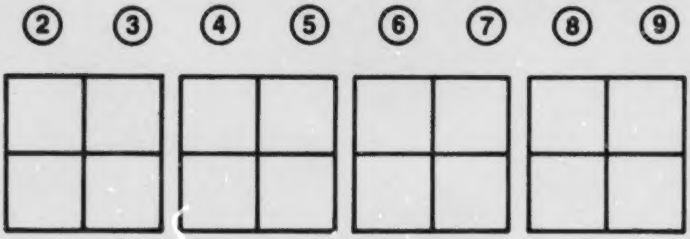
Teacher: Minicomputer set[†]; weighted checkers; colored chalk

Student: Paper; colored pencils; Worksheets N01, N02, and N03

DESCRIPTION OF LESSON

Exercise 1

Display four Minicomputer boards and a set of weighted checkers.



Write this list of numbers on the board.

28 36 46 56 240 420 1,600

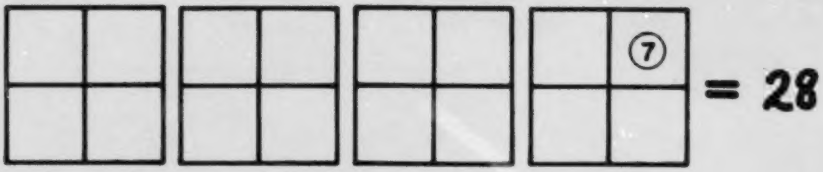
[†] A teacher's Minicomputer set consists of four demonstration Minicomputer boards and a sufficient number of demonstration Minicomputer checkers.

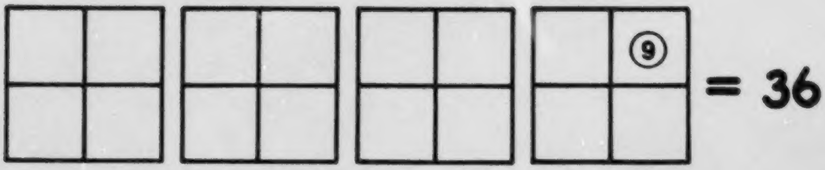
N3

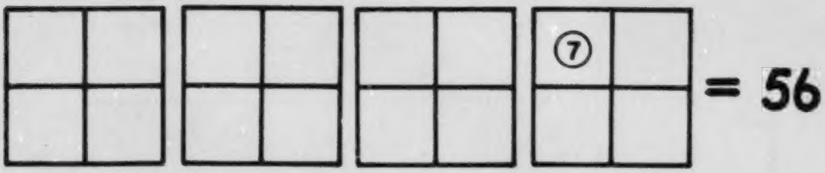
T: From this list, find all the numbers that can be put on the Minicomputer using exactly one of these checkers.

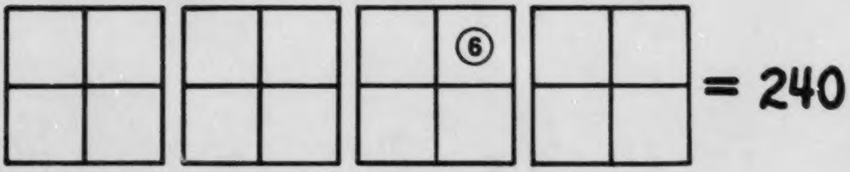
Invite students to put numbers on the Minicomputer using exactly one of the checkers. Each time, ask the rest of the class to check that the number on the Minicomputer is indeed one of the numbers in the list.

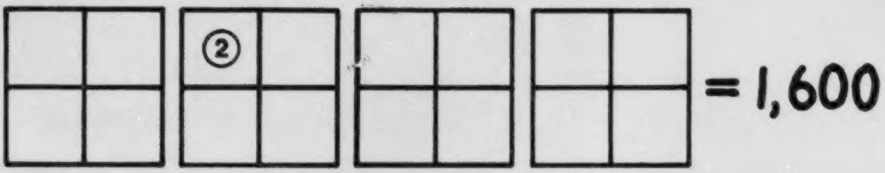
These five numbers: 28; 36; 56; 240; and 1,600 can be put on the Minicomputer using exactly one of the displayed checkers. The configurations shown below show one way (there may be others) for each number.





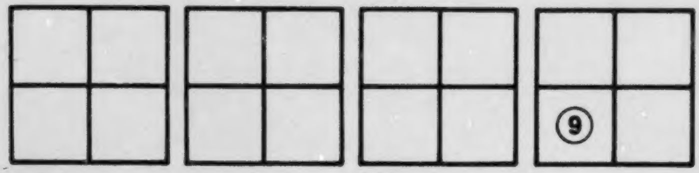






T: Can you put 46 on the Minicomputer using a ⑦-checker and a ⑨-checker?

Ask a student to place just one checker; for example, the student might choose to place a ⑨-checker.



T: Which number is on the Minicomputer?

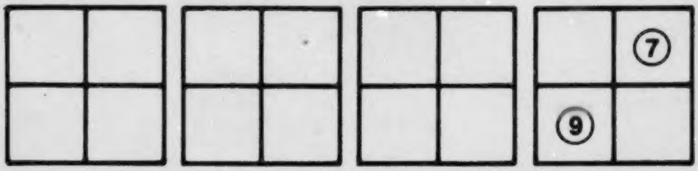
S: 18, because 9 x 2 = 18.

T: How much smaller is 18 than 46?

S: 28 smaller.

T: Who can place the other checker?

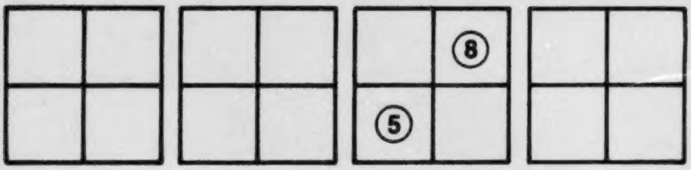
Invite a student to place the ⑦-checker.



= 46

Repeat the above activity with this problem.

Put 420 on the Minicomputer using a ⑤-checker and an ⑧-checker.



= 420

Exercise 2

Tell the class that this is a detective story about a secret number, Zig.

First clue

T: Zig is a positive prime number between 50 and 100. Which numbers could Zig be?

On the board list the numbers that Zig could be as students volunteer them. Organizing the list of numbers in order should help students to be more systematic in their search for primes. The class should conclude that Zig could be 53, 59, 61, 67, 71, 73, 79, 83, 89, or 97.

Second clue

Write this information on the board.

$$\text{Zig} =_3 10$$

T: What new information about Zig does this clue give us?

S: Zig is 1 more than a multiple of 3 just like 10; 10 = 9 + 1.

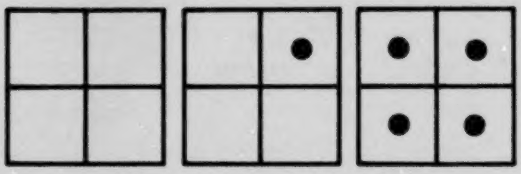
S: The difference between Zig and 10 is a multiple of 3.

Allow several minutes for individual checking of the possibilities for Zig.

The class should conclude that Zig could be 61, 67, 73, 79, or 97.

Third clue

Display three Minicomputer boards with this configuration of checkers.

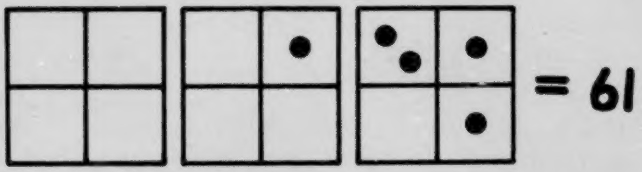


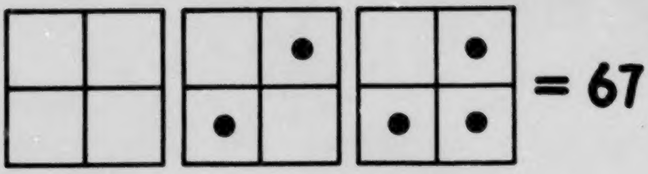
T: Which number is on the Minicomputer?

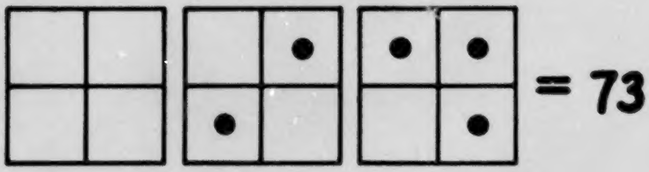
S: 55.

T: Good. If you move exactly one of these checkers to another square, you can get Zig. Which numbers could Zig be?

Invite students to show possibilities for Zig by moving one checker on the Minicomputer. The numbers that Zig could be are shown below.







The class should conclude that Zig could be 61, 67, or 73.

N 3

Fourth clue

Write this information on the board.

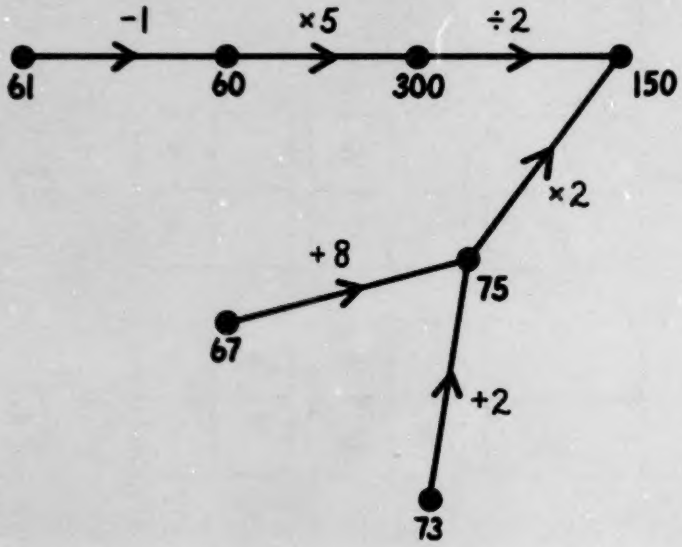
$$d_1(\text{Zig}, 150) = 3$$

T: What new information about Zig does this clue give us?

S: The one-digit distance from Zig to 150 is 3.

S: We must draw roads from Zig to 150.

Instruct students to draw some appropriate roads (using only +, -, x, ÷ a whole number from 1 to 9) on their papers. Remind them they are looking for shortest such roads to find the one-digit distance. It may be necessary to help some students get started as the one-digit distance relation is a relatively new idea. After several minutes of independent work, ask several students to draw their pictures on the board.



S: The one-digit distance from 61 to 150 is 3, and the other two numbers are each a distance of 2 from 150. Zig is 61.

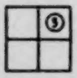
The class should conclude that Zig is 61.

Worksheets N01, N02, and N03 are available for individual work.

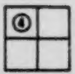
Name ____________________ N 01 *

Put each of these numbers on the Minicomputer using exactly one of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

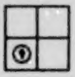


= 20

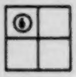


= 32

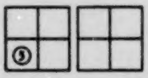
Another solution is the ⑧ -checker on the 4-square.



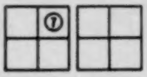
= 18



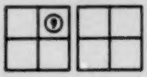
= 48



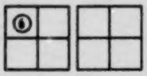
= 100



= 280



= 360



= 480

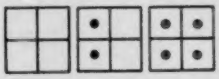
Name ____________________ N 02 **

First clue

Paf is a prime number larger than 60 and smaller than 80.

Paf could be 61, 67, 71, 73 or 79.

Second clue


= 85

Paf can be shown on these Minicomputer boards by moving exactly one checker to another square.

Paf could be 67, 73, or 79.

Third clue

$Paf =_4 9$

Paf is 73.

Name ____________________ N 03 ***

Rif and Raf are secret numbers.

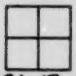
First clue

Rif is a prime number larger than 100 and smaller than 120.

Rif could be 101, 103, 107, 109, or 113.

Second clue

Raf can be put on the ones' board of the Minicomputer using just a ③ -checker and a ⑦ -checker.



Raf could be 10, 13, 19, 31, 17, 20, 26, 38
34, 40, 52, 59, 62, 68, 80

Third clue

$Rif + Raf =_3 29$ and $Rif =_{10} Raf$

Rif is 107 and Raf is 17.

3.2 Exponents and Prime Factorization #1

CAPSULE LESSON SUMMARY

Review exponential notation. Write multiples and divisors of a given number as products of prime numbers. Write various numbers as products of prime numbers making use of exponential notation.

MATERIALS

Teacher: None
Student: Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

Write this expression on the board.

$$2^3$$

T: Which number is this? What calculation does 2^3 (read: two to the third power) indicate?

S: $2^3 = 2 \times 2 \times 2 = 8$.

Write this information on the board.

$$2^3=2\times2\times2=8$$

$$3^2$$

T: Which number is 3^2?

S: 9, because $3^2 = 3 \times 3 = 9$.

N9

Summarize the student's comments by writing this calculation on the board.

$$3^2 = 3 \times 3 = 9$$

Write this number story on the board.

$$2^{\square} = 511$$

T: Copy this problem on your paper and, if possible, fill in the box with a whole number.

Let the students have a few minutes to consider the problem.

S: $2^8 = 256$ and $2^9 = 512$, so there is no (whole number) solution.

S: 2 to any (whole number) power is an even number and 511 is odd, so there is no (whole number) solution to the problem.

T: Very good. There is no whole number that could go in the box. Copy these number stories on your paper and complete them whenever possible.

Write these number stories on the board.

$$6^3 = \square \qquad 5^{\square} = 610$$

$$7^3 = \square \qquad \square^{\square} = 27$$

$$2^{\square} = 512 \qquad \square^{\triangle} = 256$$

$$\square^3 = 125$$

Observe the class work, giving assistance to those who need it. Remind students that for a problem like "$\square^{\square} = 27$", the same shapes indicate that the same number must go in both "$\square$"'s. Whereas, for a problem like "$\square^{\triangle} = 81$", the different shapes indicate that different numbers may go in "$\square$" and "$\triangle$"; however, one could put the same number in "$\square$" and in "$\triangle$" if there is one that "works."

When most of the class have finished these problems, check their work collectively. Answers are provided below.

$6^3 = \boxed{216}$ $\quad 5^{\square} = 610$ (No whole number solution)

$7^3 = \boxed{343}$ $\quad \boxed{3}^{\boxed{3}} = 27$

$2^{\boxed{9}} = 512$

$\boxed{5}^3 = 125$

The last problem "$\square^{\triangle} = 256$" has three solutions:

$$\boxed{16}^{2} = 256$$
$$\boxed{4}^{4} = 256$$
$$\boxed{2}^{8} = 256$$

Exercise 2

T: Write 60 as a product of prime numbers; that is, find a multiplication calculation for 60 using only prime numbers. You may use a prime number more than once. Try to use our shorthand notation with powers when a prime number occurs more than once in the product.

Give the students a couple of minutes to work independently, and then call on a volunteer to write the prime factorization on the board.

$$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5$$

T: Give me a positive divisor of 60 and, if possible, name it as a product of prime numbers.

S: 15 is a positive divisor of 60 and 15 = 3 x 5.

Record this information on the board.

Positive Divisors of $2^2 \times 3 \times 5$

$15 = 3 \times 5$

Continue in this way until you have a list of all the positive divisors of 60.

Positive Divisors of $2^2 \times 3 \times 5$

$15 = 3 \times 5$

2

$4 = 2^2$

$20 = 2^2 \times 5$

3

$10 = 2 \times 5$

$30 = 2 \times 3 \times 5$

$60 = 2^2 \times 3 \times 5$

5

$6 = 2 \times 3$

$12 = 2^2 \times 3$

1

T: Do you notice anything interesting about our list of positive divisors of 60?

Accept any correct observation a student makes.

S: 60 has three prime divisors: 2, 3, and 5.

S: Each divisor, except 1, is the product of some combination of 2's, 3's, and 5's.

T: Let's write some multiples of 60 as a product of prime numbers. Who can give us one?

S: 120 is a multiple of 60; 120 = $2^3 \times 3 \times 5$.

Record this information on the board.

Multiples of $2^2 \times 3 \times 5$

$$120 = 2^3 \times 3 \times 5$$

Let students suggest other multiples of 60. As the multiples become very large and as students begin to discover patterns, you may prefer to record only the prime factorization.

Multiples of $2^2 \times 3 \times 5$

$$120 = 2^3 \times 3 \times 5$$

$$240 = 2^4 \times 3 \times 5$$

$$480 = 2^5 \times 3 \times 5$$

$$600 = 2 \times 5 \times 2^2 \times 3 \times 5 = 2^3 \times 3 \times 5^2$$

$$2^5 \times 3^3 \times 5^2$$

$$2^9 \times 3 \times 5 \times 7$$

T: Do you notice anything interesting about our list of multiples of 60?

Accept any correct observation a student makes.

S: Each multiple of 60 is some whole number times $2^2 \times 3 \times 5$.

Exercise 3

Write this information on the board.

120 75 48 64 540

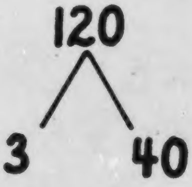
T: On your paper, write each of these numbers as a product of prime numbers. Many of you have devised your own ways to solve such problems. Let me show you one scheme for finding the prime numbers in a product for 120.

Write "120" on the board. Then allow the class to describe the procedure as follows. Note: Your class may respond somewhat differently but the end result should be the same.

T: Give me any two numbers other than 1 and 120 whose product is 120.

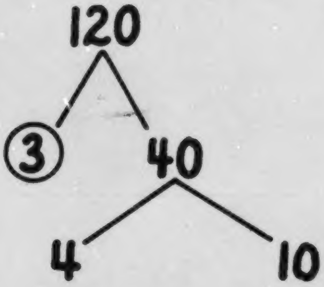
S: 3 and 40, 3 x 40 = 120.

Write this information on the board.

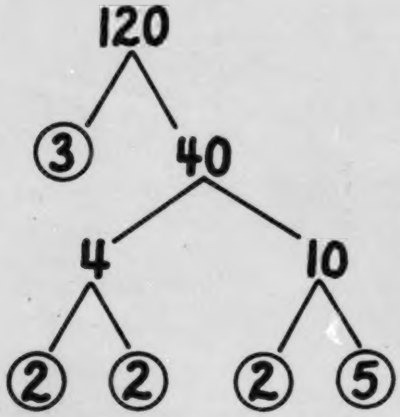


T (circling "3"): 3 is a prime number, so this branch ends. Give me two numbers other than 1 and 40 whose product is 40.

S: 4 and 10, 4 x 10 = 40.



Continue in this manner until a "tree" is complete. A branch ends at a prime number as indicated by circled endpoints here.



T: Each branch of the "tree" ends with a prime number. How does this help us to write 120 as a product of prime numbers?

S: 120 = 2 x 2 x 2 x 3 x 5; just multiply all of the ending numbers together.

S: 120 = 2^3 x 3 x 5.

N 9

Write this information on the board.

$$120 = 2 \times 2 \times 2 \times 3 \times 5 = 2^3 \times 3 \times 5$$

Let the class continue individually until most students have found some factorizations for the other four numbers. Check the work collectively. Solutions are provided below.

$$75 = 3 \times 5^2$$

$$48 = 2^4 \times 3$$

$$64 = 2^6$$

$$540 = 2^2 \times 3^3 \times 5$$

3.3 Exponents and Prime Factorization #2

CAPSULE LESSON SUMMARY

Review prime numbers and express some numbers as products of prime numbers. Given the prime factorization of a number, express some of its multiples and divisors as products of prime numbers making use of exponential notation. Draw an arrow picture for the relation "is a multiple of" and include all possible arrows between five given numbers.

MATERIALS

Teacher: Colored chalk

Student: Paper; colored pencils; Worksheets N04, N05, and N06

DESCRIPTION OF LESSON

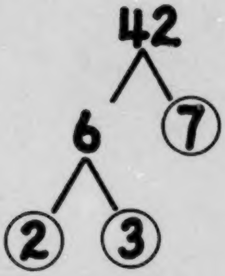
Exercise 1

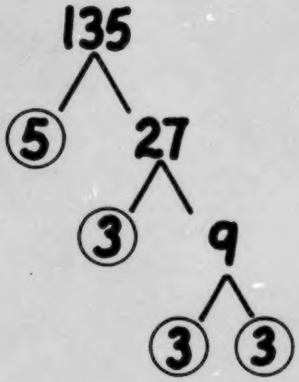
Write this list of numbers on the board.

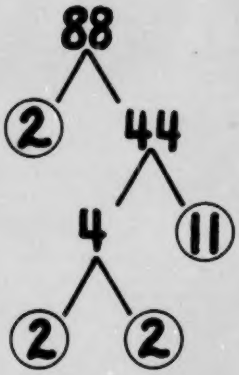
42 135 88 273 425

T: On your paper, write each of these numbers as a product of prime numbers. You may use a prime number more than once. Try to use our shorthand notation with powers when a prime number occurs more than once in the product.

If necessary with your class, review briefly the notion of prime number. Let the students work on these problems independently for several minutes, offering assistance to those students who need help getting started. Check the work collectively by inviting students to put their solutions, perhaps using "trees," on the board. Solutions and trees drawn by some of the students in one of our development classes are provided in the next illustration.



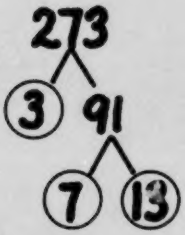


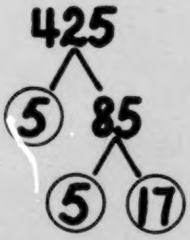


$42 = 2 \times 3 \times 7$

$135 = 3^3 \times 5$

$88 = 2^3 \times 11$





$91 = 3 \times 7 \times 13$

$425 = 5^2 \times 17$

Exercise 2

T (pointing to "$425 = 5^2 \times 17$"): Give me a positive divisor of 425 and, if possible, name it as a product of prime numbers.

S: 25 is a positive divisor of 425 and $25 = 5^2$.

Record this information on the board.

Positive Divisors of $5^2 \times 17$
$25 = 5^2$

Continue until your list includes all of the positive divisors of 425 and, except for 1, their prime factorizations.

Positive Divisors of $5^2 \times 17$

$$25 = 5^2$$

$$5$$

$$425 = 5^2 \times 17$$

$$1$$

$$17$$

$$85 = 5 \times 17$$

T: Do you notice anything interesting about this list of positive divisors of 425?

S: Each divisor, except 1, is the product of some combination of 5's or 17's.

T: Now let's write some multiples of 425 as products of prime numbers. Who can give me a multiple of 425?

S: 850 is a multiple of 425, because 2 x 425 = 850.

S: $850 = 2 \times 5^2 \times 17$.

Record this information on the board.

Multiples of $5^2 \times 17$

$$850 = 2 \times 5^2 \times 17$$

S: 2,125 is a multiple of 425, because 5 x 425 = 2,125.

T: How would we express 2,125 as a product of prime numbers?

S: 2,125 = $5^3 \times 17$.

Multiples of $5^2 \times 17$

$$850 = 2 \times 5^2 \times 17$$
$$2{,}125 = 5^3 \times 17$$

Let the students suggest other multiples of 425. As the multiples become large and as students discover patterns, record only the prime factorizations.

Multiples of $5^2 \times 17$

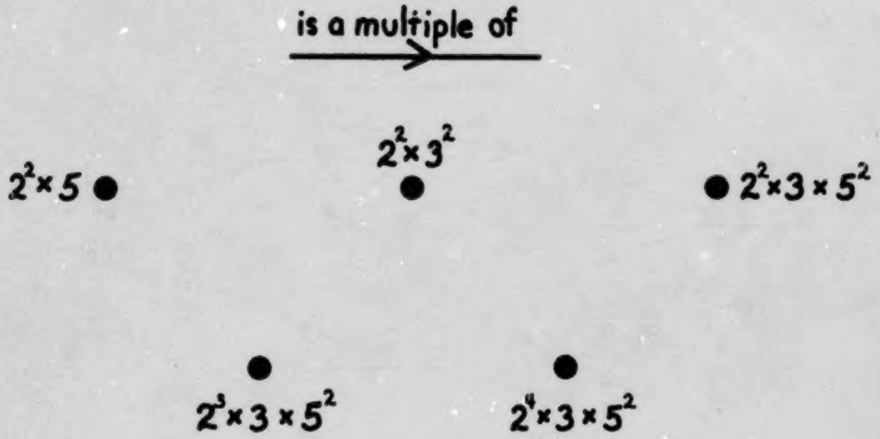
$$850 = 2 \times 5^2 \times 17$$
$$2{,}125 = 5^3 \times 17$$
$$3{,}825 = 3^2 \times 5^2 \times 17$$
$$5^4 \times 17$$
$$5^2 \times 17 \times 23$$
$$2^2 \times 5^4 \times 17^2$$

T: Do you notice anything interesting about this list of multiples of 425?

S: Each number is 425 (i.e. $5^2 \times 17$) times some whole number.

Exercise 3

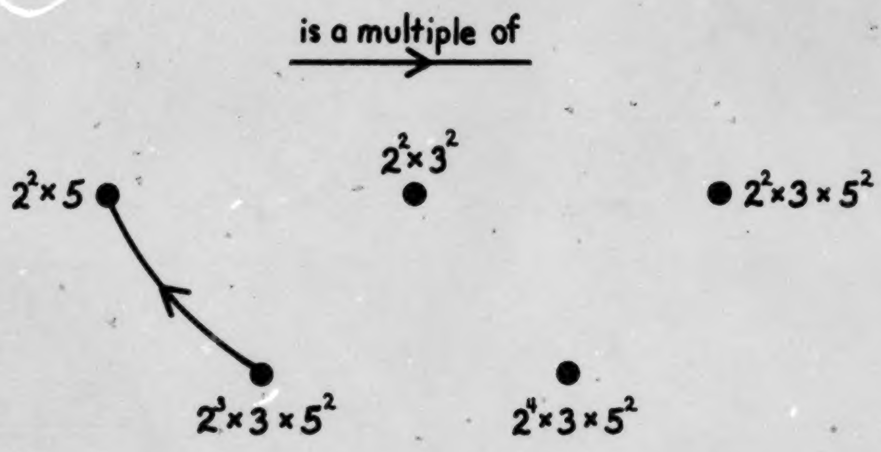
Draw this picture on the board.


is a multiple of
$2^2 \times 5$
$2^2 \times 3^2$
$2^2 \times 3 \times 5^2$
$2^3 \times 3 \times 5^2$
$2^4 \times 3 \times 5^2$

T: Who can draw a red arrow in our picture?

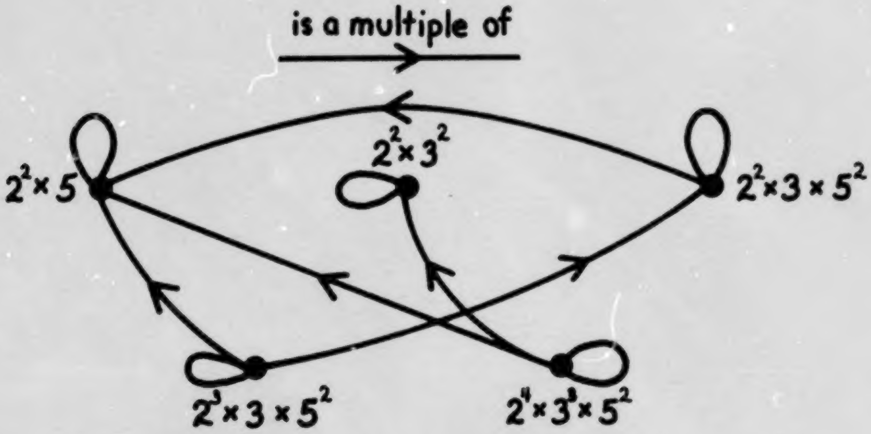
Ask the volunteer to read the information given by a red arrow.

S (drawing a red arrow): $2^3 \times 3 \times 5^2$ is a multiple of $2^2 \times 5$.

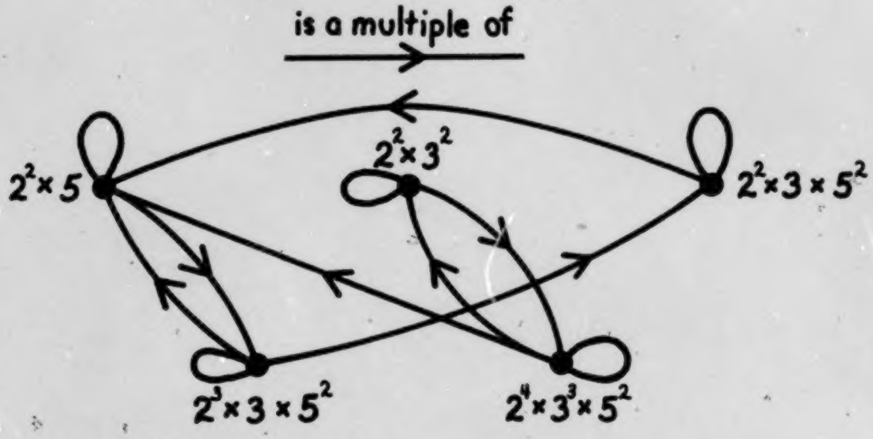

is a multiple of
$2^2 \times 5$
$2^2 \times 3^2$
$2^2 \times 3 \times 5^2$
$2^3 \times 3 \times 5^2$
$2^4 \times 3 \times 5^2$

N 22

Instruct the students to copy this picture and to draw as many red arrows as they can. If you think that this activity would be better done collectively, feel free to make this adjustment. A completed arrow picture is shown here.


is a multiple of
$2^2 \times 5$
$2^2 \times 3^2$
$2^2 \times 3 \times 5^2$
$2^3 \times 3 \times 5^2$
$2^4 \times 3^3 \times 5^2$

Draw one or two return arrows in blue on your picture. For example,


is a multiple of
$2^2 \times 5$
$2^2 \times 3^2$
$2^2 \times 3 \times 5^2$
$2^3 \times 3 \times 5^2$
$2^4 \times 3^3 \times 5^2$

T (tracing a return arrow): What could this arrow be for?

S: "Is a positive divisor of".

Worksheets N04, N05, and N06 are available for individual work.

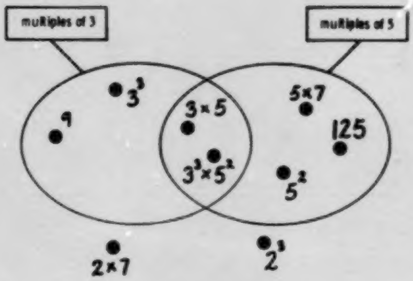
Name ____________________ **N 04** *

Put these numbers in the string picture.

5×7 3^3 125

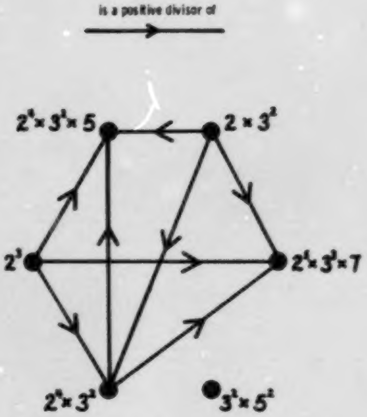
9 3×5 5^2

2^3 2×7 $3^3 \times 5^2$


multiples of 3
multiples of 5
3^3
9
3×5
$3^3 \times 5^2$
5×7
125
5^2
2×7
2^3

Name ____________________ **N 05** **

Draw all of the possible blue arrows between these dots.


is a positive divisor of
$2^4 \times 3^2 \times 5$
2×3^2
2^3
$2^4 \times 3^2 \times 7$
$2^4 \times 3^2$
$3^2 \times 5^2$

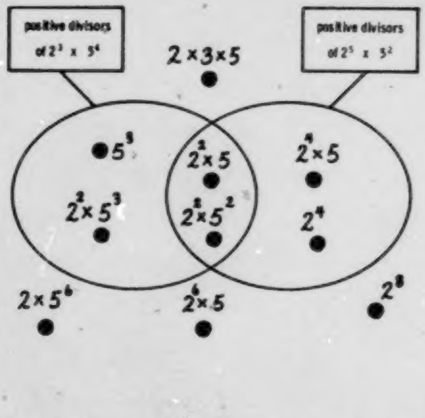
Name ____________________ **N 06** ***

Locate these numbers in the string picture.

$2^2 \times 5$ $2^4 \times 5$ 2×5^6

$2 \times 3 \times 5$ 5^3 $2^2 \times 5^2$

2^8 $2^6 \times 5$ $2^2 \times 5^3$ 2^4


positive divisors of $2^3 \times 5^4$
positive divisors of $2^5 \times 5^2$
$2 \times 3 \times 5$
5^3
$2^2 \times 5^3$
$2^2 \times 5$
$2^2 \times 5^2$
$2^4 \times 5$
2^4
2×5^6
$2^6 \times 5$
2^8

3.4 Counting Positive Divisors #1

CAPSULE LESSON SUMMARY

Express some numbers as products of prime numbers. Use string pictures as an aid for counting the number of positive divisors of $3^2 \times 7^3$.

MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils; Worksheets N07, N08, and N09

DESCRIPTION OF LESSON

Exercise 1

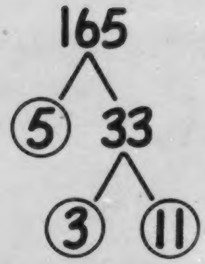
Write this list of numbers on the board.

165 225 637

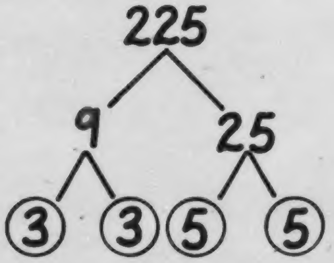
T: On your paper, write each of these numbers as a product of prime numbers. You may use a prime number more than once. Try to use our shorthand notation with powers when a prime number occurs more than once in the product.

Allow several minutes for individual work on these problems, offering assistance to those students who have trouble getting started. Check the work collectively by inviting students to put their solutions, perhaps using "trees," on the board. Solutions and trees drawn by some of the students in one of our development classes are provided in the next illustration.

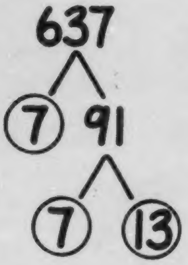
(Other trees might be used.)



$16 = 3 \times 5 \times 11$



$225 = 3^2 \times 5^2$



$637 = 7^2 \times 13$

Many students may need help with 637. You could use the following discussion to aid their thinking.

T: 63 is a multiple of 7. How do I know this?

S: 9 x 7 = 63.

T: Good. What about 630?

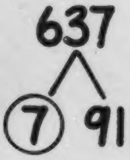
S: 90 x 7 = 630.

T: How much greater than 630 is 637?

S: 637 is 7 greater than 630.

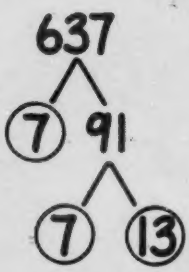
S: So 91 x 7 = 637.

Record this information in a "tree."



T: How do we go on with 91?

S: 7 x 13 = 91.



S: So, 637 = 7 x 13 x 7 or 7^2 x 13.

Whether or not you needed to lead the above discussion, continue the lesson with an examination of multiples and divisors of 637.

T (pointing to 637 = 7^2 x 13): Give me a positive divisor of 637 and, if possible, name it as a product of prime numbers.

S: 49 is a positive divisor of 637 and 49 = 7^2.

Record this information on the board.

Positive Divisors of $7^2 \times 13$

$49 = 7^2$

Continue until your list includes all the positive divisors of 637.

Positive Divisors of $7^2 \times 13$

$49 = 7^2$

7

$91 = 7 \times 13$

$637 = 7^2 \times 13$

1

13

T: Do you notice anything interesting about this list of positive divisors of $7^2 \times 13$?

S: Each divisor, except 1, is the product of some combination of 7's or 13's.

T: Yes. Now, let's write some multiples of 637 as products of prime numbers. Who can give us a multiple of 637?

S: $2 \times 7^2 \times 13$ is a multiple of 637.

Record this information on the board.

$$\underline{\text{Multiples of } 7^2 \times 13}$$

$$2 \times 7^2 \times 13$$

Encourage the students to observe patterns by recording only the prime factorizations.

$$\underline{\text{Multiples of } 7^2 \times 13}$$

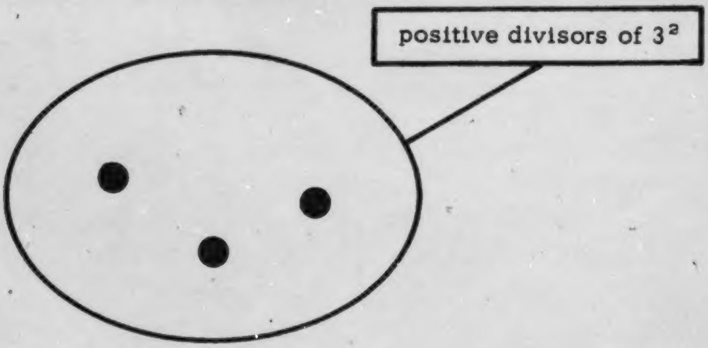
$$2 \times 7^2 \times 13$$
$$2^2 \times 7^2 \times 13$$
$$2^3 \times 7^2 \times 13$$
$$5^8 \times 7^2 \times 13$$
$$7^5 \times 13$$
$$3 \times 7^7 \times 13$$
$$7^3 \times 13^2$$

Exercise 2

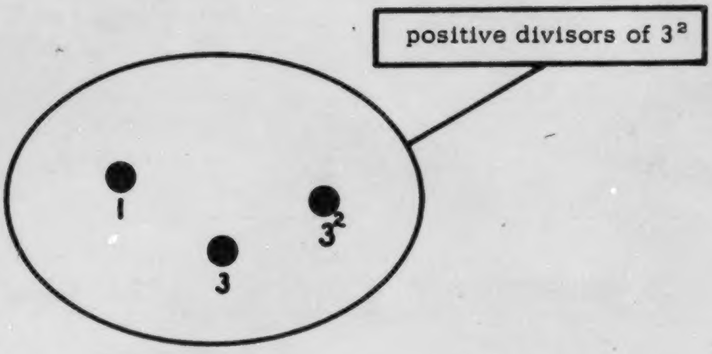
T: How many positive divisors does 3^2 have?

S: Three.

Draw this string picture on the board.


positive divisors of 3^2

Invite students to label the dots.


positive divisors of 3^2
1
3
3^2

T: What are some other numbers that have exactly three positive divisors?

Give the students a few minutes to think about the situation. Encourage students to express the numbers with exactly three positive divisors as products of prime numbers.

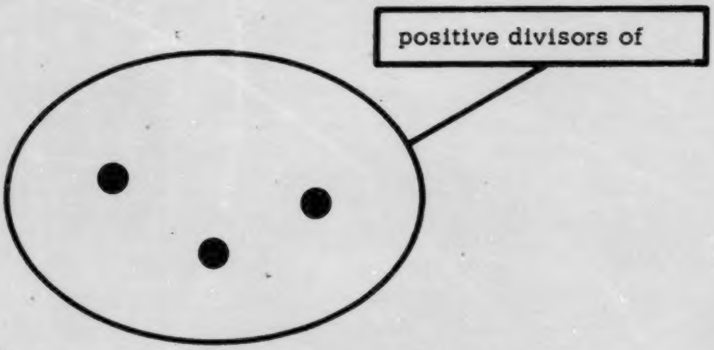
S: 4 has three positive divisors.

T: Yes, can you express 4 as a product of prime numbers?

S: $4 = 2 \times 2 = 2^2$.

N 30

Alter your string picture (as shown below) by erasing the numerals and start a list of the numbers with exactly three positive divisors.



Numbers with exactly three positive divisors

$9 = 3^2$
$4 = 2^2$

Continue to let the students suggest numbers that have exactly three positive divisors. Any disputes may be settled by listing and then counting the positive divisors of a number.

Numbers with exactly three positive divisors

$9 = 3^2$
$4 = 2^2$
$25 = 5^2$
$49 = 7^2$
$121 = 11^2$

T: Do you notice anything interesting about our list of numbers with exactly three positive divisors?

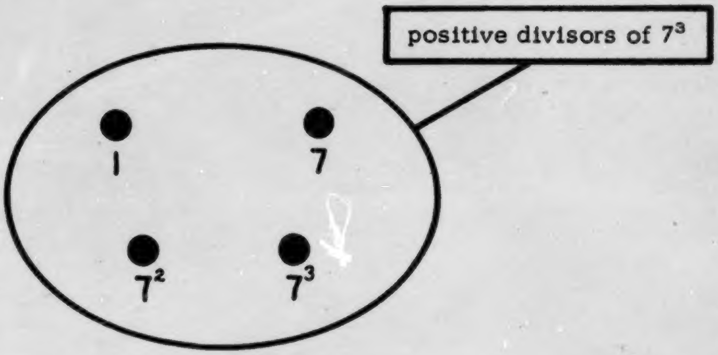
S: Each number with exactly three positive divisors is the square of a positive prime number.

A few students may incorrectly suggest that any number squared has exactly three positive divisors. In that case, choose a non-prime number such as 6 and note that $6^2 = 36$ and 36 has more than three positive divisors.

T: How many positive divisors does 7^3 have?

S: Four.

Draw this string picture on the board and ask volunteers to label the dots.



T: Name some other numbers with exactly four positive divisors.

List these numbers on the board as students suggest them. NOTE: Any positive prime number to the third power has exactly four positive divisors. Also, any number that is the product of two distinct prime numbers has exactly four positive divisors (for example, $15 = 3 \times 5$ and the positive divisors of 15 are 1, 3, 5, and 15). Accept both types of numbers, but be certain that students recognize numbers of the former type.

Numbers with exactly four positive divisors

$343 = 7^3$
$8 = 2^3$
$125 = 5^3$
$27 = 3^3$
$1{,}331 = 11^3$

$6 = 2 \times 3$
$15 = 3 \times 5$
$35 = 5 \times 7$

S: Any prime number to the third power has exactly four positive divisors.

T: Good. So 3^2 has exactly three positive divisors and 7^3 has exactly four positive divisors? How many positive divisors does $3^2 \times 7^3$ have?

Record the students' estimates on the board. If a student suggests that $3^2 \times 7^3$ has twelve divisors, ask for an explanation; otherwise, continue the lesson by drawing the string picture shown in the next illustration.

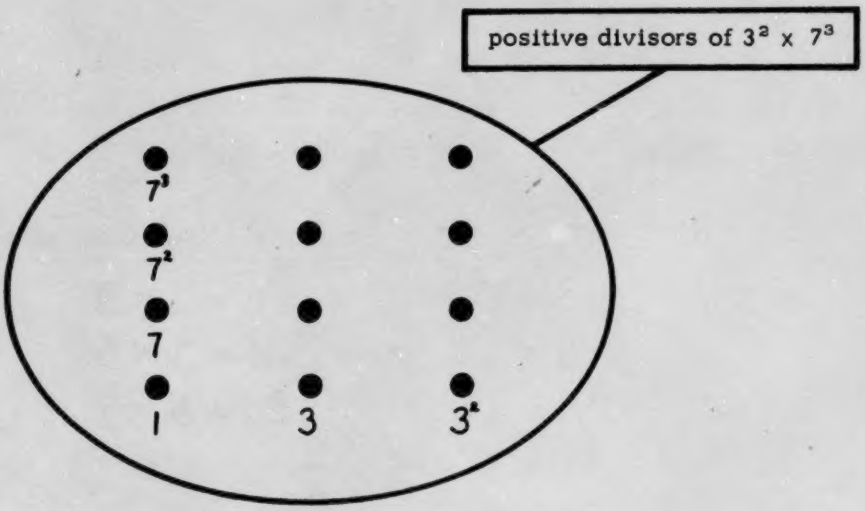
S: Twelve.

T: How did you conclude that $3^2 \times 7^3$ has exactly twelve positive divisors?

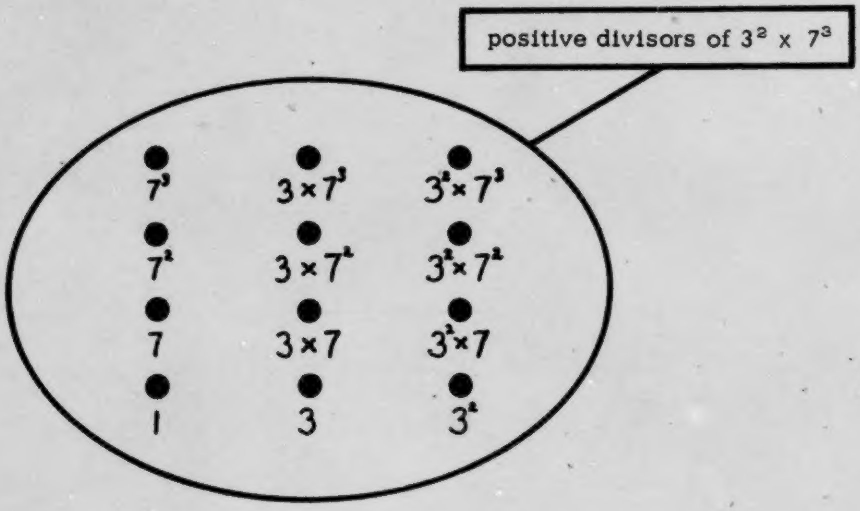
S: I looked at the exponents. I increased each exponent by one and then multiplied $3 \times 4 = 12$.

T: Let's use a string picture to examine your result.

Draw this string picture on the board.



Invite students to label the dots. Encourage them to use a scheme for labeling the dots as suggested by the array of dots shown below.


positive divisors of $3^2 \times 7^3$
7^3
3×7^3
$3^2 \times 7^3$
7^2
3×7^2
$3^2 \times 7^2$
7
3×7
$3^2 \times 7$
1
3
3^2

T: Is there any positive divisor of $3^2 \times 7^3$ we left out?

S: No.

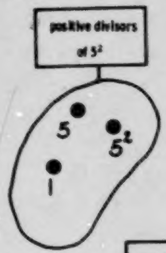
T: How many positive divisors does $3^2 \times 7^3$ have?

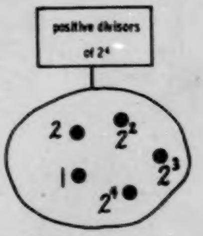
S: Twelve.

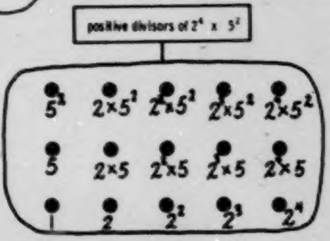
Worksheets N07, N08, and N09 are available for individual work.

Name ____________________ **N 07** *

Label the dots.


positive divisors of 5^2
5
5^2
1


positive divisors of 2^4
2
2^2
2^3
1
2^4

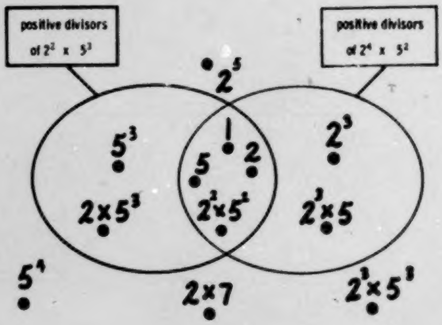

positive divisors of 2^4 x 5^2
5^2
2×5^2
$2^2\times5^2$
$2^3\times5^2$
$2^4\times5^2$
5
2×5
$2^2\times5$
$2^3\times5$
$2^4\times5$
1
2
2^2
2^3
2^4

Other arrangements of the numbers within each string are possible. Our choice for the lower string suggests a counting method and allows a quick check that all the positive divisors of $2^4 \times 5^2$ are included.

Name ____________________ **N 08** **

Put these numbers in the string picture.

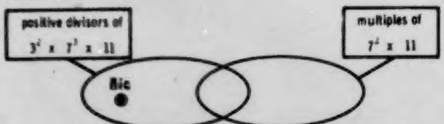
1	2	$2^2\times5^2$
5	5^3	2×5^3
2^3	5^4	$2^3\times5$
2^5	2×7	$2^3\times5^3$


positive divisors of 2^2 x 5^3
positive divisors of 2^4 x 5^2
2^5
5^3
2×5^3
1
5
2
$2^2\times5^2$
2^3
$2^3\times5$
5^4
2×7
$2^3\times5^3$

Name ____________________ **N 09** ***

Bic is a secret number.

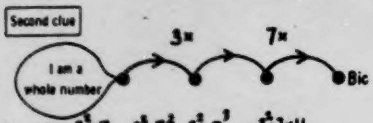
First clue


positive divisors of 3^2 x 7^3 x 11
multiples of 7^2 x 11
Bic

Bic could be 1, 3, 3^2, 7, 7^2, 7^3, 11, 3×7, 3×7^2, 3×7^3, 3×11, $3^2\times7$, $3^2\times7^2$, $3^2\times7^3$, $3^2\times11$, 7×11, $3\times7\times11$, $3^2\times7\times11$

(Hint: There are between fifteen and twenty possibilities for Bic.)

Second clue


I am a whole number
3×
7×
Bic

Bic could be $3^2\times7$, $3^2\times7^2$, $3^2\times7^3$, or $3^2\times7\times11$.

Third clue

$$\text{Bic} \sqcap 5\times7^3 = 7^2$$

Bic is $3^2\times7^2$.

3.5 Counting Positive Divisors #2

CAPSULE LESSON SUMMARY

Express a number as a product of prime numbers and count its positive divisors. Use string pictures to count the number of positive divisors of $r \cdot p^2 \cdot q^4$ where r, p, and q are distinct prime numbers. Find the probability that a positive divisor of $2^3 \times 3 \times 5^3$ is a multiple of 2.

MATERIALS

Teacher: Colored chalk
Student: Paper; pencil

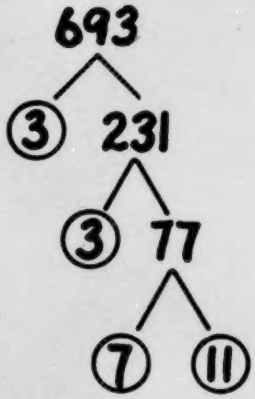
DESCRIPTION OF LESSON

Exercise 1

Give the following problem to the class for individual work on scratch paper.

T: Express 693 as a product of prime numbers.

Allow a few minutes for the individual work before asking a student to put a solution on the board. Find someone who has used a "tree" to facilitate the solution.



(Other trees might be used.)

$$693 = 3^2 \times 7 \times 11$$

T: Name some multiples of 693 and write each of those multiples as a product of prime numbers.

As students suggest multiples of 693, record them in a list on the board. For example,

$$\underline{\text{Multiples of } 3^2 \times 7 \times 11}$$

$$\begin{array}{r} 2 \times 3^2 \times 7 \times 11 \\ 3^3 \times 7 \times 11 \\ 2^2 \times 3^2 \times 7 \times 11 \\ 2^2 \times 3^3 \times 7 \times 11 \\ 3^2 \times 7^3 \times 11^2 \end{array}$$

T: What do you notice about this list of multiples of 693?

S: Each number is some whole number times $3^2 \times 7 \times 11$.

T: Now let's examine the positive divisors of 693. Before we do, predict how many positive divisors $3^2 \times 7 \times 11$ has.

Record the students' responses on the board. If a student answers "twelve", ask for an explanation; otherwise, continue the lesson by listing the positive divisors.

S: Twelve, I added one to each exponent then multiplied $3 \times 2 \times 2 = 12$.

T: Let's check your result by listing the positive divisors of $3^2 \times 7 \times 11$.

List the positive divisors of $3^2 \times 7 \times 11$ on the board as students suggest them. Try to build your list so that a pattern is suggested as in the next illustration.

Positive Divisors of $3^2 \times 7 \times 11$

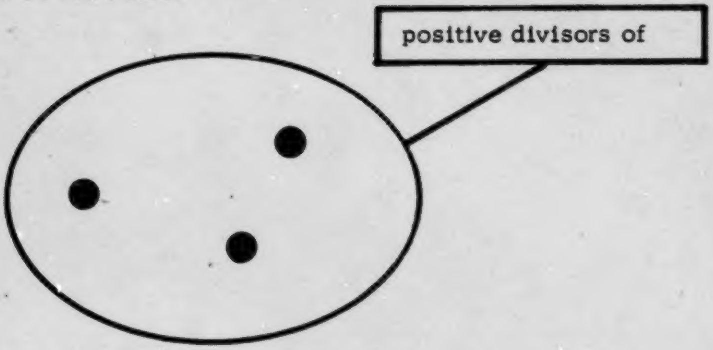
1	3	3^2
7	3×7	$3^2 \times 7$
11	3×11	$3^2 \times 11$
7×11	$3 \times 7 \times 11$	$3^2 \times 7 \times 11$

T: How many positive divisors does $3^2 \times 7 \times 11$ have?

S: Twelve.

Exercise 2

Draw this string picture on the board.



T: Name some numbers with exactly three positive divisors.

S: 4.

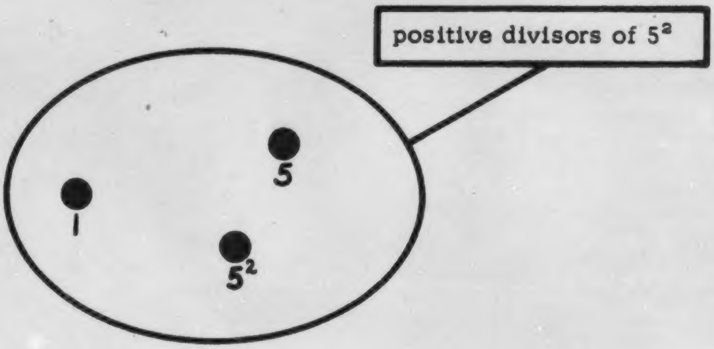
S: 25.

S: 9.

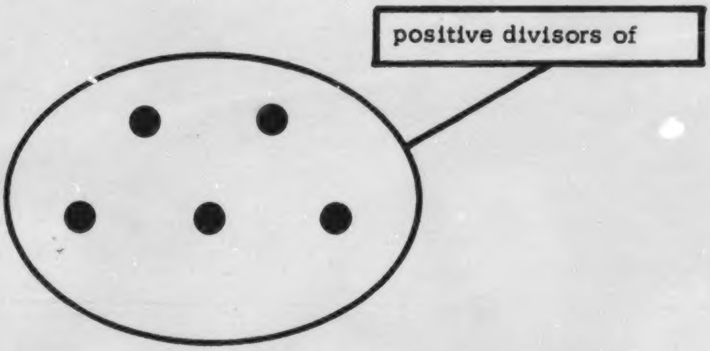
N 32

S: Any square of a prime number has exactly three positive divisors.

Complete the label for the string with one of the numbers suggested by your class and ask students to label the dots appropriately. For example,



Draw this string picture next to the first one.



T: Name some numbers that have exactly five positive divisors.

S: 81.

T: Good. Can you express 81 as a product of prime numbers?

S: $81 = 3^4$.

T: Are there other numbers with exactly five positive divisors? You may name such a number as a product of primes if you wish.

S: 2^4 or 16.

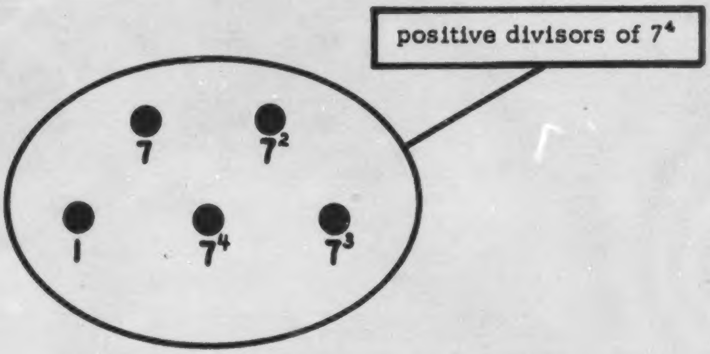
S: 5^4.

S: 11^4.

S: 7^4.

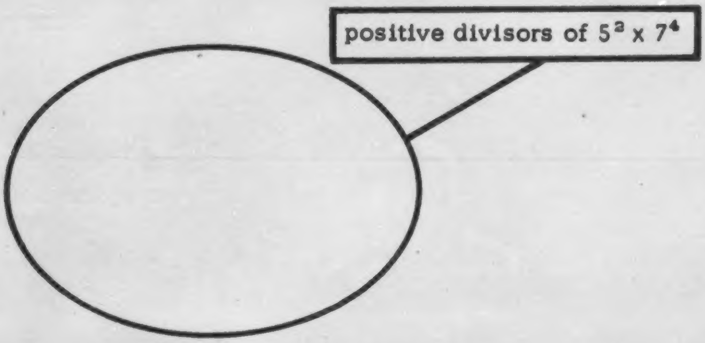
S: Any prime number to the fourth power has exactly five positive divisors.

Choose one of the suggested numbers to complete the label for the string. In the illustration below, 7^4 is used. Ask students to label the dots appropriately.


positive divisors of 7^4
7
7^2
1
7^4
7^3

T: 5^2 has three positive divisors and 7^4 has five positive divisors. How many positive divisors does $5^2 \times 7^4$ have?

While the students are considering your question, draw this third string picture on the board below the other two.

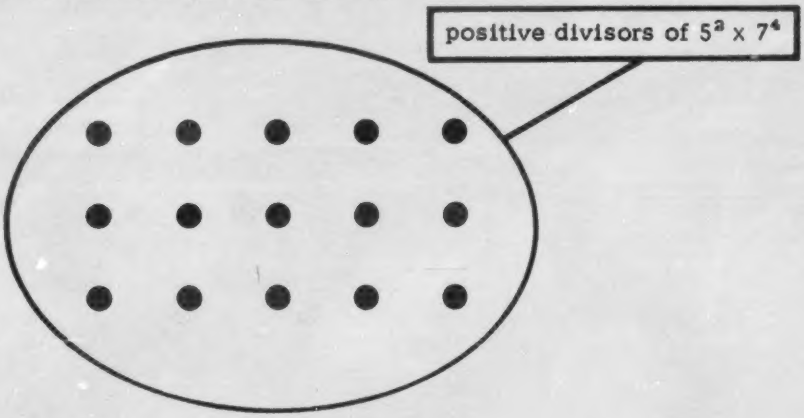

positive divisors of $5^2 \times 7^4$

S: Fifteen.

Many students will have discovered a rule to calculate the number of positive divisors; however, for the benefit of those who have not, continue as follows.

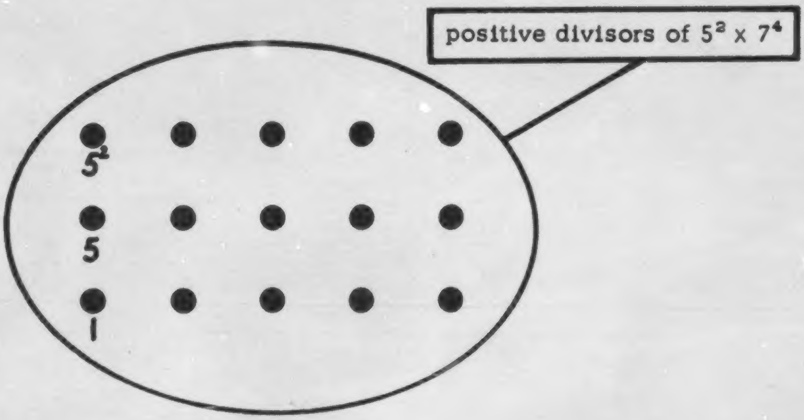
T (pointing to the string labeled "positive divisors of $5^2 \times 7^4$"): How could we arrange the dots in this picture?

S: Three rows because 5^2 has three divisors and five columns because 7^4 has five divisors (or vice versa).

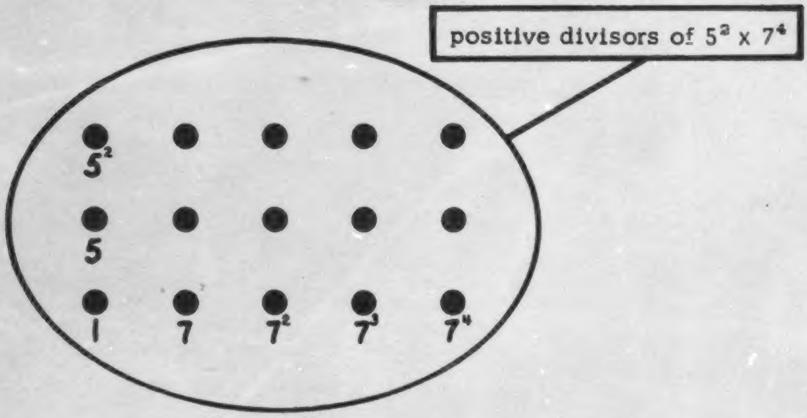

positive divisors of $5^2 \times 7^4$

T: How could we label the dots along the side?

Invite a student to label the dots.


positive divisors of $5^2 \times 7^4$
5^2
5
1

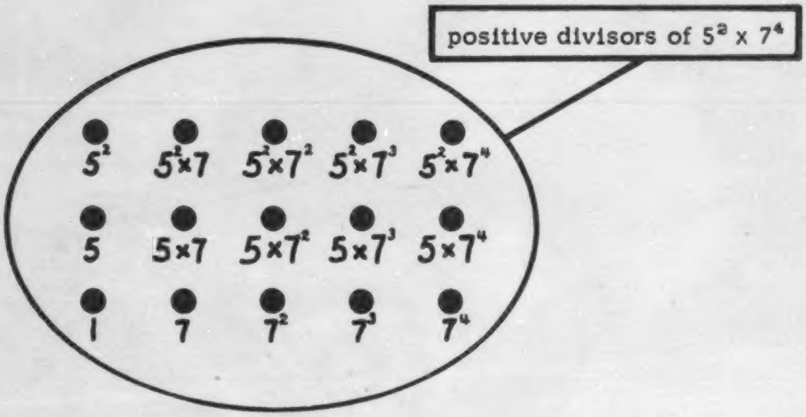
T: And what about the dots along the bottom?


positive divisors of $5^2 \times 7^4$
5^2
5
1
7
7^2
7^3
7^4

T (pointing to the dot in the second row and the third column): How could we label this dot?

S: 5×7^2.

Of course other answers would be correct, but to make best use of the array label the dot "5×7^2". Continue in this manner until the picture is complete.


positive divisors of $5^2 \times 7^4$
5^2
$5^2 \times 7$
$5^2 \times 7^2$
$5^2 \times 7^3$
$5^2 \times 7^4$
5
5×7
5×7^2
5×7^3
5×7^4
1
7
7^2
7^3
7^4

T: How many positive divisors does $5^2 \times 7^4$ have?

S: Fifteen.

S: Add 1 to each exponent and multiply;
$(2 + 1) \times (4 + 1) = 3 \times 5 = 15$.

T: Very good. Can you predict the number of positive divisors of
$3 \times 5^2 \times 7^4$?

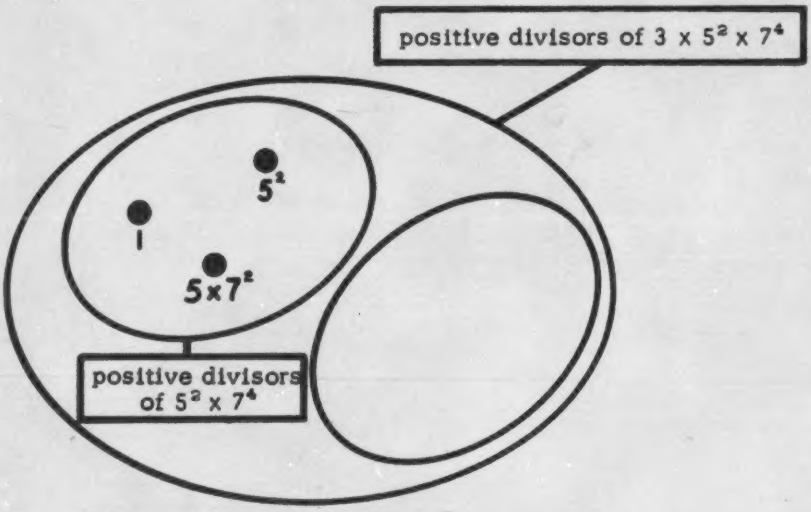
Let the students have a minute to consider this situation.

S: It has twice as many positive divisors as $5^2 \times 7^4$. All of the positive divisors of $5^2 \times 7^4$ are also positive divisors of $3 \times 5^2 \times 7^4$ and $3 \times$ any positive divisor of $5^2 \times 7^4$ is a positive divisor of $3 \times 5^2 \times 7^4$.

S: 3 has two positive divisors, 5^2 has three positive divisors, and 7^4 has five positive divisors; $2 \times 3 \times 5 = 30$.

S: Add 1 to each exponent and then multiply. $3 = 3^1$ so
$(1 + 1) \times (2 + 1) \times (4 + 1) = 2 \times 3 \times 5 = 30$.

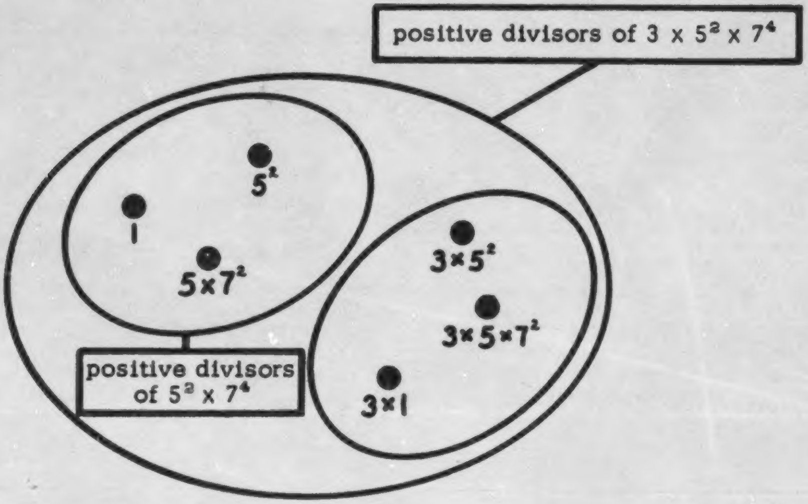
Summarize the discussion by drawing this string picture and asking students to put a few divisors in the appropriate regions.


positive divisors of $3 \times 5^2 \times 7^4$
5^2
1
5×7^2
positive divisors of $5^2 \times 7^4$

T (pointing to the string for positive divisors of $5^2 \times 7^4$): All of the positive divisors of $5^2 \times 7^4$ are in this string. Which divisors of $3 \times 5^2 \times 7^4$ are in this string (point to the unlabeled string)?

S: Those of the form $3 \times$ a positive divisor of $5^2 \times 7^4$. For example, 5×7^2 is a positive divisor of $5^2 \times 7^4$, so $3 \times 5 \times 7^2$ goes in that string.

Put $3 \times 5 \times 7^2$ in the string picture together with a few other positive divisors of the form suggested.


positive divisors of $3 \times 5^2 \times 7^4$
5^2
1
5×7^2
positive divisors of $5^2 \times 7^4$
3×5^2
$3 \times 5 \times 7^2$
3×1

T: How many positive divisors does $3 \times 5^2 \times 7^4$ have?

S: Thirty; there are fifteen in each string.

Exercise 3

NOTE: This is an optional exercise that you may choose to do if your class has many very capable students.

N 32

Write this number on the board.

$$2^3 \times 3 \times 5^3$$

T: How many positive divisors does $2^3 \times 3 \times 5^3$ have?

S: 32, because $(3 + 1) \times (1 + 1) \times (3 + 1) = 4 \times 2 \times 4 = 32$.

T: Good. Imagine that each positive divisor of $2^3 \times 3 \times 5^3$ is written on a slip of paper and that I place those slips of paper in a box. If I randomly choose one slip of paper from the box, what is the probability that I will get a multiple of 2?

There are several ways to approach this problem. Your students might suggest that the divisors of $2^3 \times 3 \times 5^3$ which are also multiples of 2 be counted and compared to 32. Or they might suggest that the divisors of $2^3 \times 3 \times 5^3$ which are not multiples of 2 be counted. The latter approach requires noting that the divisors of $2^3 \times 3 \times 5^3$ which are not multiples of 2 are just the divisors of 3×5^3. Since there are eight ($2 \times 4 = 8$) divisors of 3×5^3, there must be $32 - 8$ or 24 divisors of $2^3 \times 3 \times 5^3$ which are multiples of 2. Comparing this number to 32, the probability is $\frac{24}{32}$ or $\frac{3}{4}$.

If your students suggest counting the divisors of $2^3 \times 3 \times 5^3$ which are multiples of 2 directly, help them to be systematic by first counting divisors with a factor of 2, then divisors with a factor of 2^2, and finally divisors with a factor of 2^3. Lists of these divisors are given here.

2: 2, 2×3, 2×5, 2×5^2, 2×5^3, $2 \times 3 \times 5$, $2 \times 3 \times 5^2$, $2 \times 3 \times 5^3$

2^2: 2^2, $2^2 \times 3$, $2^3 \times 5$, $2^2 \times 5^2$, $2^2 \times 5^3$, $2^2 \times 3 \times 5$, $2^2 \times 3 \times 5^2$, $2^2 \times 3 \times 5^3$

2^3: 2^3, $2^3 \times 3$, $2^3 \times 5$, $2^3 \times 5^2$, $2^2 \times 5^3$, $2^3 \times 3 \times 5$, $2^3 \times 3 \times 5^2$, $2^3 \times 3 \times 5^3$

Note that there are eight numbers in each list, so altogether there are twenty-four multiples of 2 which are divisors of $2^3 \times 3 \times 5^3$.

The students should conclude that the probability that a divisor of $2^3 \times 3 \times 5^3$ is a multiple of 2 is $\frac{24}{32}$ or $\frac{3}{4}$.

You might wish to pose other questions of this nature; for example, what is the probability that a positive divisor of $2^3 \times 3 \times 5^3$ is:

a multiple of 3?	$(\frac{16}{32} = \frac{1}{2})$
a multiple of 6?	$(\frac{12}{32} = \frac{3}{8})$
a multiple of 10?	$(\frac{18}{32} = \frac{9}{16})$

4

POSITIONAL SYSTEMS

This chapter reviews and extends previous work with various positional systems. Three of the lessons give special attention to the binary (Base Two) system with problems involving binary calculations and the binary number line. These lessons provide support to the work in combinatorics (see Chapter 6 of the World of Numbers Strand) and probability (see Chapter 1 of the Probability and Statistics Strand). The last two lessons of the chapter introduce a new system, Base $\hat{2}$. Here comparison is made to the binary system and some unique characteristics of this system are investigated.

CAPSULE LESSON SUMMARIES

Section 4.1:	Review the binary abacus and binary writing. Do some addition calculations in the binary system. Label a number line using binary notation.
Section 4.2:	Review the binary abacus and binary writing. Label a number line using binary notation. Use binary notation to list some numbers that belong to a given interval of the number line.
Section 4.3:	Review the binary abacus and binary writing. Label a number line using binary notation. Discover a secret number given a sequence of clues involving the location of the number within intervals on a binary number line.
Section 4.4:	Review several different base abaci and writing in different bases. Present a calculation and determine in which base system the calculation was done. Discover a rule for distinguishing even and odd numbers written in base three writing. (Worksheets N10, N11)

Section 4.5: Introduce the rule of the base $\hat{2}$ abacus. Practice putting and reading numbers on it and making plays. Observe the effect of moving checkers to the left on the abacus. Starting with 10,000 checkers on the one's board make plays to put 10,000 in standard configuration on the base $\hat{2}$ abacus.

Section 4.6: Review the rule of the base $\hat{2}$ abacus. Practice putting and reading numbers on it and making plays. Observe the effect of moving checkers to the left on the abacus. Discover the base $\hat{2}$ code for $\frac{1}{3}$.
(Worksheets N12, N13, N14, N15)

4.1 Binary Writing #1

CAPSULE LESSON SUMMARY

Review the binary abacus and binary writing. Do some addition calculations in the binary system. Label a number line using binary notation.

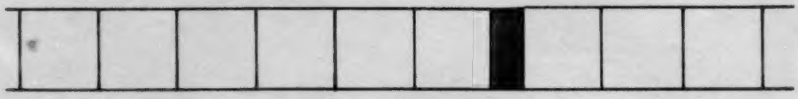
MATERIALS

Teacher: Colored chalk; demonstration Minicomputer checkers
Student: Paper; pencil

DESCRIPTION OF LESSON

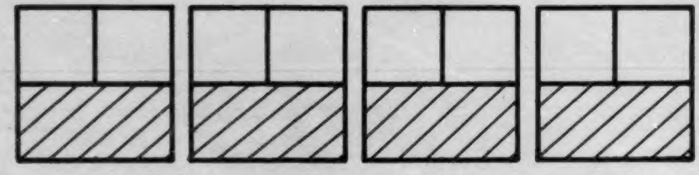
Exercise 1

Draw this part of an abacus on the chalkboard or use Minicomputer boards.[†]



T: Do you remember the binary abacus and its rule?

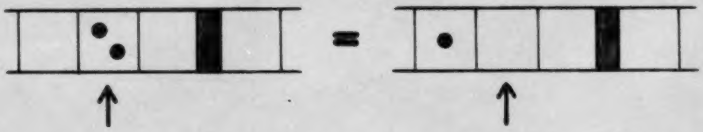
[†] If your chalkboard is not magnetic, a magnetic abacus can be constructed using your demonstration Minicomputer boards. Cover the red and white square with paper and use only the brown and purple squares on each board.



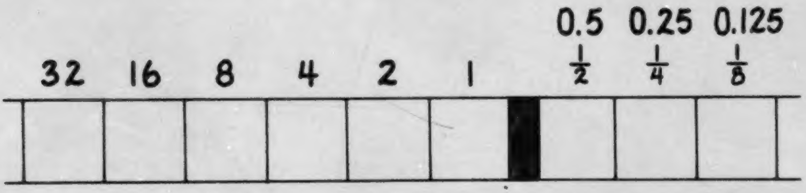
N4

Invite a student to explain the binary abacus to the class. The class should be reminded that the rule is:

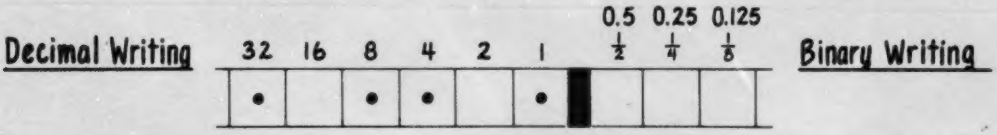
> Two checkers on a board represent the same number as one checker on the next board to the left.



Ask the students to label the boards of the abacus.


0.5
0.25
0.125
32
16
8
4
2
1
$\frac{1}{2}$
$\frac{1}{4}$
$\frac{1}{8}$

Put (draw) this configuration of checkers on the abacus and write the two headings close to the abacus.


Decimal Writing
0.5
0.25
0.125
32
16
8
4
2
1
$\frac{1}{2}$
$\frac{1}{4}$
$\frac{1}{8}$
Binary Writing

T: Which number is this?

S: 45.

Write "45" to the left of the abacus under "Decimal Writing".

T: How do we write this number in binary writing?

Ask a student to write the number on the board.

Decimal Writing		32	16	8	4	2	1	0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
45	=	●		●	●		●				=	101101

NOTE: You will notice that whenever we write numbers in the binary code we use "1" and whenever we write numbers in the decimal code, we use "|". This device serves as a visual reminder to students that we are writing in a code different from the decimal code; it has no mathematical significance.

Continue this activity with a few more examples, such as the ones below.

Decimal Writing		32	16	8	4	2	1	0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
32	=		●	●	●	●	●	●	●	● ●	=	100000

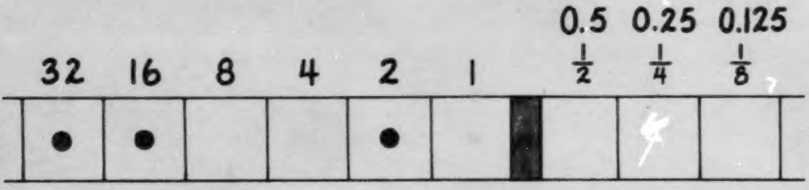
Decimal Writing		32	16	8	4	2	1	0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
38.75	=	●			●	●		●	●		=	100110.11

Decimal Writing		32	16	8	4	2	1	0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
63.5	=	●	●	●	●	●	●	●			=	111111.1

Observe that when putting a number in the binary abacus one can always do it with at most one checker on a board. Discuss why this is so and how then the binary writing of numbers uses only "0" and "1".

N4

Ask a student to put 50 on the abacus using at most one checker on any board.

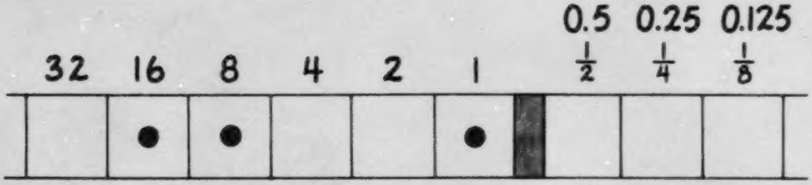

32
16
8
4
2
1
0.5
$\frac{1}{2}$
0.25
$\frac{1}{4}$
0.125
$\frac{1}{8}$

T: On your paper, write 50 in binary writing.

Ask a student to write the number on the board.

Decimal Writing		Binary Writing
50	=	110010

T: I'll move each checker on the abacus one board to the right. Now which number is on the abacus?


32
16
8
4
2
1
0.5
$\frac{1}{2}$
0.25
$\frac{1}{4}$
0.125
$\frac{1}{8}$

S: 25, moving the checkers one board to the right halves the number.

T: Good. Write 25 in binary writing.

Decimal Writing		Binary Writing
50	=	110010
25	=	11001

Repeat this activity three times. Each time move the checkers one board to the right and then ask the students to write the number using binary notation.

Decimal Writing		32	16	8	4	2	1		0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
12.5	=			●	●				●			=	1100.1

Decimal Writing		32	16	8	4	2	1		0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
6.25	=				●	●				●		=	110.01

Decimal Writing		32	16	8	4	2	1		0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
3.125	=					●	●				●	=	11.001

T: Do you see an easy way to calculate one-half of a number when it's given in binary writing?

S: If you are writing in binary, just move the (binary) point one place to the left.

Exercise 2

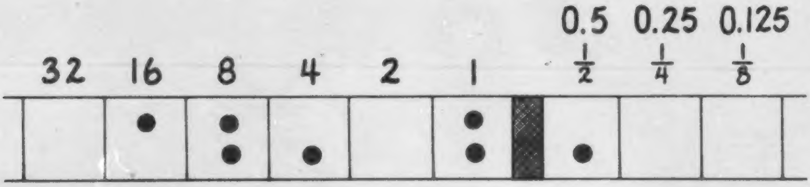
Write this addition problem on the board.

$$\begin{array}{r} 11001 \\ +\ 1101.1 \\ \hline \end{array}$$

T: I have written an addition problem in the binary code. How do you suggest we solve this problem?

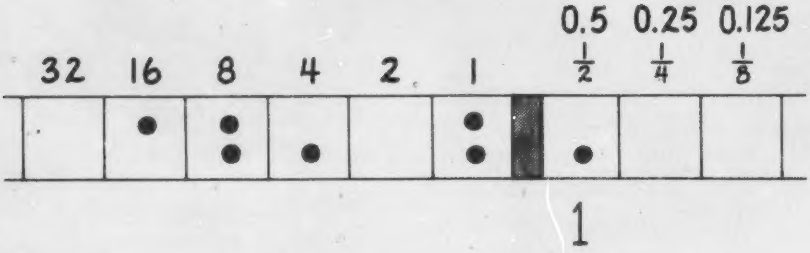
Some students might suggest rewriting the problem in the decimal system, adding in that system, and then converting back to binary. Indicate that such a method is overly cumbersome, but that rewriting into the decimal system might provide a way to check your work. Others may be able to solve the problem in the binary system directly (without the support of the binary abacus). For the benefit of those who need the visual support of the binary abacus, recommend that the class solve the problem in the binary system and make use of the abacus as follows.

Put the addends on the abacus, using checkers of a different color for each. In this illustration, **11001** is in red checkers and **1101.1** is in blue checkers.



T: Look carefully at each board of the abacus. There is only one checker on the $\frac{1}{2}$'s board, so no play need be made with that checker.

As you point this out to the students, write the corresponding information in the appropriate places under the abacus and in the addition calculation.

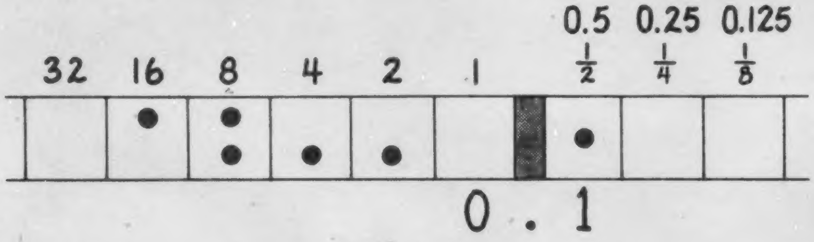


$$\begin{array}{r} 11001 \\ +\ 1101.1 \\ \hline 1 \end{array}$$

T: Now look at the ones' board. There are two checkers on this board so we can make a move; this is the same as adding **1** and **1**.

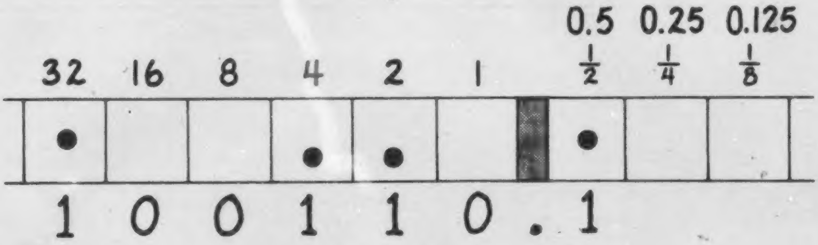
Make the play and add the corresponding information to the appropriate places under the abacus and in the addition calculation.

NOTE: After the initial placement of checkers on the abacus, the colors are no longer relevant. Therefore, the pictures from here on use black checkers.


0.5
0.25
0.125
32
16
8
4
2
1
$\frac{1}{2}$
$\frac{1}{4}$
$\frac{1}{8}$
0 . 1

$$\begin{array}{r} 110\overset{1}{0}1\\ +\ 1101.1\\ \hline 0.1\end{array}$$

Continue working from the abacus to the calculation until you have reached this configuration and have completed the calculation.


0.5
0.25
0.125
32
16
8
4
2
1
$\frac{1}{2}$
$\frac{1}{4}$
$\frac{1}{8}$
1 0 0 1 1 0 . 1

$$\begin{array}{r} \overset{1}{}\overset{1}{1}10\overset{1}{0}1\\ +\ 1101.1\\ \hline 100110.1\end{array}$$

If you feel that it is necessary, check the calculation using decimal writing.

Binary Writing		Decimal Writing
11001	=	25
+ 1101.1	=	+ 13.5
100110.1	=	38.5

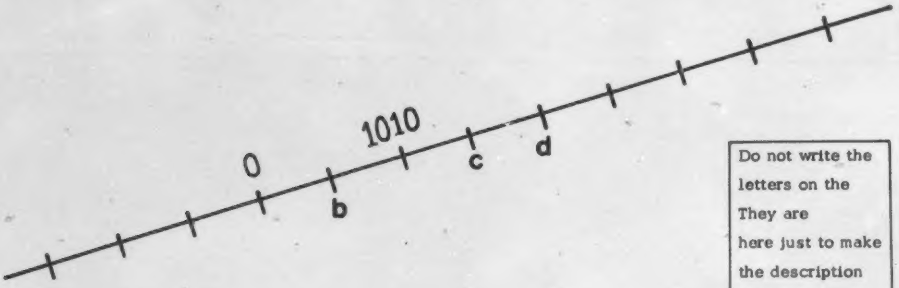
N4

Continue the activity with these problems. Allow students a chance to do the calculations on their papers and then check the solutions collectively. Answers are in the boxes.

$$\begin{array}{r} 101010 \\ +\ 11101 \\ \hline \boxed{1000111} \end{array} \qquad \begin{array}{r} 110011.001 \\ +\ 11001.101 \\ \hline \boxed{1001100.110} \end{array} \qquad \begin{array}{r} 111.011 \\ +\ 10.101 \\ \hline \boxed{1010.000} \end{array}$$

Exercise 3

Draw this part of a number line with labels in binary writing on the board.



Do not write the letters on the They are here just to make the description of the lesson easier to follow.

T: This is part of a number line with the marks labeled in binary writing. How should we label this mark (point to b)? Remember, we want to use binary writing.

Allow a few minutes for the students to study the given information. As with calculations in the binary system, some students might suggest converting to decimal writing, determining the appropriate decimal label, and then converting back. Indicate that this method would be unnecessarily involved. If necessary, ask students to notice that the mark (b) is halfway between **0** and **1010**. Then ask about halving in the binary system.

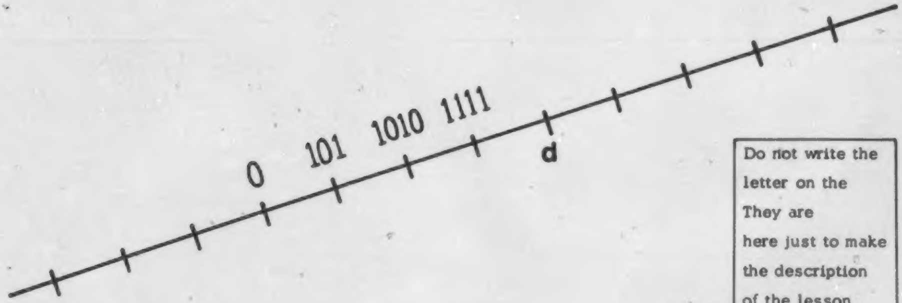
S (pointing to b): This number is **101** because it is one-half of **1010.**

The class could do the "one-half of **1010**" calculation on the binary abacus to verify this result.

T (pointing to c): How should we label this mark? Again, we want to use binary writing.

Allow a couple minutes for the students to study the situation. They should observe that the first mark to the right of **0** is for **101** and the second mark to the right of **0** is for **1010** (**101** + **101**). So to find labels for the other marks in consecutive order, they can continue adding **101**.

S: We just add **101** to **1010**; **1010** + **101** = **1111.**



Do not write the letter on the They are here just to make the description of the lesson easier to follow.

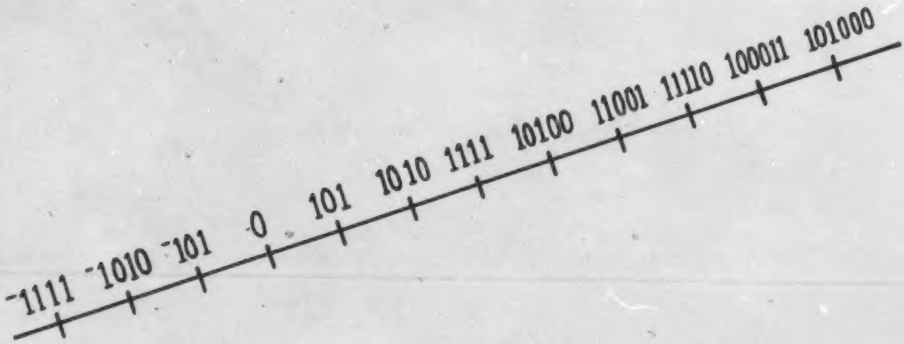
If necessary, explain yourself that on this number line we label consecutive marks counting by **101** (in the binary system). So, each time you want to label a mark add **101** to the number at the mark directly to its left .

T (pointing to d): How should we label this mark?

S: Since **1111** + **101** = **10100**, label it "**10100**".

N 4

Invite students to finish labeling the marks on the number line on the board. A complete labeling is shown below.


-1111
-1010
-101
0
101
1010
1111
10100
11001
11110
100011
101000

4.2 Binary Writing #2

CAPSULE LESSON SUMMARY

Review the binary abacus and binary writing. Label a number line using binary notation. Use binary notation to list some numbers that belong to a given interval of the number line.

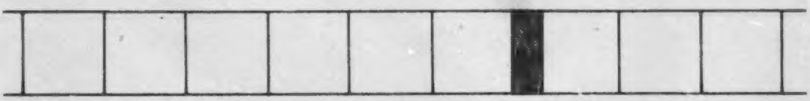
MATERIALS

Teacher: Colored chalk; demonstration Minicomputer checkers
Student: Paper; pencil

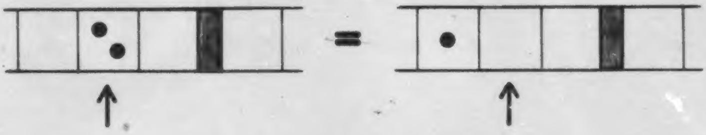
DESCRIPTION OF LESSON

Exercise 1

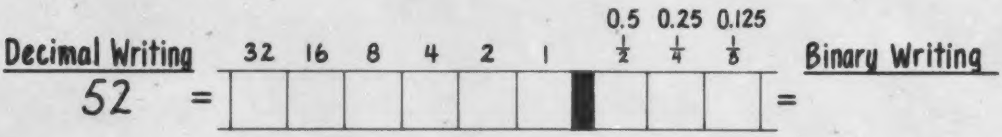
Display this part of an abacus (draw it on the chalkboard or use Minicomputer boards as suggested in Lesson N4.1) and briefly review the rule for the binary abacus.



Two checkers on a board represents the same number as one checker on the next board to the left.



Label the boards of the abacus and put the two headings with indicated information as shown below close to the abacus.



N 10

T: On your paper, write this number, 52, in binary writing. Use the abacus as an aid.

If necessary, point out to students that in order for the binary abacus to aid in binary writing one puts on the number with at most one checker on a board. After a couple minutes, ask a student to put 52 on the binary abacus and to write it in binary writing.

Decimal Writing		32	16	8	4	2	1		0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
52	=	●	●		●			▮				=	110100

NOTE: Recall that whenever we write numbers in the binary code we use "1" rather than "I".

Continue this activity using the problems below. Answers are in the boxes.

Decimal Writing		32	16	8	4	2	1		0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
$\boxed{26}$	=		●	●		●		▮				=	11010

Decimal Writing		32	16	8	4	2	1		0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
13	=			●	●		●	▮				=	$\boxed{1101}$

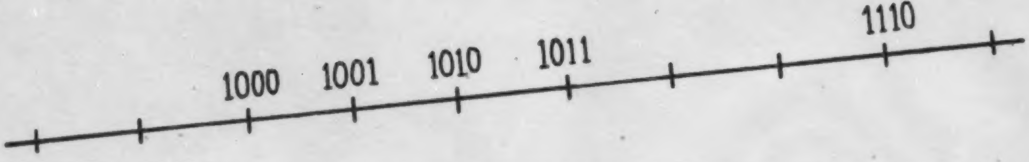
Decimal Writing		32	16	8	4	2	1		0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
8.125	=			●				▮			●	=	$\boxed{1000.001}$

Decimal Writing		32	16	8	4	2	1		0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
$\boxed{5}$	=				●		●	▮				=	101

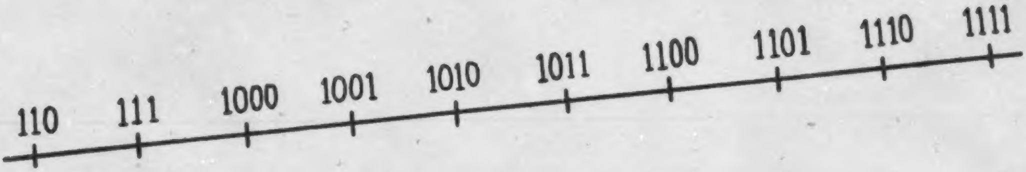
Decimal Writing		32	16	8	4	2	1		0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$		Binary Writing
$\boxed{\frac{5}{8}}$	=							▮	●		●	=	0.101

Exercise 2

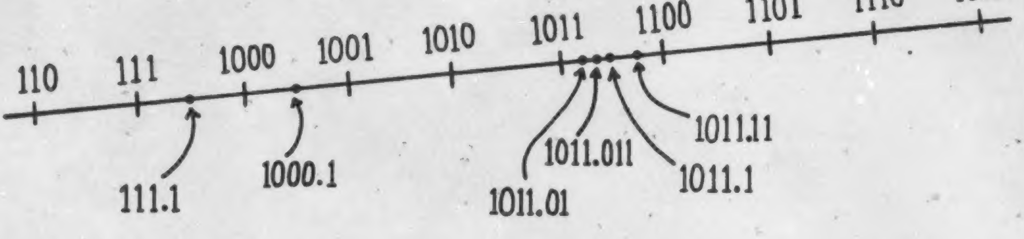
Draw this part of a number line with labels in binary writing on the board.


1000
1001
1010
1011
1110

Invite students to label any mark they wish until all the marks are labeled. The important observation to make is that consecutive marks have labels which differ by **1** ; note **1000** and **1001** .

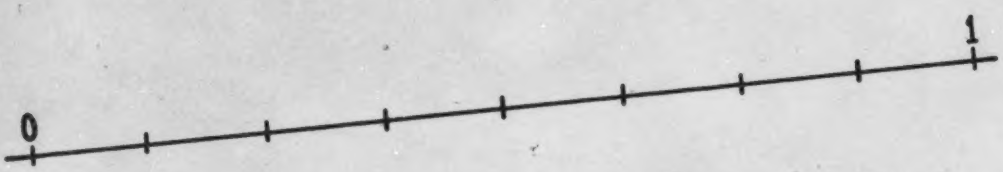

110
111
1000
1001
1010
1011
1100
1101
1110
1111

Add the points shown below to your drawing and invite students to label them.


110
111
1000
1001
1010
1011
1100
1101
1110
1111
111.1
1000.1
1011.01
1011.011
1011.1
1011.11

Exercise 3

Draw this part of a number line on the board.



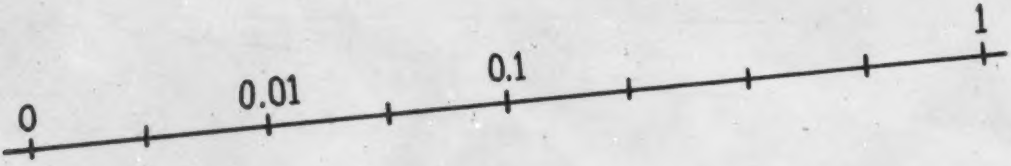
T: Copy this part of a number line and label it in binary writing.

Allow several minutes for independent work before collectively labeling the number line. Let the students label the marks in any order they wish. Perhaps they will observe that one method involves repeated halving; thus label the mark halfway between **0** and **1** first, the mark halfway between **0** and **0.1** ($\frac{1}{2}$) second, the mark halfway between **0** and **0.01** ($\frac{1}{4}$) third, and then label the other marks by adding.

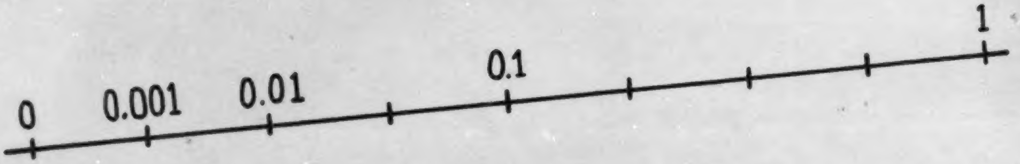
S (pointing to the appropriate mark): This mark is halfway between **0** and **1**; so it is for $\frac{1}{2}$. In binary writing that is "**0.1**".

S (pointing to the appropriate mark): Halfway between **0** and **0.1** the mark is for **0.01**; you just move the point over one place to the left.

S: **0.01** is $\frac{1}{4}$ in binary writing and that mark is one-fourth of the way from **0** to **1**.

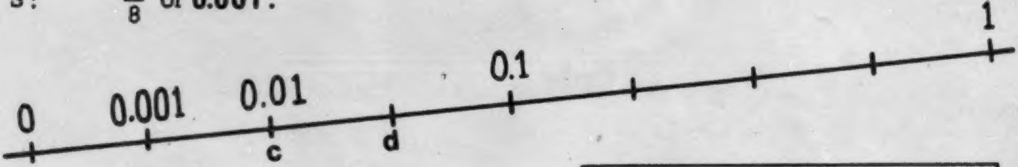


S (pointing to the appropriate mark): Halfway between **0** and **0.01** the mark is for **0.001**. That is $\frac{1}{8}$ in binary writing and that mark is one-eighth of the way from **0** to **1**.



T: The marks on this number line show counting by how much?

S: $\frac{1}{8}$ or **0.001**.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

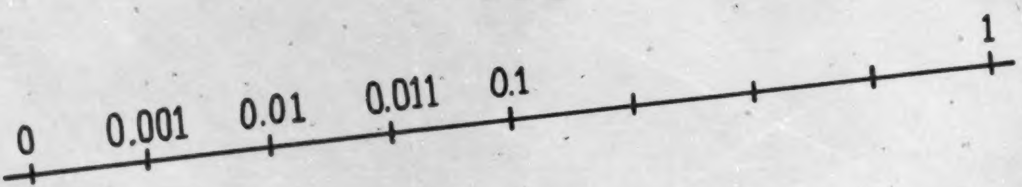
T (pointing to d): How much larger is this number than (point to c) **0.01**?

S: **0.001** larger or $\frac{1}{8}$ larger.

T: Calculate "**0.01 + 0.001**".

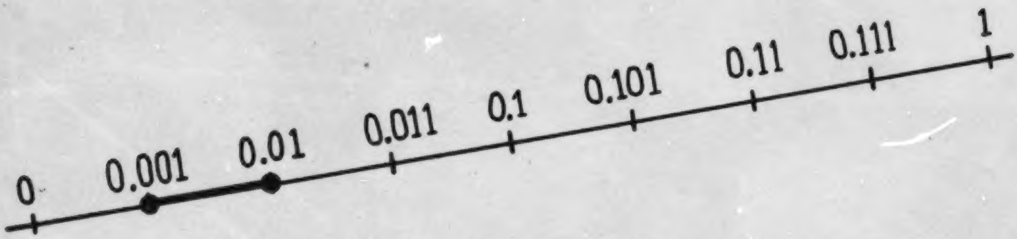
Let the students do this calculation individually or, if you prefer, do the calculation collectively at the board.

$$\begin{array}{r} 0.01 \\ +\ 0.001 \\ \hline 0.011 \end{array}$$



N 10

Continue in this manner until the number line is completely labeled; then add the red segment shown in the illustration below.



T: Give me in binary writing a number that belongs to the red segment but is not one of the endpoints.

Allow a few minutes for the class to study the situation and for independent work. Encourage faster students to find several numbers that belong to the red segment. Make a list of suggested numbers on the board. Several numbers suggested by students in our development classes are given below.

0.0011
0.0011101
0.00101010101 . . .
0.001111

T: Do you notice anything interesting about these numbers?

S: Each number starts "**0.001**".

S: Any number starting with "**0.001**" and followed by as many "**0**"'s and "**1**"'s as you wish will belong to the red segment. Also **0.01** belongs to the red segment, but it is an endpoint.

Choose another segment, for example from **0.101** to **0.11**, and repeat this activity. The students should conclude that any number starting with "**0.101**" and also the endpoint **0.11** belongs to this red segment.

Write this number on the board.

0.1111

T: Where does **0.1111** belong on this number line?

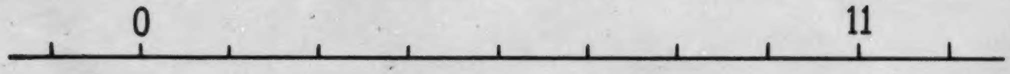
S: **0.1111** is between **0.111** and **1**.

Ask the students to locate a few more numbers as between two given marks on the number line, for example:

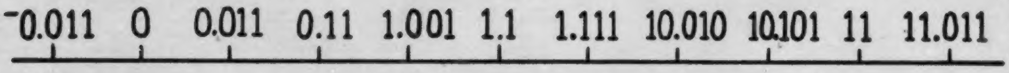
0.0110101	between **0.011** and **0.1**
0.11011	between **0.11** and **0.111**
0.000101	between **0** and **0.001**

If you wish, the following problem could be assigned to be done in class or as homework.

Label this part of the number line in binary writing.



Solution:


⁻0.011
0
0.011
0.11
1.001
1.1
1.111
10.010
10.101
11
11.011

4.3 Binary Writing #3

CAPSULE LESSON SUMMARY

Review the binary abacus and binary writing. Label a number line using binary notation. Discover a secret number given a sequence of clues involving the location of the number within intervals on a binary number line.

MATERIALS

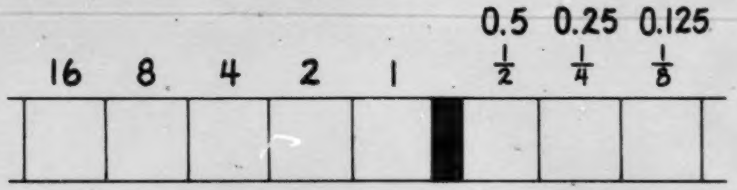
Teacher: Meter stick; colored chalk; demonstration Minicomputer checkers

Student: Paper; pencil

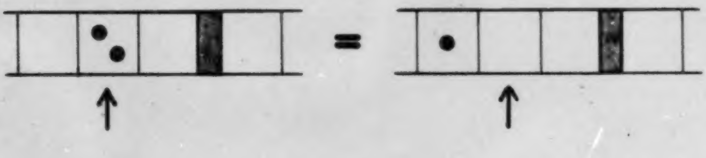
DESCRIPTION OF LESSON

Exercise 1

Display this part of an abacus (draw it on the chalkboard or use Minicomputer boards as suggested in Lesson N4.1) and briefly review the rule for the binary abacus. Use the rule to label the boards of the abacus.



Two checkers on a board represents the same number as one checker on the next board to the left.



Write this information close to the abacus.

Binary Writing	Decimal Writing
1000100	

T: On your paper, write this number in decimal writing.

If necessary, point out to students that they can first put the number on the binary abacus. After a couple minutes ask a student to put the number on the abacus and to write it in decimal writing.

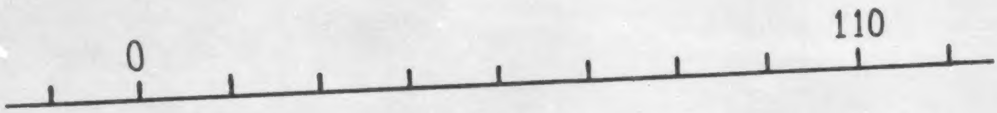
Binary Writing	Decimal Writing
1000100	68

Continue this activity using the problems suggested below. Answers are in the boxes.

Binary Writing	Decimal Writing
$\boxed{100010}$	34
10001	$\boxed{17}$
1000.1	$\boxed{8.5}$
100.01	$\boxed{4.25}$
$\boxed{101}$	5
0.101	$\boxed{\frac{5}{8} = 0.625}$
0.0101	$\boxed{\frac{5}{16} = 0.3125}$

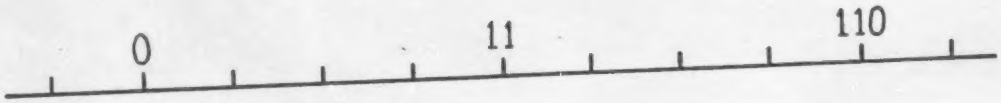
Exercise 2

Draw this part of a number line with labels in binary writing on the board.

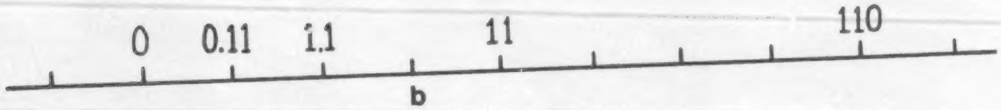


T: I would like to label the other marks on this number line in binary writing.

Invite students to label the marks in any order they choose; however, presumably a likely first choice will be to label the midpoint of **0** and **110**.



If your students are hesitant, encourage them to recall the ease of halving in the binary system and to label the midpoints of segments from **0** to some other labeled point. If necessary, the abacus can be used to verify calculations. Continue until the mark to the right of **0** is labeled.



Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

T (tracing the segment from **0** to **0.11**): The marks on this number line show counting by how much? Remember we are thinking of the binary writing of numbers here.

S: **0.11.**

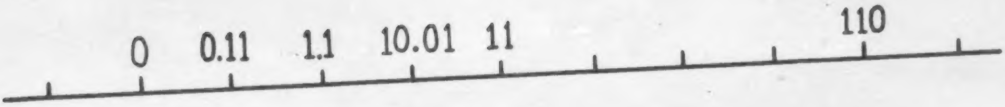
T (pointing to b): How much larger is this number than **1.1** ?

S: **0.11** larger.

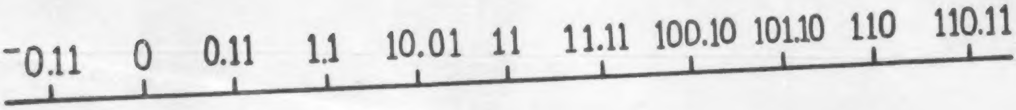
T: Calculate " **1.1 + 0.11**".

The students may do this calculation individually or, if you prefer, collectively with someone at the board.

$$\begin{array}{r} 1 \\ 1.1 \\ +\ 0.11 \\ \hline 10.01 \end{array}$$

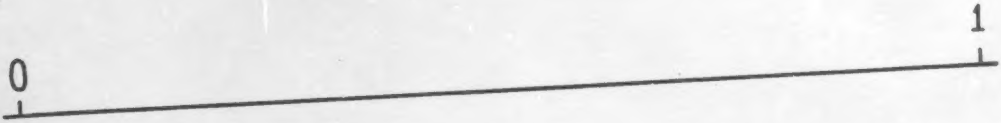

0
0.11
1.1
10.01
11
110

Continue until all the marks on the number line are labeled.


-0.11
0
0.11
1.1
10.01
11
11.11
100.10
101.10
110
110.11

Exercise 3

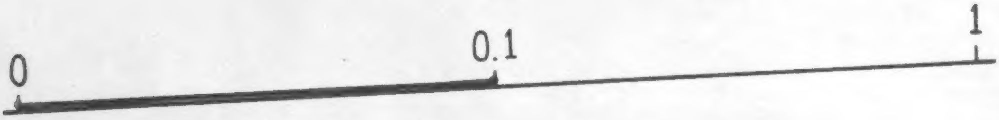
Draw this part of a number line with labels in binary writing on the board. Be sure your number line is at least a meter long.



T: I am thinking of a secret number. My number is located somewhere in this part of the binary number line.

I will give you some clues to help you locate the number.

Add this blue segment to your number line.



T: My secret number is in this blue segment. What does this clue tell you about the blue segment?

S: The blue segment starts at 0 and ends at the midpoint of **0** and **1**, **0.1**.

T: Good. So what does the clue say about my secret number?

Let the class have a few minutes to consider your question.

S: Your secret number is between **0** and **0.1** .

S: Or it could be one of the endpoints **0** or **0.1**.

T: Yes. If we were to write my secret number in binary writing, how would we start?

S: "**0.0**".

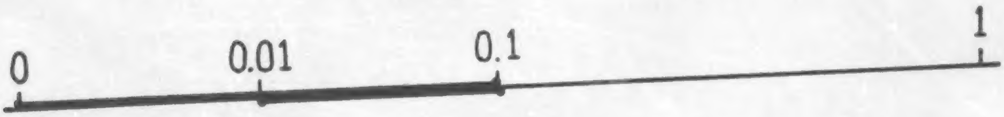
S: Or just "**0.1**".

Record the students' suggestions on the board in the following manner.

0.0. . . or 0.1

T: The three dots indicate that there could be a sequence of **0**'s and **1**'s that we have not yet determined.

The second clue is that my secret number is also in this red segment. What do you notice?



S: The red segment starts at **0.01** and ends at **0.1**. It is the right half of the blue segment.

S: The secret number is between **0.01** and **0.1**.

S: Or it is one of the endpoints **0.01** or **0.1** .

T: What more does this clue tell us about the binary writing of my secret number?

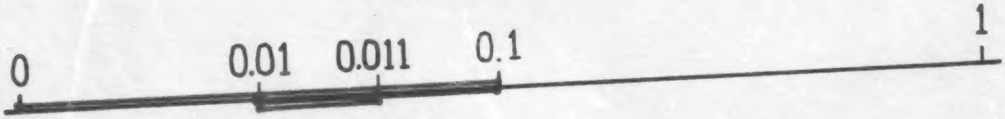
S: It still could be just "**0.1**", or it starts with "**0.01**".

Record the students' suggestions on the board.

0.0. . . or 0.1
0.01. . . or 0.1

T: The third clue is that my secret number is in this smaller blue segment.

Draw a blue segment as shown below and let the students comment.



S: This time you took the left half of the red segment.

S: The secret number is between **0.01** and **0.011**.

S: Or it could be one of the endpoints, **0.01** or **0.011**.

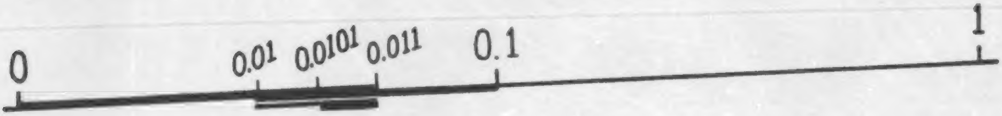
S: The secret number cannot be **0.1**.

T: Good. What more does this clue tell us about the binary writing of the secret number?

S: It starts with "**0.010**" or it is just "**0.011**".

0.0. . .	or	0.1
0.01. . .	or	0.1
0.010. . .	or	0.011

T: The fourth clue is this red segment. What do you think?



S: You took the right half of the blue segment — left, right, left, right.

S: The secret number is between **0.0101** and **0.011**.

S: Or it is one of the endpoints, **0.0101** or **0.011**.

T: What about the writing of this number?

S: It starts with "**0.0101**" or it is just "**0.011**".

0.0. . .	or	0.1
0.01. . .	or	0.1
0.010. . .	or	0.011
0.0101. . .	or	0.011

N 14

If the class has not yet caught on to your game, continue to give clues each time taking either the left or right half of the previous segment. When some of the students have discovered this strategy, ask:

T: Can you predict what my next clue will be?

S: A blue segment which is the left half of the red segment.

S: You always choose the left half with blue, then the right half with red.

S: Your clues go on and on. The segments get shorter and shorter.

T: Yes. If we could go on forever, we would find the secret number. Do you know what my secret number is?

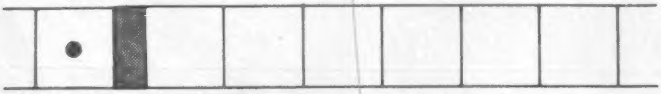
Let the students have some time to consider the situation.

S: It is 0.01010101 . . . .

S: It looks like $\frac{1}{3}$ on the number line.

If at this point no student has come forth with $\frac{1}{3}$, put 1 on the binary abacus and calculate $\frac{1}{3} \times 1$ as indicated in the following dialogue.

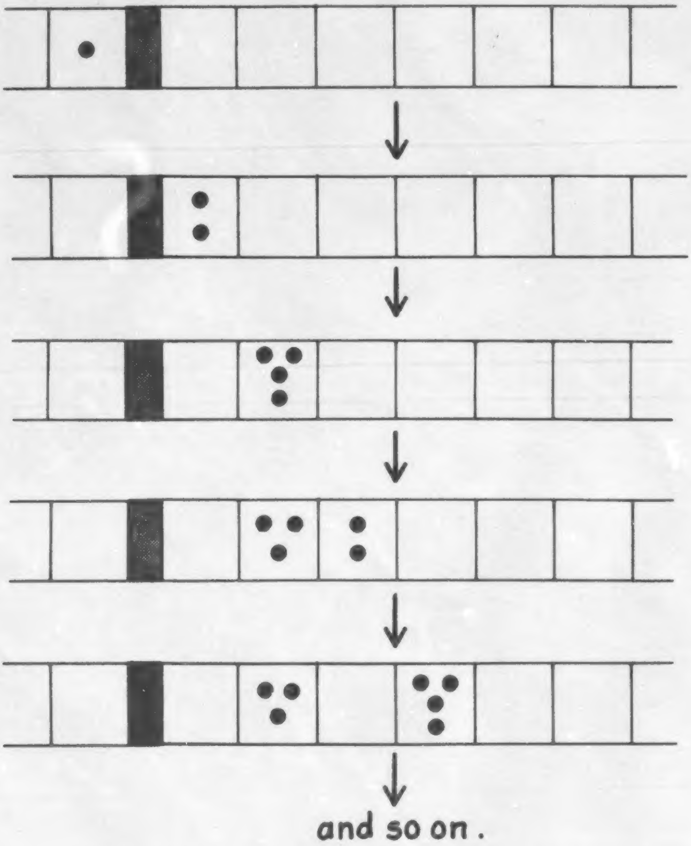
T: You have suggested that "0.010101 . . ." is the binary writing for $\frac{1}{3}$. Let's verify this on the binary abacus. I will put 1 on the abacus.



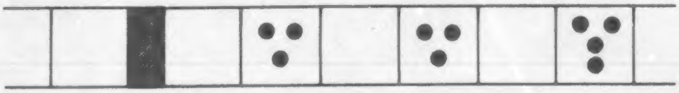
T: How can we calculate "$\frac{1}{3} \times 1$"?

S: Try to get three checkers on any board with checkers on it. Then just leave one checker on each board where there are three.

Invite students to make the appropriate backward plays as indicated in the sequence of pictures below.



Continue until this configuration is on the abacus.



S: We can go on and on.

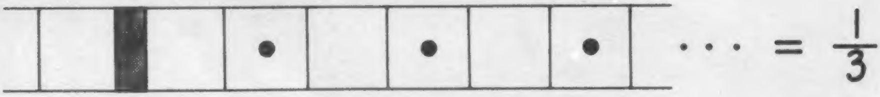
N 14

T: Which number is on the abacus?

S: 1.

T: Who can show $\frac{1}{3}$?

Let a student remove two out of every three checkers, leaving this configuration on the abacus.



T: Write $\frac{1}{3}$ in binary writing.

$$\frac{1}{3} = 0.010101\ldots$$

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4.4 Review of Positional Systems

CAPSULE LESSON SUMMARY

Review several different base abaci and writing in different bases. Present a calculation and determine in which base system the calculation was done. Discover a rule for distinguishing even and odd numbers written in base three writing.

MATERIALS

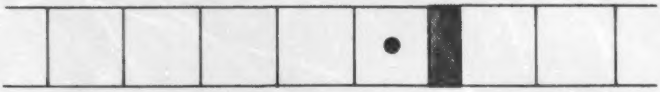
Teacher: Demonstration Minicomputer checkers; weighted checkers

Student: Paper; pencil; Worksheets N10 and N11

DESCRIPTION OF LESSON

Exercise 1

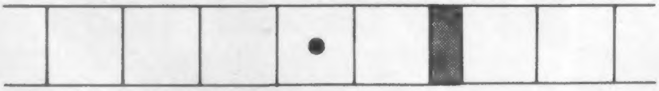
Display this part of an abacus (draw it on the chalkboard or use Minicomputer boards as suggested in Lesson N4.1) and put on one checker as shown below.



T: Which number is on the abacus?

S: 1.

Move the checker one square to the left.



N 23

T: Now which number is on the abacus?

Allow the students to respond without commenting yourself. Someone should observe that the question cannot be answered until more is known about the abacus.

S: We don't know what number it is until we know the rule.

T: That's right. Which rules are you familiar with?

S: Two checkers on a board is the same as one checker on the next board to the left.

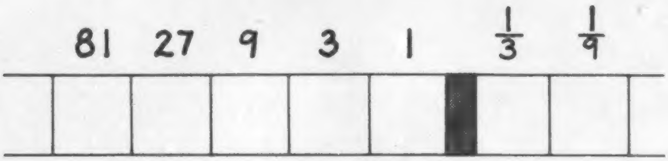
S: That would be the binary or base two abacus. We also know about base three and base twelve.

T: Suppose this were the base three abacus. What is the rule?

S: Three checkers on a board is the same as one checker on the next board to the left.

T: How would we label the boards of the base three abacus?

Label the boards with student assistance.



T: If we were conserving checkers, what is the most we would ever need to put on one board?

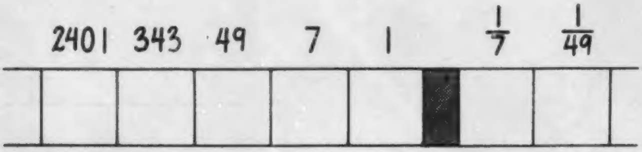
Pause a minute for students to consider the question.

S: Two checkers, because if there were three checkers we could make a play, exchanging the three checkers for one checker on the next board to the left.

T: Yes. How many digits do we need for writing in base three?

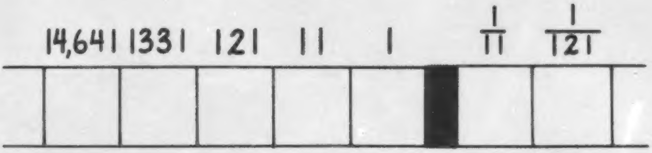
S: Three digits: 0, 1, and 2.

Repeat this activity for base seven and for base eleven. A base seven abacus is illustrated below.



The students should conclude that at most six checkers are needed on one board of the base seven abacus and that seven digits (0, 1, 2, 3, 4, 5, 6) are needed for writing in base seven.

For base eleven the situation is analogous.



At most ten checkers are needed on one board of a base eleven abacus.

T: How many digits do we need for writing in base eleven?

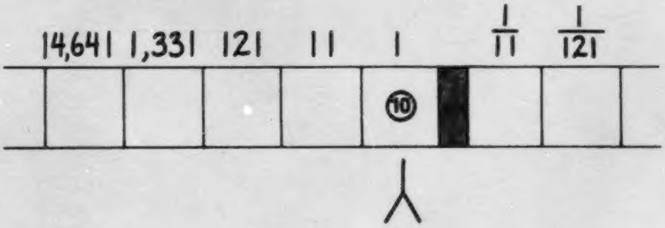
S: Eleven digits.

T: Yes, do you remember those digits?

S: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and a special one.

T: In a previous lesson, we invented a digit for the situation where ten checkers were on a board. Do you recall which symbol we used?

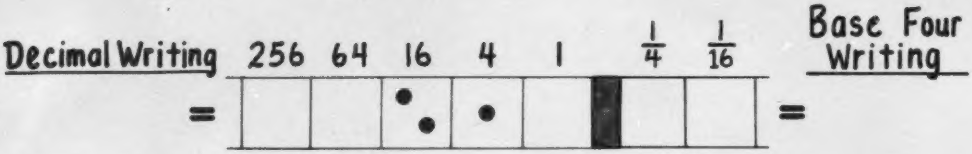
Invite a student to write the symbol on the chalkboard.


14,641
1,331
121
11
1
$\frac{1}{11}$
$\frac{1}{121}$
⑩
⅄

NOTE: The symbol above is the symbol invented by the students in one of our development classes and probably is not the symbol your students chose to use. Make the appropriate adjustments in the remainder of the lesson by replacing "⅄" with your class's symbol.

T: Now suppose this is the base four abacus. Let's label the boards.

Invite a student to label the boards. Then put this configuration on the abacus and write the two headings close to the abacus.


Decimal Writing
=
256
64
16
4
1
$\frac{1}{4}$
$\frac{1}{16}$
Base Four Writing
=

T: Which number is this?

S: 36.

Write "36" under "Decimal Writing".

T: How do we write this number in base four writing?

Decimal Writing		256	64	16	4	1		$\frac{1}{4}$	$\frac{1}{16}$		Base Four Writing
36	=			●●	●		▮			=	210

Continue this activity with a few more problems such as the ones given here. Vary the problem by sometimes giving the base four writing and asking the students to write the number in decimal writing. Answers are in the boxes.

Decimal Writing		256	64	16	4	1		$\frac{1}{4}$	$\frac{1}{16}$		Base Four Writing
$\boxed{30}$	=			●	●●●	●●	▮			=	132

Decimal Writing		256	64	16	4	1		$\frac{1}{4}$	$\frac{1}{16}$		Base Four Writing
$85\frac{6}{16}$	=		●	●	●	●	▮	●	●	=	$\boxed{1111.11}$

Decimal Writing		256	64	16	4	1		$\frac{1}{4}$	$\frac{1}{16}$		Base Four Writing
$\boxed{120\frac{9}{16}}$	=		●	●●●	●●		▮	●●	●	=	1320.21

Continue this activity using base twelve.

Decimal Writing			1,728	144	12	1		$\frac{1}{12}$	$\frac{1}{144}$		Base Twelve Writing
125	=				⑩	⑤	▮			=	$\boxed{⅄ 5}$

Decimal Writing			1,728	144	12	1		$\frac{1}{12}$	$\frac{1}{144}$		Base Twelve Writing
$\boxed{200\frac{1}{4}}$	=			●	④	⑧	▮	③		=	148.3

N 23

Exercise 2

Write this information on the board.

Calculation	Base
12 + 13 = 30	☐

T: Here is an addition problem that is correct. Can you tell me the base this problem is written in?

Let the students have several minutes to study the situation and to experiment independently before leading a collective discussion.

S: It cannot be in decimal notation because $12 + 13 \neq 30$.

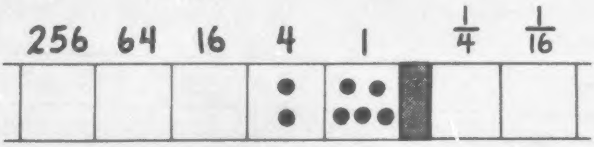
S: It cannot be in base two because 0 and 1 are the only two digits in base two.

S: Also, it cannot be in base three because there is no digit 3 in base three.

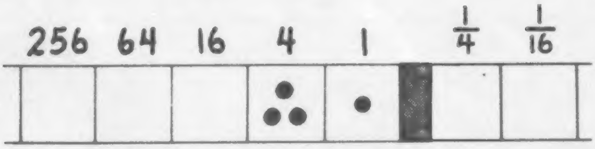
Some students may rely strictly on the abacus as an aid to finding a solution.

S: Maybe it is written in base four. Let's use a base four abacus to do the problem.

Relabel the abacus so that it is a base four abacus and ask a student to put checkers on the abacus.



Ask a student to make plays to obtain:


256
64
16
4
1
$\frac{1}{4}$
$\frac{1}{16}$

T: In base four **12 + 13 = 31**, not **30**.

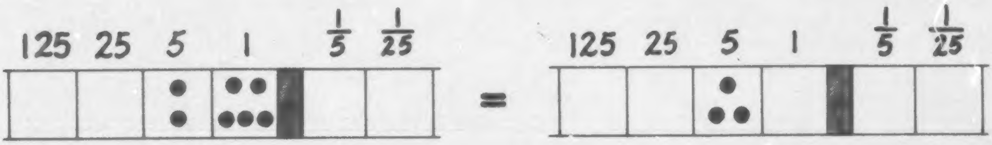
S: I tried something different. I wrote the problem this way.

$$\begin{array}{r} 12 \\ +\,13 \\ \hline 30 \end{array}$$

Then I asked in which base is 2 + 3 some number ending in "0". That's base five because "**2 + 3 = 10**" in base five.

Once base five is suggested, use the abacus to verify that is indeed the correct base.

In base five:


125
25
5
1
$\frac{1}{5}$
$\frac{1}{25}$
=
125
25
5
1
$\frac{1}{5}$
$\frac{1}{25}$

Calculation	Base
12 + 13 = 30	5

Ask the students to work out these problems. Answers are in the boxes.

Calculation	Base
3 × 4 = 13	9
7 × 8 = 56	10
2 × 2 = 4	Any base greater than 4

The last problem may require some discussion as many answers are possible.

Exercise 3

T: In decimal or base ten writing, how do we recognize odd or even numbers?

S: Look at the one's digit. If it is even, then the number is even; if it is odd, then the number is odd.

T: Yes. Let's examine the situation for base three writing and see if we can find a rule.

Write the list of decimal numbers below on the board and ask the students to write each number using base three writing on their paper. When most of the students have finished, check the work collectively before continuing. Answers are in boxes.

Decimal Writing	Base Three Writing
12	110
10	101
32	1012
40	1111
5	12
7	21
13	111
27	1000

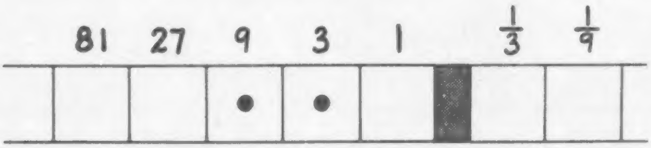
T: Do you notice anything interesting about the base three writing for odd numbers and for even numbers?

S: We cannot use the rule of looking at the last digit.

T: Yes, but can we find a new rule? I'll give you a hint. Think about putting each number on a base three abacus.

Allow a few minutes for examining the situation. If no good response is forthcoming, make the following suggestion.

Ask a student to put 12 on the abacus.



T: How many checkers did we use?

S: Two checkers.

Record this information in a third column.

Decimal Writing	Base Three Writing	Number of Checkers
12	110	2
10	101	
32	1012	
40	1111	
5	12	
7	21	
13	111	
27	1000	

N 23

Continue to fill in the third column. Most students will not need the support of the base three abacus to determine the number of checkers. When the column is complete, continue your questioning.

Decimal Writing	Base Three Writing	Number of Checkers
12	110	2
10	101	2
32	1012	4
40	1111	4
5	12	3
7	21	3
13	111	3
27	1000	1

T: What do you notice?

S: The even numbers require an even number of checkers and the odd numbers require an odd number of checkers.

T: How can we explain this?

S: Each time you put a checker on (to the left of the bar) the base three abacus, you add an odd number. If an odd number of checkers are used, then the sum of an odd number of odd numbers is odd. The sum of an even number of odd numbers is even.

S: When a number is written in base three writing, we can tell whether it is odd or even by the sum of its digits (the number of checkers on the abacus).

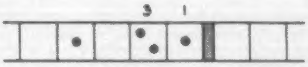
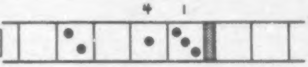
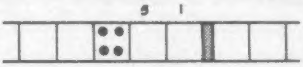
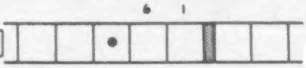
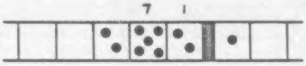
Let the students describe the rule in their own way, then summarize the situation for them.

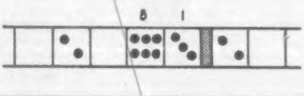
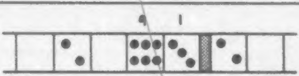
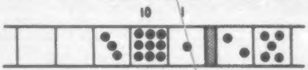
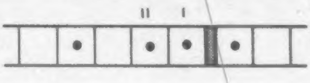
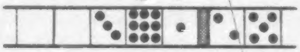
T: So, in base three writing, a number is odd if and only if the sum of its digits is odd. The number is even if and only if the sum of its digits is even.

Worksheets N10 and N11 are available for individual work.

Name ____________________ **N 10 ***

Put each number on the abacus and fill in the boxes.

Decimal Writing 34		Base Three Writing 1021
135		Base Four Writing 2013
100		Base Five Writing 400
36		Base Six Writing 100
$135\frac{1}{7}$		Base Seven Writing 252.1

Decimal Writing $1075\frac{1}{4}$		Base Eight Writing 2063.2
$1515\frac{2}{9}$		Base Nine Writing 2063.2
391.25		Base Ten Writing 391.25
$1343\frac{1}{11}$		Base Eleven Writing 1011.1
$541\frac{29}{144}$		Base Twelve Writing 391.25

Name ____________________ **N 11 ****

Fill in the boxes showing the base for each calculation.

Calculation	Base
15 + 12 = 30	7
2 × 2 = 10	4
22 + 20 = 112	3
6 × 8 = 48	10
100 + 10 = 110	2 Any whole number larger than 2 is also a solution.
3 × 24 = 132	5

4.5 Base $\hat{2}$ Abacus #1

CAPSULE LESSON SUMMARY

Introduce the rule of the base $\hat{2}$ abacus. Practice putting and reading numbers on it and making plays. Observe the effect of moving checkers to the left on the abacus. Starting with 10,000 checkers on the one's board make plays to put 10,000 in standard configuration on the base 2 abacus.

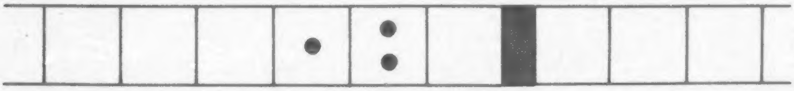
MATERIALS

Teacher:	Colored chalk; demonstration Minicomputer checkers; weighted checkers
Student:	Paper; pencil

DESCRIPTION OF LESSON

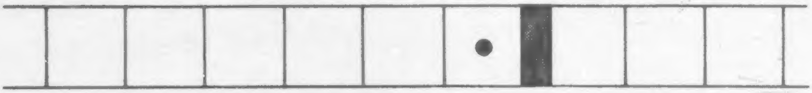
Exercise 1

Display this part of an abacus (draw it on the chalkboard or use Minicomputer boards as suggested in Lesson N4.1) and put on this configuration of checkers.



T: This is a new kind of abacus and the number on it now is 0. In fact, anytime there is one checker on a board and two checkers on the next board to the right, the result is 0. I could just remove the checkers.

Remove the checkers. Then put on one checker as shown.

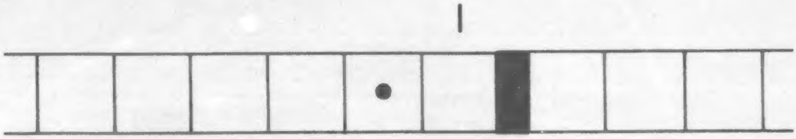


N 27

T: Which number is this?

S: 1.

Label the board "1" and put the one checker on the next board to the left.



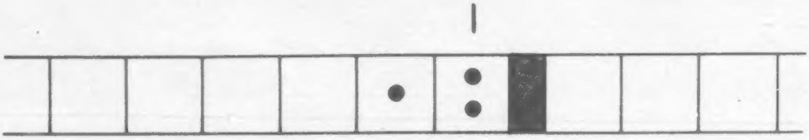
T: Do you know which number this is?

The students will need time to study this situation.

S: $\hat{2}$.

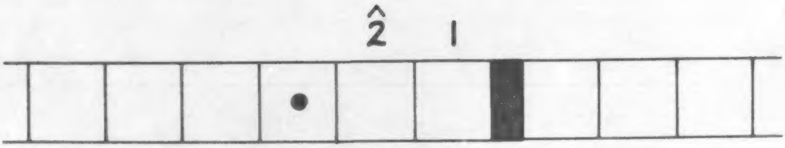
T: How did you get $\hat{2}$?

Invite a student to give an explanation. You may need to do some prompting with such a new situation.



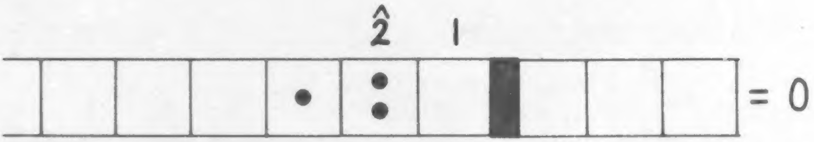
S: This is 0. There are two checkers on the ones' board, that's 2 and $\hat{2} + 2 = 0$. So the other checker must be for $\hat{2}$; it is on the $\hat{2}$'s board.

Label the board "$\hat{2}$" and move the one checker to the next board to the left.

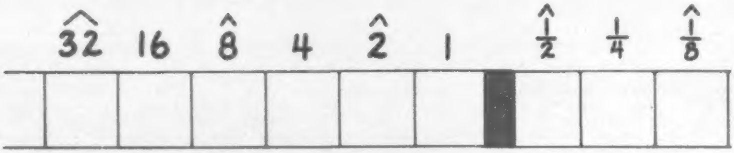


T: Which number is this?

S: 4 because $4 + \hat{4} = 0$.



Label the board "4" and continue in this manner until each board is labeled.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\widehat{\frac{1}{2}}$
$\frac{1}{4}$
$\widehat{\frac{1}{8}}$

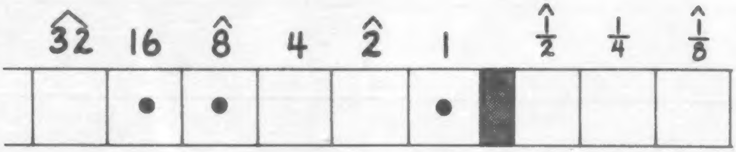
T: Which abacus does this one remind you of?

S: It's like the binary abacus, except every other board is for a negative number. Instead of 2, 8, 32, and so on, here it is $\hat{2}$, $\hat{8}$, $\widehat{32}$, and so on.

T: Just like the binary abacus, we can always put numbers on this abacus with at most one checker on a board.

N 27

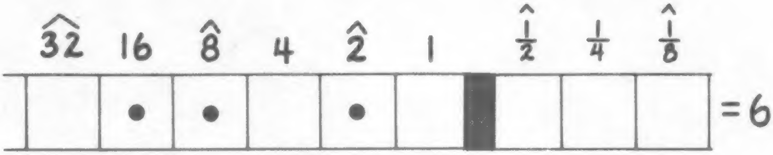
Display this configuration of checkers and ask the students to write the number on their paper.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$

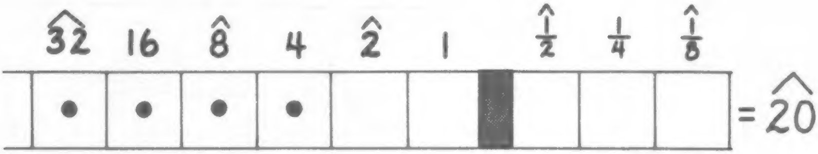
S: $1 + \hat{8} + 16 = 9.$

T: Good. Now put 6 on the abacus. Remember, try to do it with at most one checker on a board.

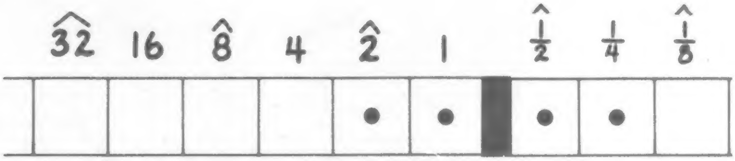
Invite a student to put on the checkers.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$
$= 6$

T: Put $\widehat{20}$ on the abacus.

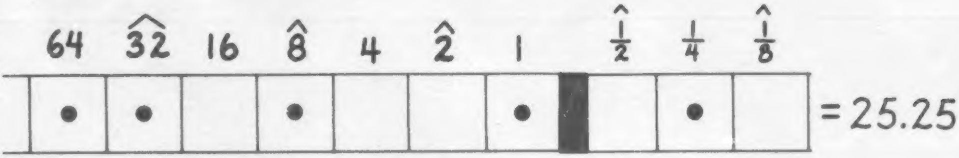

$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$
$= \widehat{20}$

Display this configuration of checkers and ask for the number.

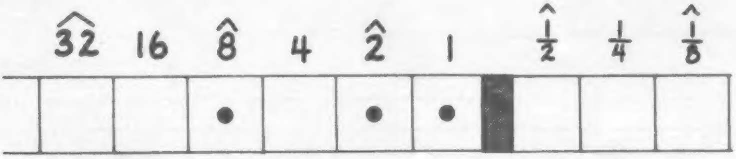

$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$

S: $\widehat{1.25}$, because $\hat{2} + 1 = \hat{1}$ and $\frac{\hat{1}}{2} + \frac{1}{4} = \frac{\hat{1}}{4}$.

T: Put 25.25 on the abacus.

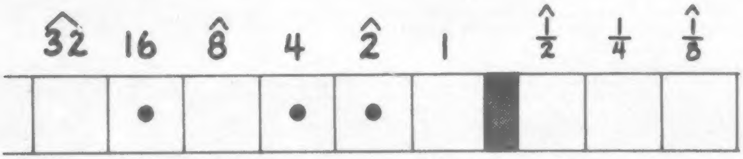

64
$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\frac{\hat{1}}{2}$
$\frac{1}{4}$
$\frac{\hat{1}}{8}$
= 25.25

Put this configuration of checkers on the abacus and ask for the number.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\frac{\hat{1}}{2}$
$\frac{1}{4}$
$\frac{\hat{1}}{8}$

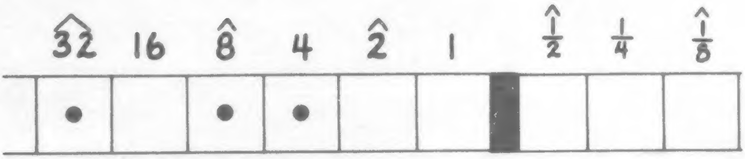
S: $\hat{9}$.

Move each checker one board to the left and ask for the new number.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\frac{\hat{1}}{2}$
$\frac{1}{4}$
$\frac{\hat{1}}{8}$

S: 18.

Very quickly move each checker one board to the left and ask for the new number.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\frac{\hat{1}}{2}$
$\frac{1}{4}$
$\frac{\hat{1}}{8}$

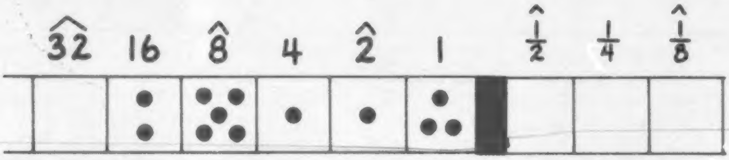
S: $\hat{36}$.

S: On the binary abacus, moving checkers one board to the left doubles the number. Here it is almost the same except it also changes from positive to negative or from negative to positive.

S: Moving the checkers one board to the left is the same as multiplying by $\hat{2}$.

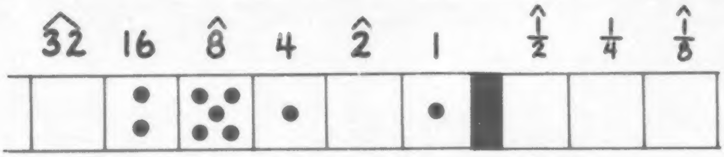
Exercise 2

Display this configuration of checkers.


$\hat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$

T: Show the same number with fewer checkers.

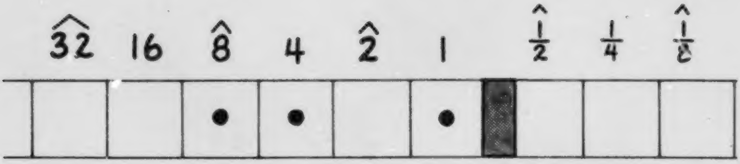
Invite a student to make a play, taking off checkers that represent 0.


$\hat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$

S: One checker wipes out two checkers to the right.

T: Can we make another play now?

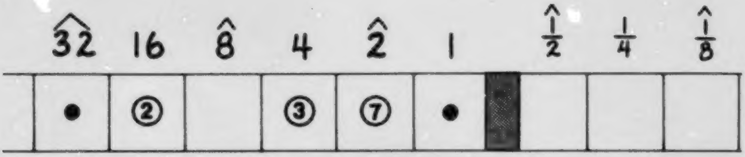
S: Two checkers on the 16's board wipe out four checkers on the $\hat{8}$'s board.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$

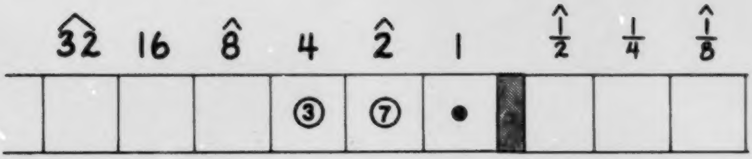
T: Which number is on the abacus?

S: $\hat{3}$.

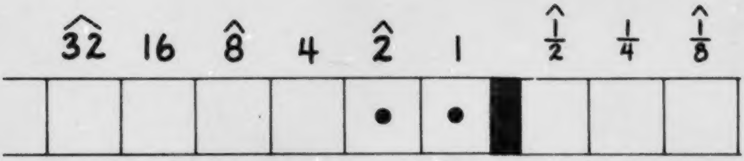
Display this configuration of checkers and ask students to make plays until there is at most one checker on a board.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$
②
③
⑦

S: One checker on the $\widehat{32}$'s board wipes out two checkers on the 16's board.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$
③
⑦

S: Three checkers on the 4's board wipe out six checkers on the $\hat{2}$'s board.

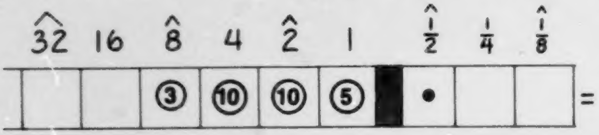

$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$

: Which number is on the abacus?

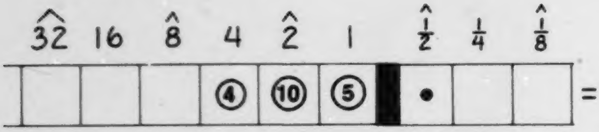
: $\hat{1}$.

isplay this configuration of checkers and call on students to make plays.

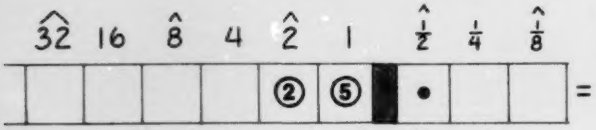
ontinue to make plays until there is at most one checker on a board.



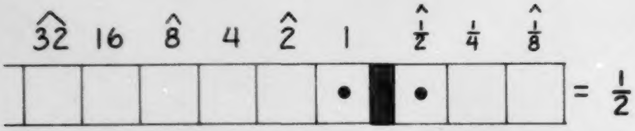
$\hat{32}$ 16 $\hat{8}$ 4 $\hat{2}$ 1 $\hat{\frac{1}{2}}$ $\frac{1}{4}$ $\hat{\frac{1}{8}}$
③ ⑩ ⑩ ⑤ •
=



$\hat{32}$ 16 $\hat{8}$ 4 $\hat{2}$ 1 $\hat{\frac{1}{2}}$ $\frac{1}{4}$ $\hat{\frac{1}{8}}$
④ ⑩ ⑤ •
=



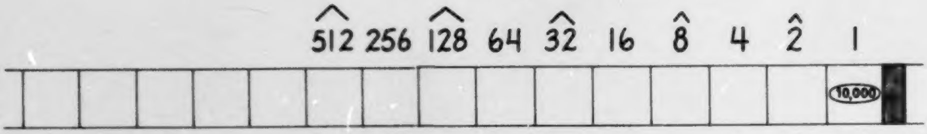
$\hat{32}$ 16 $\hat{8}$ 4 $\hat{2}$ 1 $\hat{\frac{1}{2}}$ $\frac{1}{4}$ $\hat{\frac{1}{8}}$
② ⑤ •
=



$\hat{32}$ 16 $\hat{8}$ 4 $\hat{2}$ 1 $\hat{\frac{1}{2}}$ $\frac{1}{4}$ $\hat{\frac{1}{8}}$
• •
= $\frac{1}{2}$

kercise 3

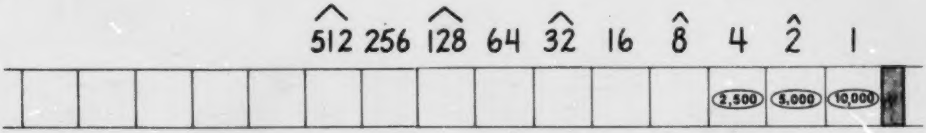
edraw the abacus as shown below and put on 10,000.



$\hat{512}$ 256 $\hat{128}$ 64 $\hat{32}$ 16 $\hat{8}$ 4 $\hat{2}$ 1
10,000

: I'd like to show 10,000 with fewer checkers. Can you think of a play to make?

Give the students some time to think about this situation. If no one makes a suggestion, put these checkers on the abacus.



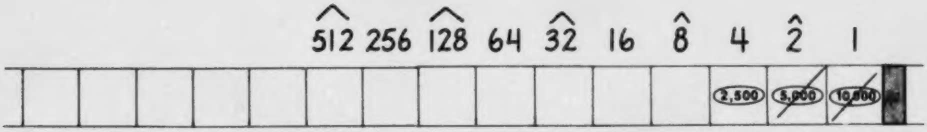
$\widehat{512}$ 256 $\widehat{128}$ 64 $\widehat{32}$ 16 $\hat{8}$ 4 $\hat{2}$ 1
2,500
5,000
10,000

T: Which number is on the abacus?

S: It is still 10,000 because 2,500 checkers on the 4's board wipe out 5,000 checkers on the $\hat{2}$'s board.

T: Do you see a play we could make that would result in fewer checkers on the abacus?

S: 5,000 checkers on the $\hat{2}$'s board wipe out 10,000 checkers on the 1's board.



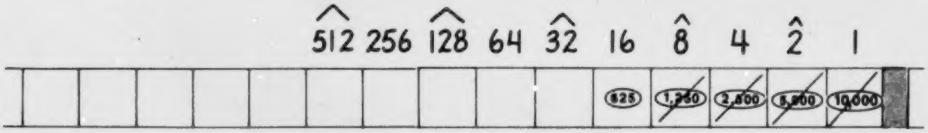
$\widehat{512}$ 256 $\widehat{128}$ 64 $\widehat{32}$ 16 $\hat{8}$ 4 $\hat{2}$ 1
2,500
5,000
10,000

T: How could we go on to get 10,000 with still fewer checkers?

S: 1,250 checkers wipe out 2,500 checkers.

T: Yes, but we want to always keep 10,000 on the abacus.

S: 625 checkers wipe out 1,250 checkers, so put 1,250 checkers on the $\hat{8}$'s board and 625 checkers on the 16's board.



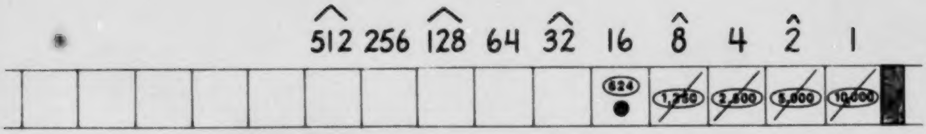
$\widehat{512}$ 256 $\widehat{128}$ 64 $\widehat{32}$ 16 $\hat{8}$ 4 $\hat{2}$ 1
625
1,250
2,500
5,000
10,000

: Now, how can we go on?

: We have an odd number of checkers; the same method won't work.

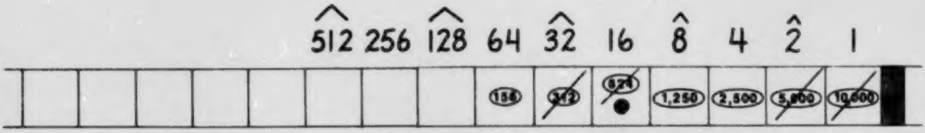
t this point, you may need to give another hint.

: We can think of 625 as 624 + 1.

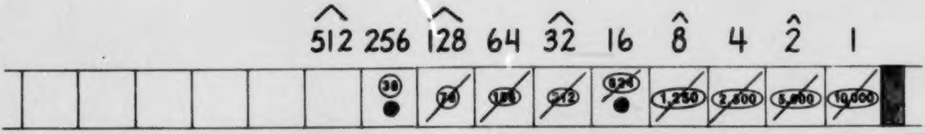

$\widehat{512}$ 256 $\widehat{128}$ 64 $\widehat{32}$ 16 $\hat{8}$ 4 $\hat{2}$ 1
624
1,250
2,500
5,000
10,000

: 156 checkers wipe out 312 checkers.

: And, 312 checkers wipe out 624 checkers.


$\widehat{512}$ 256 $\widehat{128}$ 64 $\widehat{32}$ 16 $\hat{8}$ 4 $\hat{2}$ 1
156
312
624
1,250
2,500
5,000
10,000

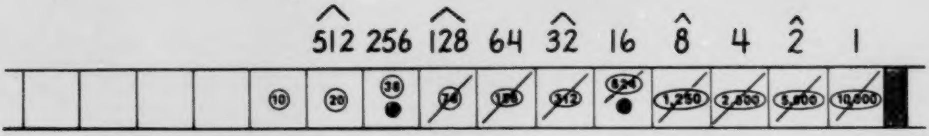
Continue making plays until the configuration shown below is reached.


$\widehat{512}$ 256 $\widehat{128}$ 64 $\widehat{32}$ 16 $\hat{8}$ 4 $\hat{2}$ 1
38
78
156
312
624
1,250
2,500
5,000
10,000

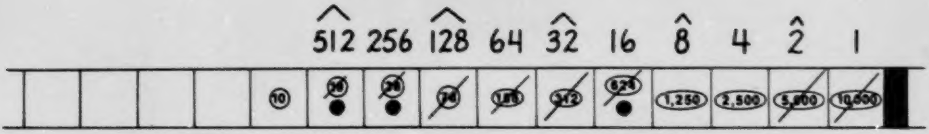
: We cannot put 19 checkers on the next board because 19 is an odd number.

: Since 10 checkers wipe out 20 checkers, put on 20 checkers and 10 checkers.

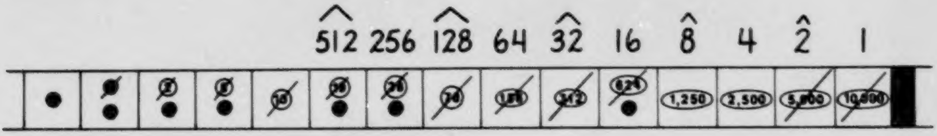
If necessary, make this suggestion yourself.


512
256
128
64
32
16
8
4
2
1
10
20
38
1,250
2,500
5,000
10,000

S: Now we have 19 checkers to wipe out 38 checkers and one checker left over.


512
256
128
64
32
16
8
4
2
1
10
1,250
2,500
5,000
10,000

Continue making plays until each board has at most one checker left of it.


512
256
128
64
32
16
8
4
2
1
1,250
2,500
5,000
10,000

T: On your paper, write 10,000 using base 2 writing.

When most students have finished, invite a student to put the result on the chalkboard.

Decimal Writing		Base $\hat{2}$ Writing
10,000	=	111101100010000

.6 Base $\hat{2}$ Abacus #2

CAPSULE LESSON SUMMARY

Review the rule of the base $\hat{2}$ abacus. Practice putting and reading numbers on it and making plays. Observe the effect of moving checkers to the left on the abacus. Discover the base $\hat{2}$ code for 1/3.

MATERIALS

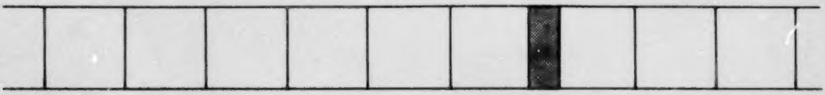
Teacher: Colored chalk; demonstration Minicomputer checkers; multiplication checkers

Student: Paper; pencil; Worksheets N12, N13, N14, and N15

ESCRIPTION OF LESSON

xercise 1

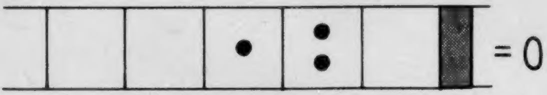
isplay this part of an abacus (draw it on the chalkboard or use Minicomputer oards as suggested in Lesson N4.1).



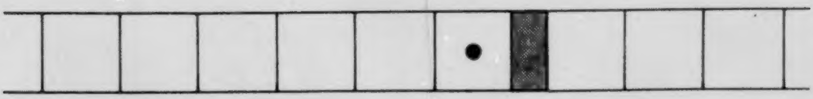
: Do you remember the rule for the base $\hat{2}$ abacus?

nvite a student to explain the rule for the base $\hat{2}$ abacus.

One checker on a board wipes out two checkers on the next board to the right.

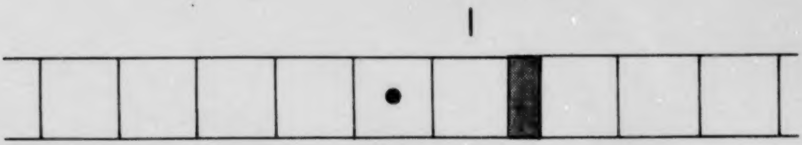


Put one checker on the abacus as shown here.



T: This is the number 1.

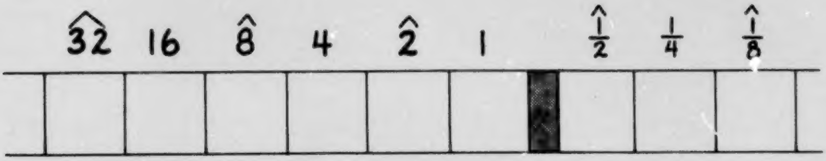
Label the board "1" and move the checker to the next board to the left.



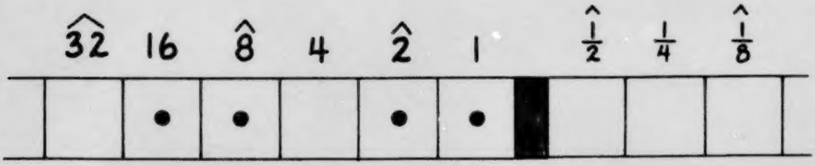
T: Which number is this?

S: $\hat{2}$.

Label the board "$\hat{2}$" and continue in this manner until each board is labeled.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$

Display this configuration of checkers.

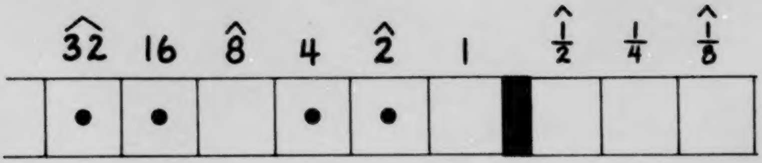

$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$

N 29

T: Which number is on the abacus?

S: 7.

Move each checker one board to the left.

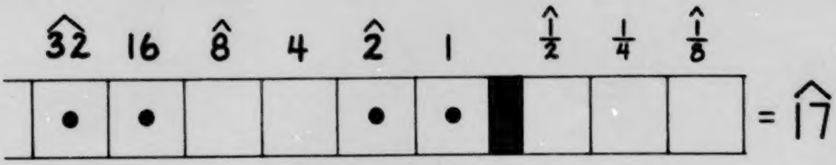

$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\widehat{\tfrac{1}{2}}$
$\tfrac{1}{4}$
$\widehat{\tfrac{1}{8}}$

T: Now which number is on the abacus?

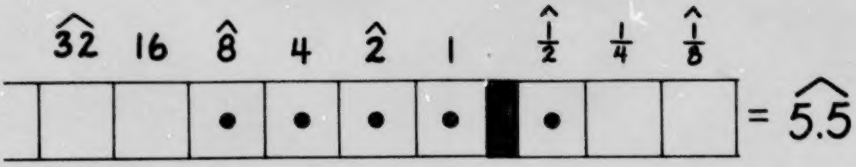
S: $\widehat{14}$; just double and change from positive to negative.

S: $\hat{2} \times 7 = \widehat{14}$.

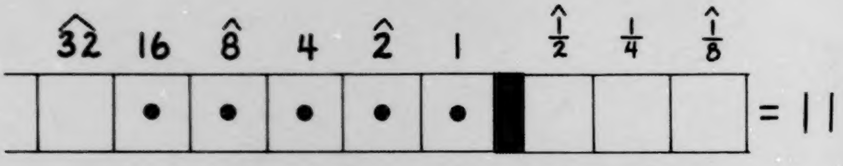
Remove the checkers and ask a student to put $\widehat{17}$ on the abacus.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\widehat{\tfrac{1}{2}}$
$\tfrac{1}{4}$
$\widehat{\tfrac{1}{8}}$
= $\widehat{17}$

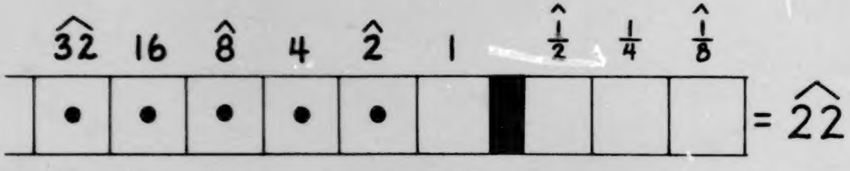
Ask a student to put $\widehat{5.5}$ on the abacus.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\widehat{\tfrac{1}{2}}$
$\tfrac{1}{4}$
$\widehat{\tfrac{1}{8}}$
= $\widehat{5.5}$

Move each checker one board to the left and ask for the new number.


$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$
= 11

Repeat the activity by moving each checker one board to the left.

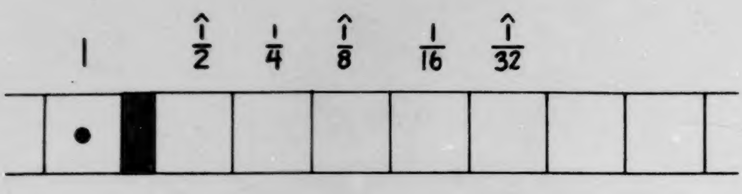

$\widehat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$
= $\widehat{22}$

Exercise 2

T: What could we do to show $\frac{1}{3}$ on this abacus?

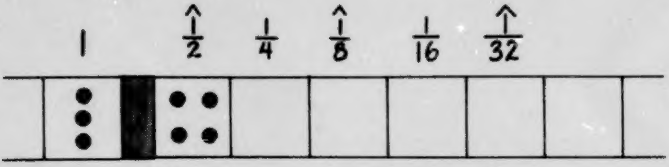
S: Put 1 on the abacus and then make plays to get three checkers on any board with checkers on it. Then $\frac{1}{3}$ will be shown by looking at just one of the three checkers on each board.

Put 1 on the abacus.

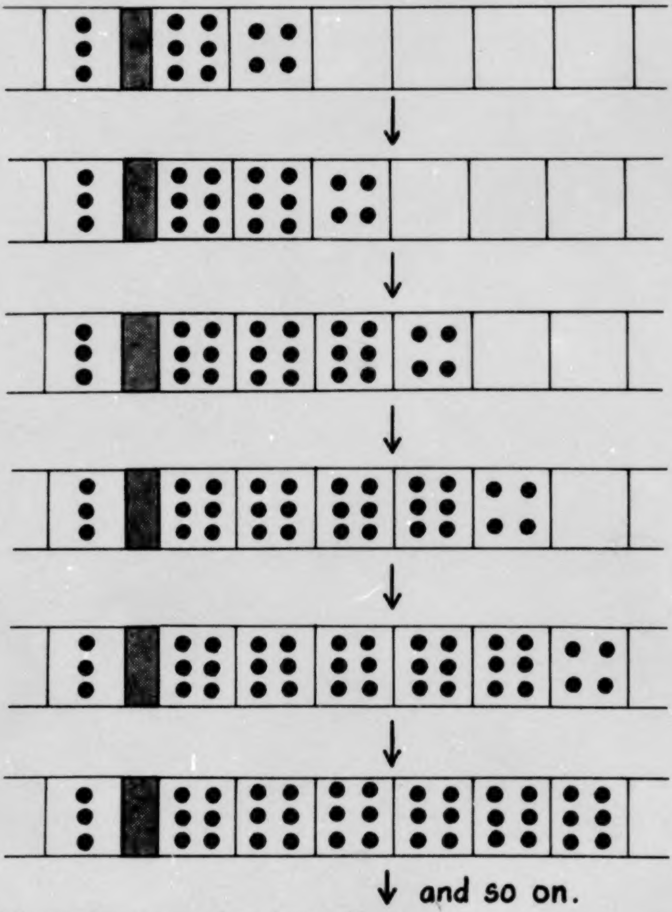

1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$
$\frac{1}{16}$
$\hat{\frac{1}{32}}$

T: Which play shall we make?

If no suggestion is forthcoming, make this play yourself. Two checkers on the one's board wipes out four checkers on the $\frac{\hat{1}}{2}$'s board.


1
$\frac{\hat{1}}{2}$
$\frac{1}{4}$
$\frac{\hat{1}}{8}$
$\frac{1}{16}$
$\frac{\hat{1}}{32}$

Continue adding checkers and forming groups of three checkers until someone comments that this procedure will never end.

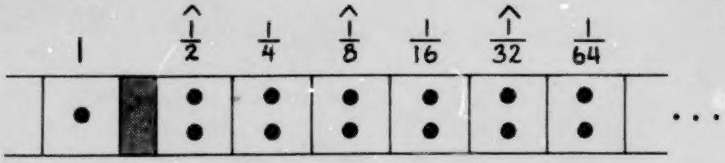


T: Which number is on the abacus?

S: 1.

T: Who can show $\frac{1}{3}$?

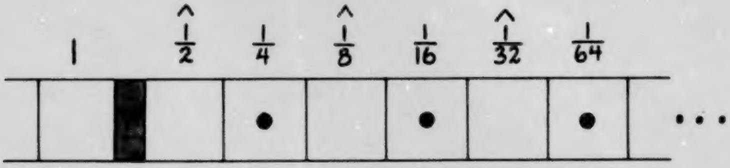
Let a student remove two out of every three checkers, leaving this configuration on the abacus.


1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$
$\frac{1}{16}$
$\hat{\frac{1}{32}}$
$\frac{1}{64}$
...

T: What must we do now so that there is at most one checker on any board?

S: Make some plays; one checker wipes out two checkers on the board to the right.

Ask students to make plays until this configuration is obtained.


1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$
$\frac{1}{16}$
$\hat{\frac{1}{32}}$
$\frac{1}{64}$
...

T: Which number is this?

S: $\frac{1}{3}$.

T: On your paper, write the base $\hat{2}$ writing for $\frac{1}{3}$.

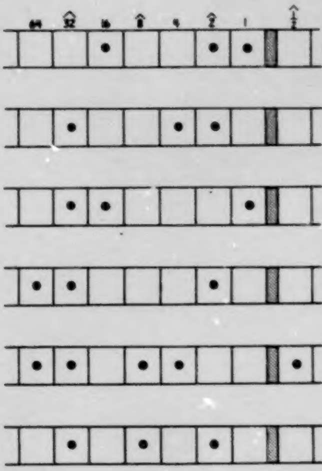
$$\frac{1}{3} = 0.010101\ldots = 0.\overset{*}{\overline{01}}$$

Worksheets N12, N13, N14 and N15 are available for individual work.

Name

N 12 *

Complete.



64
$\hat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$

= 15

= ⁻30

= ⁻15

= 30

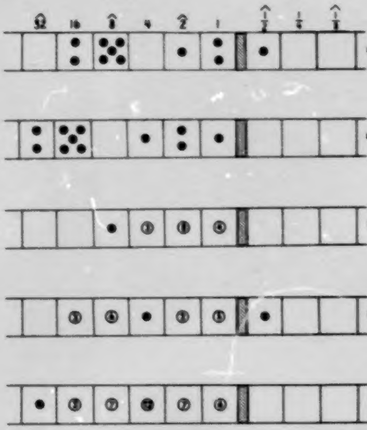
= 27½

= ⁻42

Name

N 13 **

Complete.



$\hat{32}$
16
$\hat{8}$
4
$\hat{2}$
1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$

= ⁻8½

= 17

= ⁻8

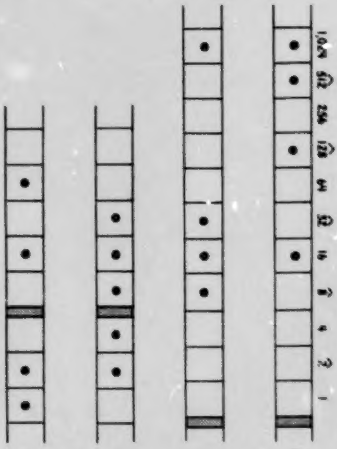
= 4½

= 0

Name

N 14 ***

Complete using at most one checker on a board.



1,024
$\hat{512}$
256
$\hat{128}$
64
$\hat{32}$
16
$\hat{8}$
4
$\hat{2}$
1

= 400

= 1,000

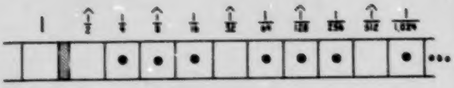
= 2¾

= ⁻9½

Name

N 15 ****

Find a Base $\hat{2}$ code name for $\frac{1}{5}$.



1
$\hat{\frac{1}{2}}$
$\frac{1}{4}$
$\hat{\frac{1}{8}}$
$\frac{1}{16}$
$\hat{\frac{1}{32}}$
$\frac{1}{64}$
$\hat{\frac{1}{128}}$
$\frac{1}{256}$
$\hat{\frac{1}{512}}$
$\frac{1}{1{,}024}$
•••

$\frac{1}{5}$ = 0.011101110111··· code $\hat{2}$

5

RATIONAL NUMBERS[†]

In this chapter, a variety of situations provide opportunities to further explore the rational numbers and to develop algorithms for operating with rational numbers. The situations involve rational numbers on the number line, arrow diagrams, Guess My Rule games with rational numbers, string pictures, and patterns in sets of calculations. Some of the many properties being reinforced by the activities in the lessons include:

- $\frac{4}{5}\times$ is the composite function $4\times$ followed by $\div 5$ (or "against" $5\times$) and the converse of $\frac{4}{5}\times$ is $\frac{5}{4}\times$ or $\div\frac{4}{5}$.
- the composition of $\frac{2}{3}\times$ followed by $\frac{5}{7}\times$ is $\frac{10}{21}\times$, and $\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$.
- $\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8} = 1\frac{1}{8}$.
- a rational number may have many fractional names and, if a decimal, many decimal names.
- $\frac{a}{b} = \frac{c}{d}$ if and only if $a \times d = b \times c$ if and only if $\frac{a}{c} = \frac{b}{d}$.
- $\frac{1}{3} + \frac{1}{5} = \frac{3 + 5}{3 \times 5} = \frac{8}{15}$ and
  $\frac{2}{3} + \frac{4}{5} = \frac{(2 \times 5) + (4 \times 3)}{3 \times 5} = \frac{10 + 12}{15} = \frac{22}{15} = 1\frac{7}{15}$.

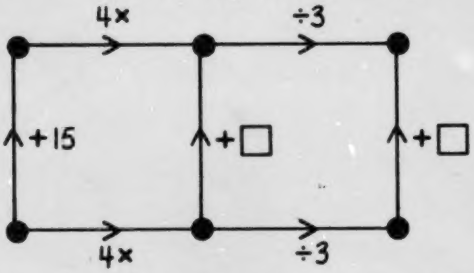
[†] The rational numbers include integers and all numbers which have fractional names with integer numerators and positive integer denominators; e.g., $\frac{3}{4}$, $\frac{81}{732}$, $\frac{9}{5}$, $\frac{^-1}{2}$, $\frac{^-7}{3}$. Decimal numbers are also rational numbers since, for example, $0.75 = \frac{3}{4}$, $1.3 = \frac{13}{10}$, and so on. In communicating with students, we suggest you ignore the distinction between rational numbers and their fractional names and refer to them simply as "fractions". The lesson descriptions use "fraction" to refer to a certain kind of name for a rational number, but we feel it is better pedagogical practice at this level not to force this distinction in student discussions, and feel it is unnecessary to introduce the term "rational number" with the students.

CAPSULE LESSON SUMMARIES

Section 5.1: Find many fractional names for numbers such as $\frac{3}{8}$ and $\frac{2}{3}$. Solve addition, subtraction, multiplication, and division problems with these rational numbers. Compare the results of these calculations. Given nonstandard names for some rational numbers, such as "$\frac{2}{3} + \frac{1}{2}$" and "$\frac{2}{3} \div \frac{1}{2}$", locate them on the number line.

(Worksheets N16, N17, N18)

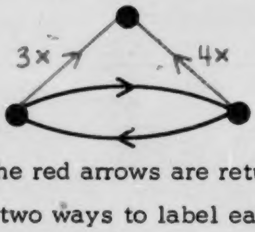
Section 5.2: Label the arrows in an arrow picture where all the arrows are for multiplication functions. Using composition of addition and multiplication functions and of addition and division functions, label arrows in a picture such as:



With the support of a hand-calculator, label arrows which complete a cycle of addition, multiplication, and division arrows.

(Worksheets N19, N20, N21)

Section 5.3: Label arrows in an arrow picture such as:



Observe that the red arrows are return arrows of each other and find two ways to label each, where one way is of the form "÷□". Use such pictures to help do calculations involving division by rational numbers

(expressed as fractions); for example, $\frac{1}{4} \div \frac{3}{4} = \frac{1}{4} \times \frac{4}{3} = \frac{4}{12} = \frac{1}{3}$. Use an arrow picture to show the equality of $n\times$ and $\div\frac{1}{n}$ (where n is a non-zero whole number). Do calculations involving division by rational numbers of the form "$\frac{1}{n}$"; for example, $6 \div \frac{1}{3} = 6 \times 3 = 18$.
(Worksheets N22, N23, N24, N25)

Section 5.4: Play a Guess My Rule game for an operation $*$ where $a * b = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{a \times b}$. Write some numbers as sums of unit fractions and sums of distinct unit fractions.

Section 5.5: Supply arrow labels in an expanding arrow picture. From the arrow picture observe that:

$$\frac{2}{3} + \frac{3}{4} = \frac{(2 \times 4) + (3 \times 3)}{3 \times 4} = \frac{17}{12}$$

Play a game between two teams where the object is to label the dots in a two-string picture using a set of non-standard names for numbers.

Section 5.6: Play a Guess My Rule Game for an operation $*$ where $a * b = (a \div b) - 1 = (a - b) \div b$. Find pairs of numbers satisfying equations such as:

$$\square * \triangle = 3$$

Observe patterns in the possible number pairs.
(Worksheets N26, N27, N28, N29)

N5

5.1 Fractions #1

CAPSULE LESSON SUMMARY

Find many fractional names for numbers such as 3/8 and 2/3. Solve addition, subtraction, multiplication, and division problems with these rational numbers. Compare the results of these calculations. Given non-standard names for some rational numbers, such as "2/3 + 1/2" and "2/3 ÷ 1/2", locate them on the number line.

MATERIALS

Teacher: Colored chalk

Student: Paper; colored pencils; Worksheets N16, N17, and N18.

DESCRIPTION OF LESSON

Exercise 1

Write " $\frac{3}{8}$ " on the board and ask:

T: Can you suggest some other fractions for $\frac{3}{8}$?

As students suggest names for $\frac{3}{8}$, write them on the board in a string of equalities.

$$\frac{3}{8}=\frac{6}{16}=\frac{12}{32}=\frac{15}{40}=\cdots$$

After you have several names for $\frac{3}{8}$ in your list, ask students to fill in the boxes for other names; for example,

$$\frac{3}{8}=\frac{6}{16}=\frac{12}{32}=\frac{15}{40}=\frac{\square}{24}=\frac{18}{\square}=\frac{\square}{32}=\frac{24}{\square}$$

S: $\frac{3}{8} = \frac{9}{24}$.

S: $\frac{3}{8} = \frac{18}{48}$.

S: $\frac{3}{8} = \frac{12}{32}$.

S: $\frac{3}{8} = \frac{24}{64}$.

Repeat the above activity with $\frac{2}{3}$. If you prefer, instruct the students first to write the names they find on their papers. Then list some of their findings on the board, and ask students to fill in the boxes for several other names.

$$\frac{2}{3} = \frac{4}{6} = \frac{8}{12} = \frac{\boxed{10}}{15} = \frac{6}{\boxed{9}} = \frac{\boxed{14}}{21} = \frac{20}{\boxed{30}} = \frac{\boxed{16}}{24}$$

Leaving your strings of equalities for $\frac{3}{8}$ and $\frac{2}{3}$ on the board, write these problems and ask the students to solve them. (Answers are in the boxes.)

$$\frac{2}{3} + \frac{3}{8} = \boxed{\frac{16}{24} + \frac{9}{24} = \frac{25}{24} = 1\frac{1}{24}}$$

$$\frac{2}{3} - \frac{3}{8} = \boxed{\frac{16}{24} - \frac{9}{24} = \frac{7}{24}}$$

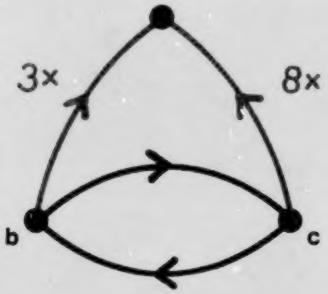
$$\frac{2}{3} \times \frac{3}{8} = \boxed{\frac{6}{24} = \frac{1}{4}}$$

$$\frac{2}{3} \div \frac{3}{8} = \boxed{\frac{2}{3} \times \frac{8}{3} = \frac{16}{9} = 1\frac{7}{9}}$$

Allow several minutes for individual work. Check the work collectively by asking students to give explanations at the board. If you prefer, these problems could be solved collectively and serve as a review of the basic operations on rational numbers.

N5

Some students may need help with the division problem. In that case, draw this arrow picture and lead a discussion similar to that suggested below.

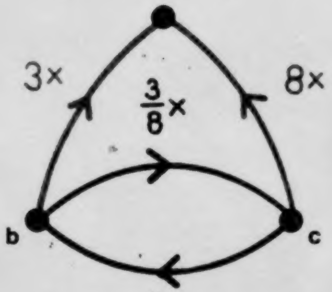


Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (tracing the arrow from b to c): What could this red arrow be for?

S: $\frac{3}{8}x$, because $3x$ followed by "against" $8x$ is $\frac{3}{8}x$.

Label the arrow "$\frac{3}{8}x$".



T (tracing the arrow from c to b): What could this red arrow be for?

S: $\frac{8}{3}x$, because $8x$ followed by "against" $3x$ is $\frac{8}{3}x$.

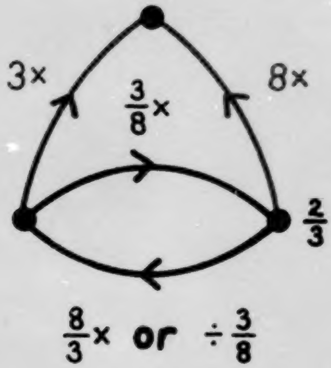
Label the arrow "$\frac{8}{3}x$".

T: What is another way to label this arrow?

Give the students a minute to examine the picture.

S: " $\div \frac{3}{8}$ " because it is the return arrow for $\frac{3}{8}\times$.

Label the arrow " $\div \frac{3}{8}$ " and then label the right most dot " $\frac{2}{3}$ ".



T: The pictures suggest that we can calculate " $\frac{2}{3} \div \frac{3}{8}$ " by doing a multiplication calculation. What multiplication calculation could we do?

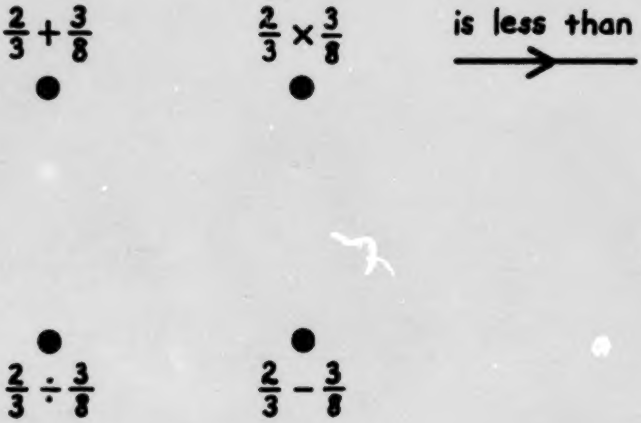
S: " $\frac{2}{3} \times \frac{8}{3}$ ".

Allow a minute for students to complete the calculation on their papers, and then invite a student to solve the problem at the board.

$$\frac{2}{3} \div \frac{3}{8} = \frac{2}{3} \times \frac{8}{3} = \frac{16}{9} = 1\frac{7}{9}$$

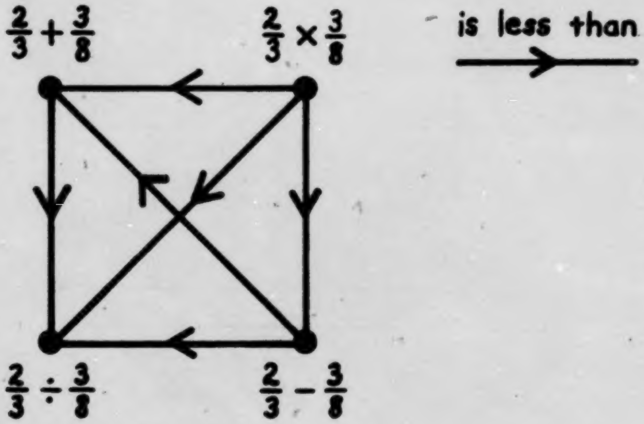
Exercise 2

Write this information on the board.


$\frac{2}{3}+\frac{3}{8}$
$\frac{2}{3}\times\frac{3}{8}$
is less than
$\frac{2}{3}\div\frac{3}{8}$
$\frac{2}{3}-\frac{3}{8}$

T: Copy this picture and draw as many red arrows as you can.

Give the students a few minutes to work independently, and then check their work collectively by asking students to draw arrows in the picture on the board. A completed arrow picture is shown below.


$\frac{2}{3}+\frac{3}{8}$
$\frac{2}{3}\times\frac{3}{8}$
is less than
$\frac{2}{3}\div\frac{3}{8}$
$\frac{2}{3}-\frac{3}{8}$

T (pointing to the arrow picture): What is the smallest number in this picture? Why?

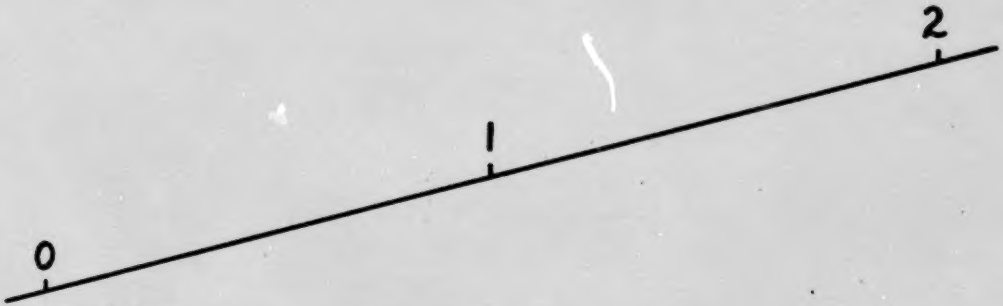
S: $\frac{2}{3} \times \frac{3}{8}$ is the smallest number because no red arrow ends at $\frac{2}{3} \times \frac{3}{8}$.

T: Good. What is the largest number in this picture?

S: $\frac{2}{3} \div \frac{3}{8}$ is the largest number; no red arrow starts at it, and there is a red arrow starting at each other number and ending at $\frac{2}{3} \div \frac{3}{8}$.

Exercise 3

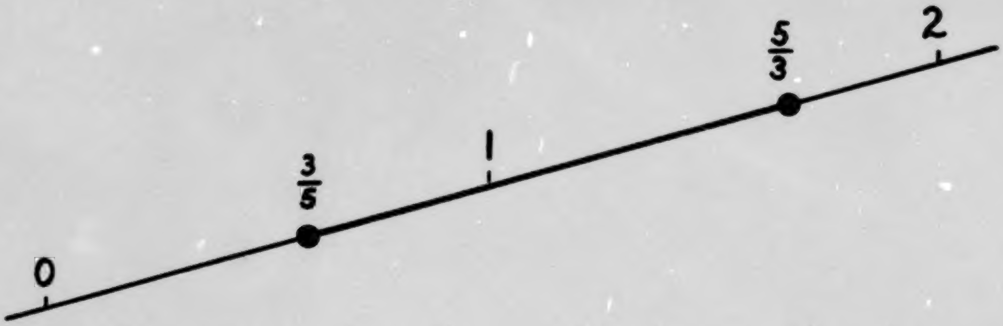
Draw this part of the number line on the board.



Invite students to locate $\frac{3}{5}$ and $\frac{5}{3}$ approximately. The class might suggest that the segment from 0 to 1 be divided into five and three pieces of equal length, respectively. For the benefit of weaker students, locate without labeling some additional numbers on the number line; for example, $\frac{1}{5}$, $\frac{4}{5}$, $\frac{5}{5}$, $\frac{6}{5}$, $\frac{2}{3}$, $\frac{3}{3}$, $\frac{4}{3}$, and $\frac{6}{3}$.

N5

Add the blue segment shown below to your drawing and list these numbers (with the given non-standard names) beneath the number line.



$\frac{3}{4} \times \frac{1}{2}$ $\frac{3}{4} + \frac{1}{2}$ $\frac{3}{4} - \frac{1}{2}$ $\frac{3}{4} \div \frac{1}{2}$

$\frac{2}{3} \times \frac{1}{2}$ $\frac{2}{3} + \frac{1}{2}$ $\frac{2}{3} - \frac{1}{2}$ $\frac{2}{3} \div \frac{1}{2}$

T: Copy these numbers and circle in blue those which are in the blue segment.

Allow several minutes for individual work and observe the students' progress. Offer assistance to those who have difficulty getting started. Many students like to find the standard name for a number before deciding whether or not the number belongs to the blue segment. Collectively check the individual work and discuss questions the students raise. Answers are provided below.

$\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$ $\boxed{\frac{3}{4} + \frac{1}{2} = \frac{5}{4} = 1\frac{1}{4}}$ $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ $\boxed{\frac{3}{4} \div \frac{1}{2} = \frac{6}{4} = 1\frac{1}{2}}$

$\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$ $\boxed{\frac{2}{3} + \frac{1}{2} = \frac{7}{6} = 1\frac{1}{6}}$ $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$ $\boxed{\frac{2}{3} \div \frac{1}{2} = \frac{4}{3} = 1\frac{1}{3}}$

Worksheets N16, N17, and N18 are available.

Name ______________ **N 16 ***

Write at least five fractions for $\frac{4}{5}$.

$\frac{8}{10}, \frac{16}{20}, \frac{32}{40}, \frac{40}{50}, \frac{12}{15}, \frac{24}{30}$

Write at least five fractions for $\frac{3}{4}$.

$\frac{6}{8}, \frac{12}{16}, \frac{24}{32}, \frac{30}{40}, \frac{15}{20}, \frac{9}{12}$

Other solutions are possible.

Complete the calculations.

$\frac{4}{5} + \frac{3}{4} = \frac{16}{20} + \frac{15}{20} = \frac{31}{20} = 1\frac{11}{20}$

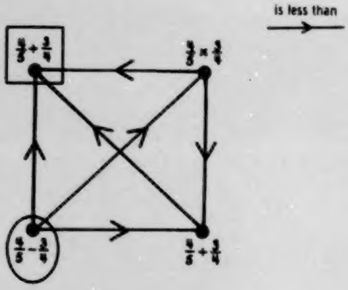
$\frac{4}{5} \times \frac{3}{4} = \frac{12}{20} = \frac{3}{5}$

$\frac{4}{5} - \frac{3}{4} = \frac{16}{20} - \frac{15}{20} = \frac{1}{20}$

$\frac{4}{5} \div \frac{3}{4} = \frac{4}{5} \times \frac{4}{3} = \frac{16}{15} = 1\frac{1}{15}$

Name ______________ **N 17 ****

Draw as many red arrows as possible.


is less than
$\frac{4}{5} + \frac{3}{4}$
$\frac{4}{5} \times \frac{3}{4}$
$\frac{4}{5} - \frac{3}{4}$
$\frac{4}{5} \div \frac{3}{4}$

Circle the smallest number in the arrow picture.

Draw a box around the largest number in the arrow picture.

Name ______________ **N 18 *****

Complete the tables.

×	$\frac{3}{4}$	$\frac{3}{2}$
$\frac{5}{3}$	$1\frac{1}{4}$	$2\frac{1}{2}$
2	$1\frac{1}{2}$	3

+	$\frac{3}{4}$	$\frac{3}{2}$
$\frac{5}{3}$	$2\frac{5}{12}$	$3\frac{1}{6}$
2	$2\frac{3}{4}$	$3\frac{1}{2}$

÷	$\frac{3}{4}$	$\frac{3}{2}$
$\frac{5}{3}$	$\frac{20}{9}$	$\frac{10}{9}$
2	$\frac{8}{3}$	$\frac{4}{3}$

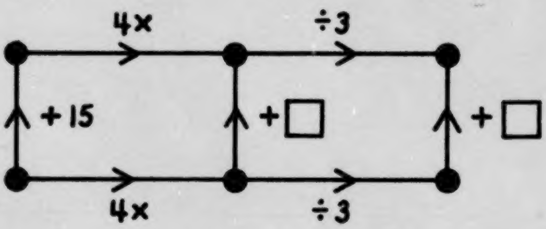
−	$\frac{3}{4}$	$\frac{3}{2}$
$\frac{5}{3}$	$\frac{11}{12}$	$\frac{1}{6}$
2	$1\frac{1}{4}$	$\frac{1}{2}$

Other names for the entries in these tables are possible.

5.2 Composition

CAPSULE LESSON SUMMARY

Label the arrows in an arrow picture where all the arrows are for multiplication functions. Using composition of addition and multiplication functions and of addition and division functions, label arrows in a picture such as:



With the support of a hand-calculator, label arrows which complete a cycle of addition, multiplication, and division arrows.

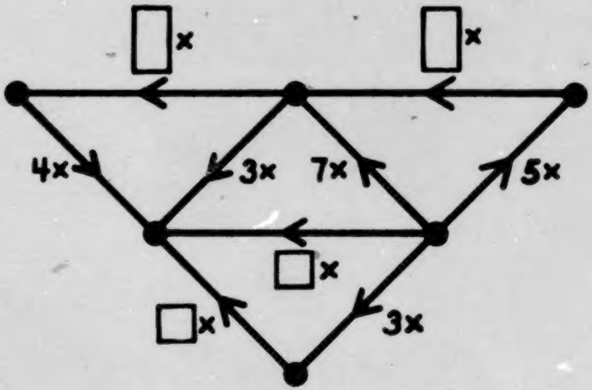
MATERIALS

Teacher: Colored chalk
Student: Hand-calculator; colored pencils; Worksheets N19, N20, and N21.

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.



Invite students to fill in the boxes for the arrows. The dialogue here assumes looking first at the "middle" unknown arrow and then working from bottom to top. Encourage students to trace the appropriate arrows as they give their explanations.

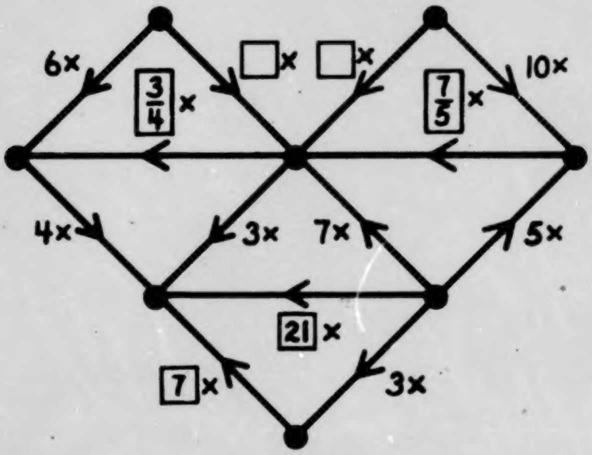
S: 7 × followed by 3 × is 21 × .

S: And 3 × followed by 7 × is 21 × .

S: "Against" 5 × followed by 7 × is $\frac{7}{5}$ × .

S: 3 × followed by "against" 4 × is $\frac{3}{4}$ × .

Extend your picture as shown in the next drawing and ask the class to fill in the boxes.



For the benefit of students who have difficulty labeling these new arrows, direct the class's attention as follows.

T (tracing the appropriate arrows): $\frac{7}{5}$x is "against" 10 x followed by times some whole number.

Write this equality on the board.

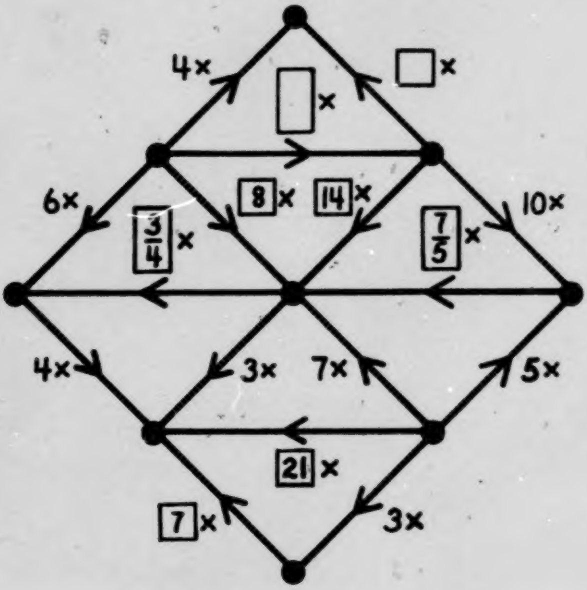
$$\frac{7}{5}x = \frac{\square}{10}x$$

S: That arrow is for 14 x because $\frac{7}{5} = \frac{14}{10}$.

Proceed in a similar manner for the remaining arrow, writing this hint on the board.

$$\frac{3}{4}x = \frac{6}{\square}x$$

Extend the arrow picture with the addition of three arrows as shown below. Let the students fill in the boxes for these new arrows.



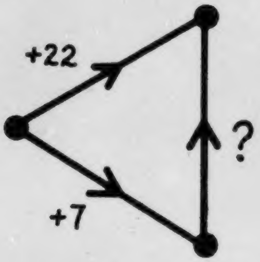
218

S: 8 x followed by "against" 14 x is $\frac{8}{14}$ x.

S: The last arrow is for 7 x because $\frac{8}{14} = \frac{4}{7}$.

Exercise 2

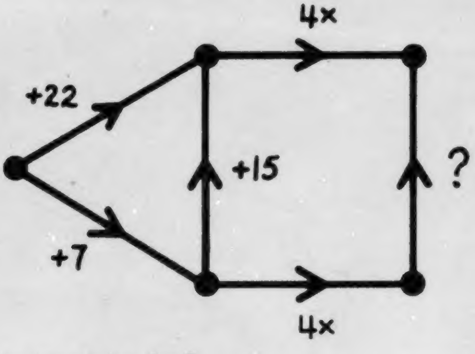
Draw this arrow picture on the board.



T: What could this arrow be for?

S: + 15; + 7 followed by + 15 is + 22.

Label the arrow "+ 15" and extend your picture.

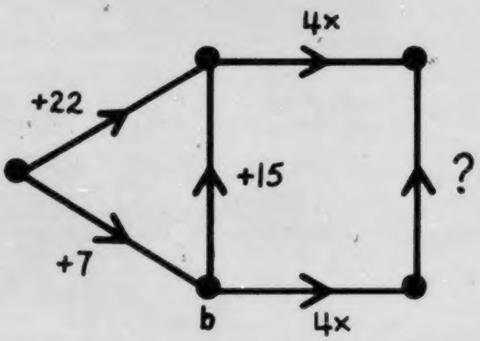


T: What could this arrow be for?

Give the class a couple of minutes to study this situation.

S: + 60.

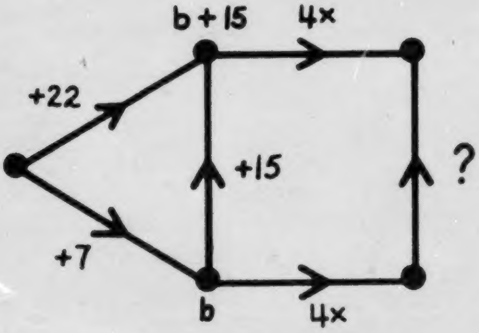
Check the student's response in the following way. Label the lower left dot "b".


4x
+22
+15
?
+7
b
4x

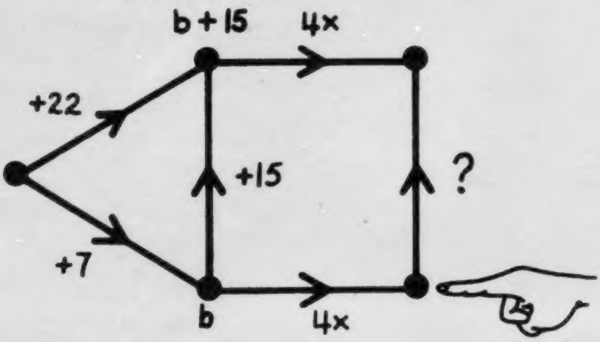
T (pointing to "b"): Suppose we put some number here. Let's call it "b". (Trace the + 15 arrow and point to the ending dot.) What can we say about the number here?

S: It is 15 more than b.

Label the dot "b + 15".


b + 15
4x
+22
+15
?
+7
b
4x

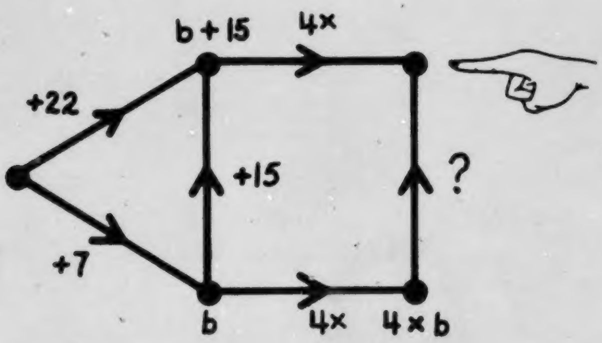
Trace the 4 x arrow starting at b and point to the ending dot.


b + 15
4x
+22
+15
?
+7
b
4x

T: What do we know about this number?

S: It is four times b.

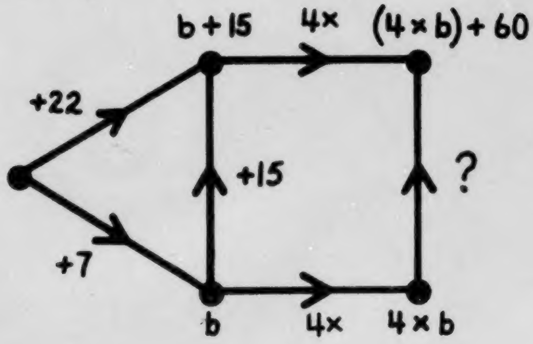
Label the dot "4 × b". Then point to the upper right dot.


b + 15
4×
+22
+15
?
+7
b
4×
4 × b

T: What about this number?

S: It is 4 × (b + 15) or (4 × b) + 60.

Label the upper right dot.


b + 15
4×
(4 × b) + 60
+22
+15
?
+7
b
4×
4 × b

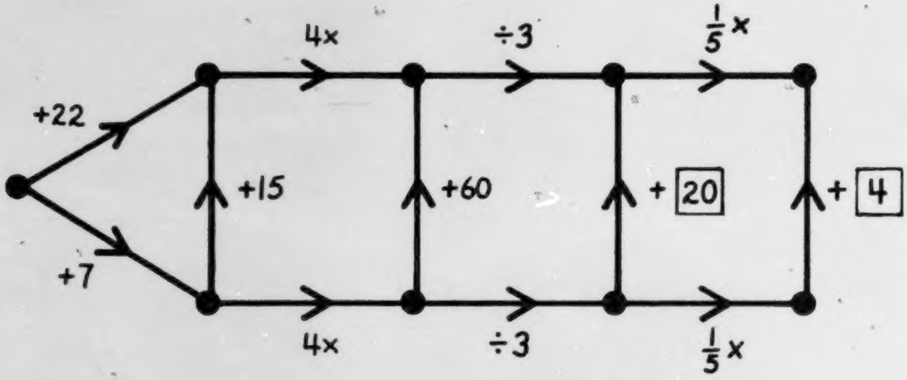
Trace the arrow from 4 × b to (4 × b) + 60.

T: What could this arrow be for?

S: + 60.

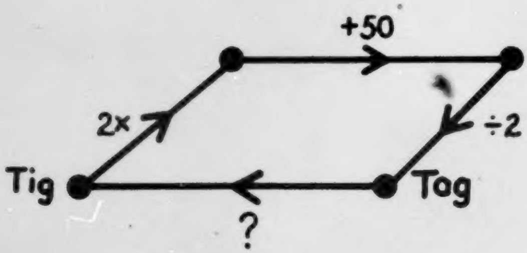
N16

Label the arrow "+ 60", erase the labels for the dots, and extend the arrow picture three new arrows at a time. With each extension, ask the students to provide the labels shown in the boxes.



Exercise 3

Draw this arrow picture on the board and provide a hand-calculator for each student.



T: Choose any number you wish for Tig and display the number on your calculator. Hide the display and then push the appropriate keys to follow the arrows until you come to Tag. Next, come back to Tig with just one operation. (Trace the arrow labeled "?".) What could this arrow be for?

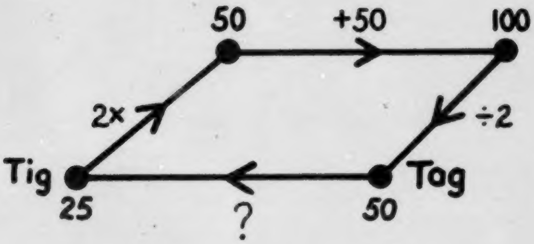
Give the students a few minutes to experiment and to solve this problem.

S: - 25.

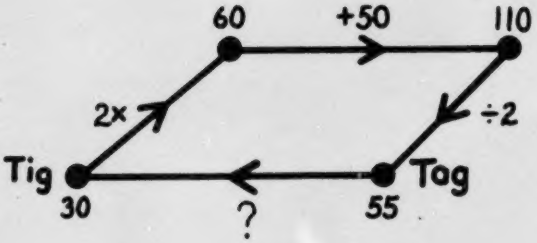
T: How do you know?

S: I followed the arrows ; 2 x followed by + 50 followed by ÷ 2 . I know that Tag will be 25 more than Tig, so the arrow from Tag to Tig could be for - 25.

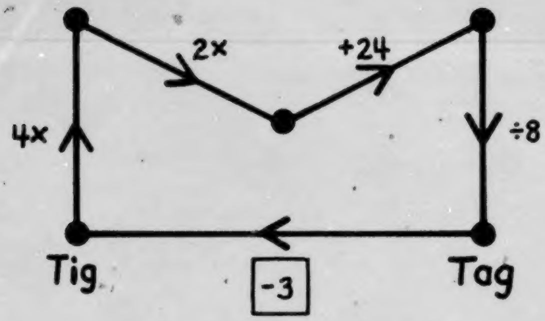
It may happen that some students will claim that, based on their choices for Tig, the arrow could be for something different from - 25. For example, in one of our development classes a student chose 25 for Tig and concluded that the arrow could be for ÷ 2.

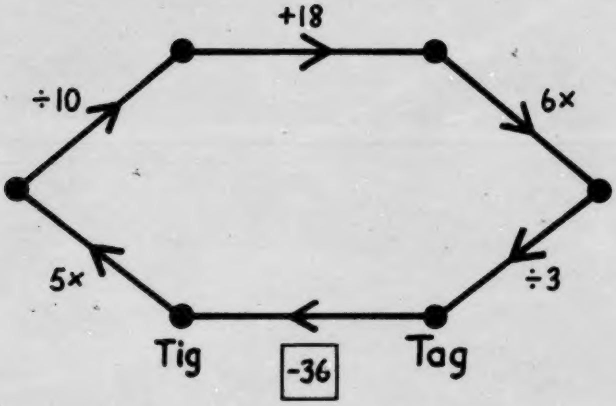


Should this situation occur in your class, stress that we are looking for an operation that works for any choice of a number for Tig. In the above example, if Tig were chosen to be 30, the arrow could no longer be for ÷ 2.



Repeat this activity using the arrow pictures shown on the following page.

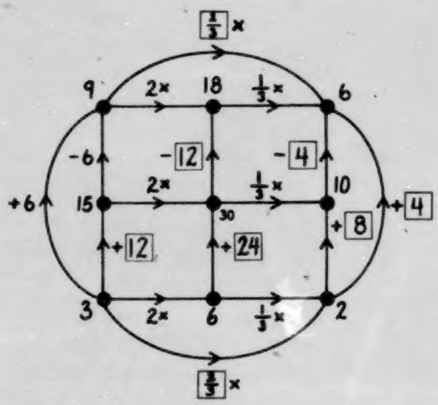

2×
+24
4×
÷8
Tig
-3
Tag


+18
÷10
6×
5×
÷3
Tig
-36
Tag

Worksheets N19, N20, and N21 are available for individual work.

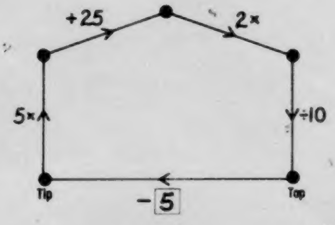
Name ______________________ **N 19** *

Label the dots and fill in the boxes for the arrows.


2/3 ×
9
2×
18
1/3 ×
6
-6
-12
-4
+6
15
2×
1/3 ×
10
30
+4
+8
+12
+24
3
2×
6
1/3 ×
2
2/3 ×

Name ______________________ **N 20** **

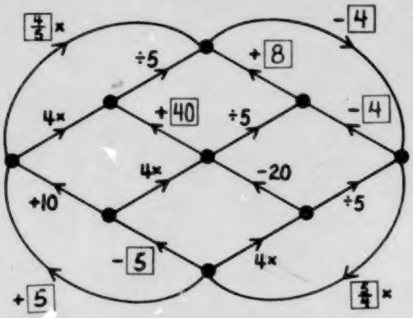
Fill in the box for the arrow and complete the table.


+25
2×
5×
÷10
Tip
-5
Top

Tip	Top
3	8
10	15
-3	2
0.4	5.4
0.2	5.2

Name ______________________ **N 21** ***

Fill in the boxes for the arrows.

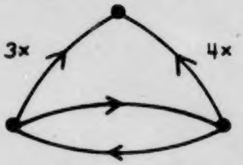

4/5 ×
-4
÷5
+8
4×
+40
÷5
-4
4×
-20
+10
÷5
-5
4×
+5
5/4 ×

N18

5.3 Fractions #2

CAPSULE LESSON SUMMARY

Label arrows in an arrow picture such as:



Observe that the red arrows are return arrows of each other and find two ways to label each, where one way is of the form "$\div\square$". Use such pictures to help do calculations involving division by rational numbers (expressed as fractions); for example, $1/4 \div 3/4 = 1/4 \times 4/3 = 4/12 = 1/3$. Use an arrow picture to show the equality of n x and ÷1/n (where n is a non-zero whole number). Do calculations involving division by rational numbers of the form "1/n"; for example, $6 \div 1/3 = 6 \times 3 = 18$.

MATERIALS

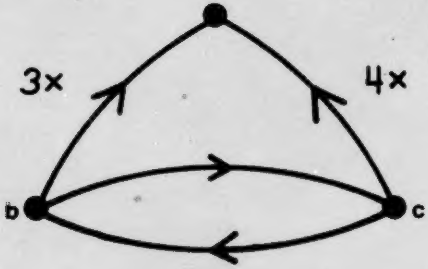
Teacher: Colored chalk

Student: Paper; colored pencils; Worksheets N22, N23, N24, and N25.

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (tracing the arrow from b to c): What could this arrow be for?

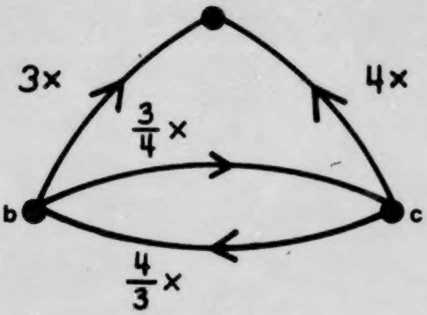
S: $\frac{3}{4}x$, because $3x$ followed by "against" $4x$ is $\frac{3}{4}x$.

Label the arrow "$\frac{3}{4}x$".

T (tracing the arrow from c to b): What could this arrow be for?

S: $\frac{4}{3}x$, because $4x$ followed by "against" $3x$ is $\frac{4}{3}x$.

Label the arrow "$\frac{4}{3}x$".



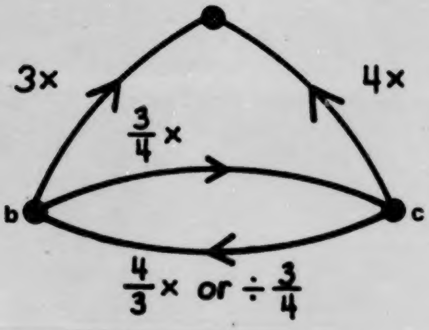
T: Does the picture suggest another way to label the $\frac{4}{3}x$ arrow?

Give the students a minute to examine the picture.

S: It is also the return arrow for $\frac{3}{4}x$, so it is for $\div\frac{3}{4}$.

If "$\div\frac{3}{4}$" is not suggested, prompt the students to observe that it is a return arrow for an already labeled arrow and suggest labeling it "$\div\square$".

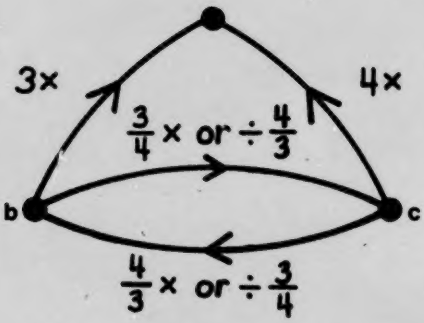
Label the arrow "$\div\frac{3}{4}$" as shown in the next illustration.


3x
4x
$\frac{3}{4}$x
b
c
$\frac{4}{3}$x or ÷$\frac{3}{4}$

T: Could we label the $\frac{3}{4}$ x arrow in another way?

S: " ÷ $\frac{4}{3}$ ", because it's the return arrow for $\frac{4}{3}$ x .

Label the arrow " ÷ $\frac{4}{3}$ ".


3x
4x
$\frac{3}{4}$x or ÷$\frac{4}{3}$
b
c
$\frac{4}{3}$x or ÷$\frac{3}{4}$

T (tracing the appropriate arrows): Dividing by $\frac{4}{3}$ is the same as multiplying by $\frac{3}{4}$. Likewise, dividing by $\frac{3}{4}$ is the same as multiplying by $\frac{4}{3}$.

Label b "28" and write this problem on the board.

$$28 \div \frac{4}{3} =$$

T: Use the arrow picture to help solve this problem.

Allow a few minutes for individual work.

S: $28 \div \frac{4}{3} = 28 \times \frac{3}{4} = 21.$

Label c "21" and summarize this information on the board.

$$28 \div \frac{4}{3} = 28 \times \frac{3}{4} = 21$$

Erase the dot labels, label c "66", and write this problem on the board.

$$66 \div \frac{3}{4} =$$

Ask the class to solve this problem using the arrow picture as an aid.

S: $66 \div \frac{3}{4} = 66 \times \frac{4}{3} = 88.$

Write these problems on the board and ask the students to solve them on their papers. Check the problems collectively. Answers are in the boxes.

$$8 \div \frac{4}{3} = \boxed{6} \qquad 12 \div \frac{3}{4} = \boxed{16} \qquad 33 \div \frac{3}{4} = \boxed{44}$$

$$\frac{3}{4} \div \frac{4}{3} = \boxed{\frac{9}{16}} \qquad \frac{8}{5} \div \frac{4}{3} = \boxed{\frac{6}{5}} \qquad \frac{7}{8} \div \frac{4}{3} = \boxed{\frac{21}{32}}$$

$$\frac{3}{4} \div \frac{3}{4} = \boxed{1} \qquad \frac{1}{3} \div \frac{3}{4} = \boxed{\frac{4}{9}} \qquad \frac{1}{4} \div \frac{3}{4} = \boxed{\frac{1}{3}}$$

Write two more problems on the board for the students to solve. Answers are in the boxes.

$\frac{3}{4}$ kg of apples costs 42¢
1 kg costs $\boxed{56¢}$

$1\frac{1}{3}$ kg blueberries costs 84¢
1 kg costs $\boxed{63¢}$

Let the students work independently for several minutes before discussing their solutions.

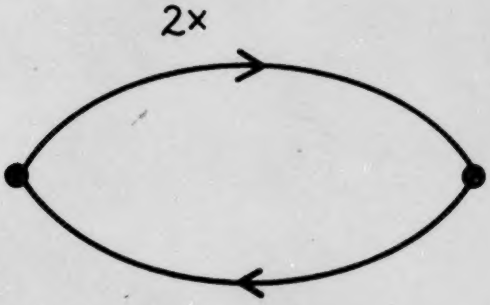
S: Since $\frac{3}{4}$ kilogram costs 42¢, $\frac{1}{4}$ kilogram costs 14¢ and 1 kilogram must cost 4 x 14 or 56¢.

S: $42 \div \frac{3}{4} = 42 \times \frac{4}{3} = 56$. So 1 kilogram of apples costs 56¢.

S: For the second problem divide 84 by $\frac{4}{3}$: $84 \div \frac{4}{3} = 84 \times \frac{3}{4} = 63$. 1 kilogram of blueberries costs 63¢.

Exercise 2

Draw this arrow picture on the board.



T: What could this blue arrow be for?

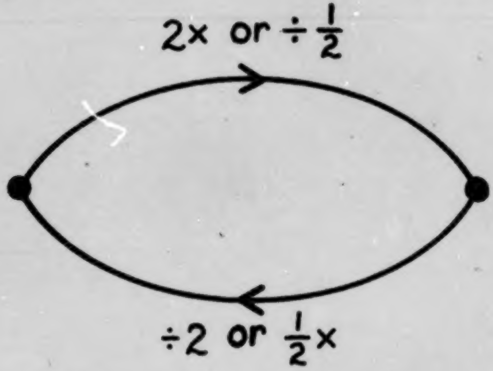
S: $\div 2$ or $\frac{1}{2}x$.

Label the blue arrow " $\div 2$ or $\frac{1}{2}\times$ ".

T: Could we label the $2\times$ arrow in another way?

S: Yes; $\div \frac{1}{2}$ because it is the return arrow for $\frac{1}{2}\times$.

Label the arrow " $\div \frac{1}{2}$ ".



T (tracing the appropriate arrows): Dividing by $\frac{1}{2}$ is the same as multiplying by 2.

Write these two problems on the board and ask the class to solve them using the arrow picture as an aid. Answers are in the boxes.

$$6 \div \frac{1}{2} = \boxed{6 \times 2 = 12}$$

$$\frac{3}{4} \div \frac{1}{2} = \boxed{\frac{3}{4} \times 2 = \frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}}$$

T: How could we calculate " $7 \div \frac{1}{3}$ "?

S: $\div \frac{1}{3}$ is the same as $3\times$, so $7 \div \frac{1}{3} = 7 \times 3 = 21$.

N 18

Write these problems on the board and ask the students to copy and solve them on their papers. Check the problems collectively. Answers are in the boxes.

$$\frac{3}{8} \div \frac{1}{3} = \boxed{\frac{3}{8} \times 3 = \frac{9}{8} = 1\frac{1}{8}}$$

$$6 \div \frac{1}{3} = \boxed{6 \times 3 = 18}$$

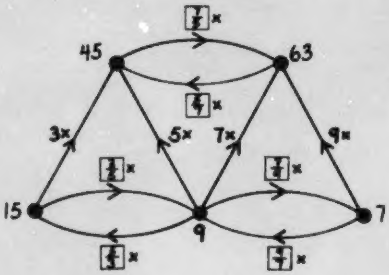
$$6 \div \frac{1}{4} = \boxed{6 \times 4 = 24}$$

$$6 \div \frac{1}{5} = \boxed{6 \times 5 = 30}$$

Worksheets N22, N23, N24, and N25 are available for individual work.

Name ______________________ **N 22 ***

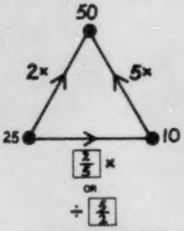
Label the dots and fill in the boxes for the arrows.



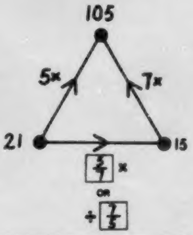
7/5 ×
45
63
5/7 ×
3×
5×
7×
9×
3/5 ×
7/9 ×
15
9
7
5/3 ×
9/7 ×

Name ______________________ **N 23 ****

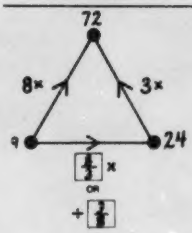
Label the dots and fill in the boxes for the blue arrows.



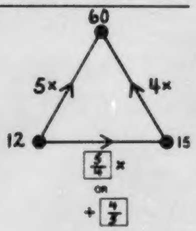
50
2×
5×
25
10
2/5 ×
OR
÷ 5/2



105
5×
7×
21
15
5/7 ×
OR
÷ 7/5



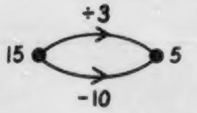
72
8×
3×
9
24
8/3 ×
OR
÷ 3/8

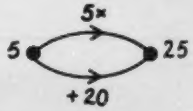


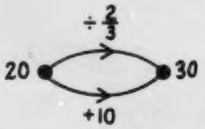
60
5×
4×
12
15
5/4 ×
OR
÷ 4/5

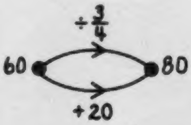
Name ______________________ **N 24 *****

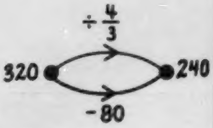
Label the dots.



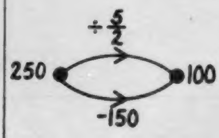








÷ 4/3
320
240
-80

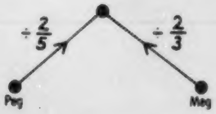


÷ 5/2
250
100
-150

Name ______________________ **N 25 ******

Peg and Meg are secret whole numbers less than 75.

First clue



÷ 2/5
÷ 2/3
Peg
Meg

Peg	3	6	9	12	15	18	21	24	27	30	33	36	39	42
Meg	5	10	15	20	25	30	35	40	45	50	55	60	65	70

Second clue

T_D	Peg	Meg
6	1	1
4	1	1

Peg is 36.

Meg is 60.

N 26

5.4 Unit Fractions

CAPSULE LESSON SUMMARY

Play a Guess My Rule game for an operation $*$ where $a * b = \frac{1}{a} + \frac{1}{b} = \frac{a + b}{a \times b}$. Write some numbers as sums of unit fractions and sums of distinct unit fractions.

MATERIALS

Teacher: None
Student: Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

Write this information on the board.

$$\begin{aligned} 2 * 5 &= \tfrac{7}{10} \\ 3 * 1 &= \tfrac{4}{3} \\ 7 * 3 &= \tfrac{10}{21} \\ 4 * 5 &= \tfrac{9}{20} \\ 7 * 4 &= \square \end{aligned}$$

T: I have a secret rule for $*$. Here are four examples of how to use $*$. With these examples as clues, try to figure out my rule. Which number do you think goes in the box?

NOTE: The rule is: $a * b = \frac{1}{a} + \frac{1}{b} = \frac{a + b}{a \times b}$. The number in the box is $\frac{11}{28}$, because $7 * 4 = \frac{1}{7} + \frac{1}{4} = \frac{7 + 4}{7 \times 4} = \frac{11}{28}$. Do not explain the rule to the class at this point.

Suggest that students whisper their guesses in your ear. Acknowledge, aloud, correct guesses and reject incorrect guesses. For example, "Yes, that's correct" or "No, $\frac{3}{7}$ is not the number I get using my rule." Let a student who guesses correctly tell the class that $\frac{11}{28}$ is in the box, but do not permit the student to give away the rule. If no one guesses correctly, put "$\frac{11}{28}$" in the box yourself.

Continue the activity with these examples.

$$2 * 9 = \boxed{\tfrac{11}{18}}$$

$$4 * 2 = \boxed{\tfrac{6}{8} = \tfrac{3}{4}}$$

$$4 * 5 = \boxed{\tfrac{9}{20}}$$

$$3 * 3 = \boxed{\tfrac{6}{9} = \tfrac{2}{3}}$$

$$9 * 9 = \boxed{\tfrac{18}{81} = \tfrac{2}{9}}$$

$$5 * 10 = \boxed{\tfrac{15}{50} = \tfrac{3}{10}}$$

$$3 * 7 = \boxed{\tfrac{10}{21}}$$

$$4 * 6 = \boxed{\tfrac{10}{24} = \tfrac{5}{12}}$$

T: Who can explain the rule?

S: To get the top number of the fraction, add the numbers; to get the bottom number, multiply the two numbers.

S: I was adding fractions; $2 * 5 = \frac{1}{2} + \frac{1}{5} = \frac{7}{10}$.

T: Yes. These are two different ways of explaining the rule.

Check several examples until it is clear that most students understand the rule. Ask the students to solve several problems (e.g. those in the next illustration) independently. Answers are in the boxes.

$$3 * 5 = \boxed{\frac{8}{15}}$$

$$6 * 12 = \boxed{\frac{18}{72} = \frac{1}{4}}$$

$$5 * \boxed{7} = \frac{12}{35}$$

Exercise 2

T: Fractions like $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{10}$, etc. (that is, one over some non-zero integer) are called unit fractions. The rule for * is to add two unit fractions.

Can you express $\frac{3}{4}$ as a sum of unit fractions?

Let the students have a few minutes to work on this problem.

S: $\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$.

Record this number sentence on the board.

T: Good. You have expressed $\frac{3}{4}$ as a sum of unit fractions, namely; $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$. Find a way to express $\frac{3}{4}$ as a sum of different (distinct) unit fractions.

S: $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, so $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$.

Record this information on the board.

$$\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{4}$$

T: Express $\frac{2}{3}$ as a sum of unit fractions. Try to do it in more than one way.

Allow a few minutes for students to work independently.

S: $\frac{2}{3} = \frac{1}{3} + \frac{1}{3}$.

S: $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$.

S: $\frac{2}{3} = \frac{1}{4} + \frac{1}{4} + \frac{1}{6}$.

Record the solutions on the board.

$$\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{4}$$

$$\frac{2}{3} = \frac{1}{3} + \frac{1}{3} = \frac{1}{2} + \frac{1}{6} = \frac{1}{4} + \frac{1}{4} + \frac{1}{6}$$

T: Express $\frac{7}{10}$ as a sum of unit fractions. Try to find several ways to do it.

S: $\frac{7}{10} = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10}$.

S: $\frac{7}{10} = \frac{1}{2} + \frac{1}{10} + \frac{1}{10}$.

S: $\frac{7}{10} = \frac{1}{2} + \frac{1}{5}$.

S: $\frac{7}{10} = \frac{1}{4} + \frac{1}{4} + \frac{1}{5}$.

N 26

List these numbers on the board.

$$\frac{2}{5} \qquad \frac{4}{5} \qquad \frac{4}{7} \qquad \frac{2}{7} \qquad \frac{9}{10} \qquad 1$$

Ask the students to work on these problems (i.e. to express each of these numbers as a sum of unit fractions) independently for a short while. Encourage students to make use of previous information whenever possible. Faster students might be encouraged to find solutions with distinct unit fractions. Then discuss some of the solutions collectively.

S: $\frac{2}{5} = \frac{6}{15}$, so $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$.

S: $\frac{2}{5} = \frac{1}{4} + \frac{1}{10} + \frac{1}{20}$.

T: What about $\frac{4}{5}$?

S: $2 \times \frac{2}{5} = \frac{4}{5}$, so $\frac{4}{5} = 2 \times (\frac{1}{4} + \frac{1}{10} + \frac{1}{20}) = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$.

S: For $\frac{4}{7}$; since $\frac{4}{7} = \frac{8}{14} = \frac{7}{14} + \frac{1}{14}$, we have $\frac{4}{7} = \frac{1}{2} + \frac{1}{14}$.

T: Good. What about $\frac{2}{7}$?

S: Divide by 2. $\frac{4}{7} = \frac{1}{2} + \frac{1}{14}$, so $\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$.

T: And $\frac{9}{10}$?

S: $\frac{9}{10} = \frac{8}{10} + \frac{1}{10} = \frac{4}{5} + \frac{1}{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{10}$.

S: $\frac{9}{10} = \frac{7}{10} + \frac{2}{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{5}$.

S: $\frac{9}{10} = \frac{1}{2} + \frac{1}{4} + \frac{1}{10} + \frac{1}{20}$.

T: What about 1?

S: Just add $\frac{1}{10}$ to any of the solutions for $\frac{9}{10}$.

$1 = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10}$.

S: $\frac{1}{2} + \frac{1}{2} = 1$.

S: $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20}$.

S: $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$.

5.5 Composition and Addition of Fractions

CAPSULE LESSON SUMMARY

Supply arrow labels in an expanding arrow picture. From the arrow picture observe that:

$$\frac{2}{3} + \frac{3}{4} = \frac{(2 \times 4) + (3 \times 3)}{3 \times 4} = \frac{17}{12}$$

Play a game between two teams where the object is to label the dots in a two-string picture using a set of non-standard names for numbers.

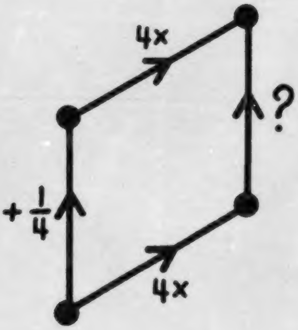
MATERIALS

Teacher: Colored chalk
Student: Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board. Allow room for later extension.



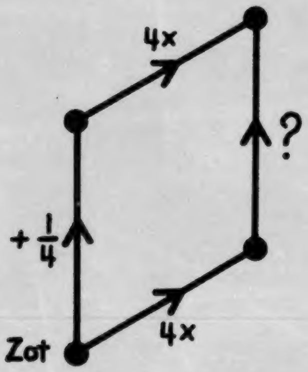
T: What could this arrow be for?

Give the class a couple of minutes to study this situation.

S: +1.

240

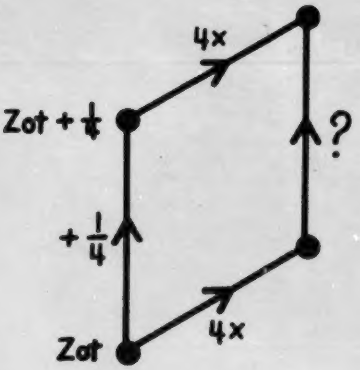
Check the student's response in the following way. Label the lower left dot "Zot".


4x
?
$+\frac{1}{4}$
4x
Zot

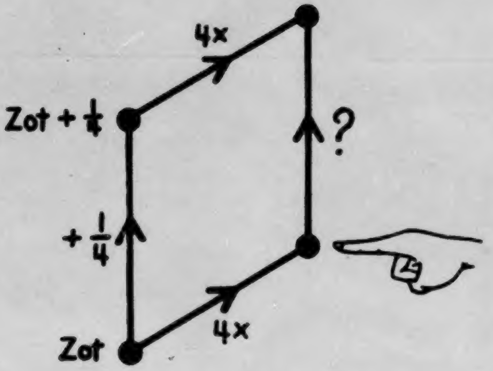
T (pointing to "Zot"): Suppose we put some number here. Let's call it "Zot". (Trace the $+\frac{1}{4}$ arrow and point to the ending dot.) What can we say about the number here?

S: It is $\frac{1}{4}$ more than Zot.

Label the dot "Zot $+\frac{1}{4}$".


4x
Zot $+\frac{1}{4}$
?
$+\frac{1}{4}$
4x
Zot

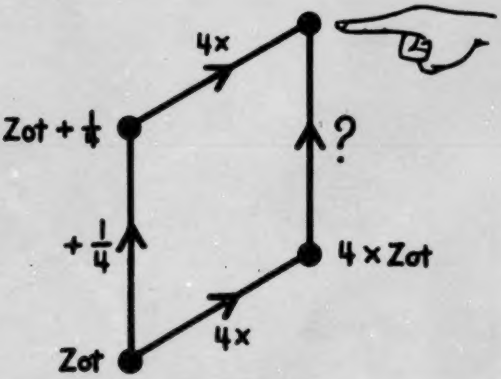
Trace the 4 x arrow starting at Zot and point to the ending dot.


4x
Zot + 1/4
?
+ 1/4
4x
Zot

T: What do we know about this number?

S: It is four times Zot.

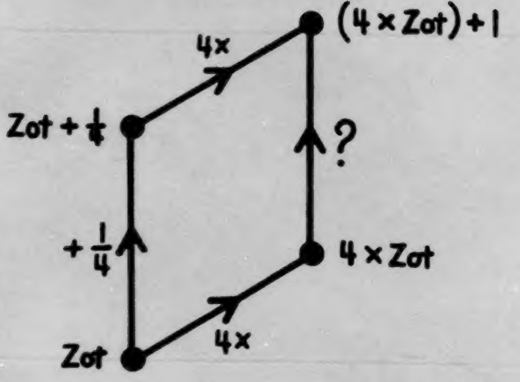
Label the dot "4 x Zot" ; then point to the upper right dot.


4x
Zot + 1/4
?
+ 1/4
4 x Zot
4x
Zot

T: What about this number?

S: It is 4 x (Zot + $\frac{1}{4}$) or (4 x Zot) + (4 x $\frac{1}{4}$) or (4 x Zot) + 1.

Label the upper right dot "(4 x Zot) + 1".

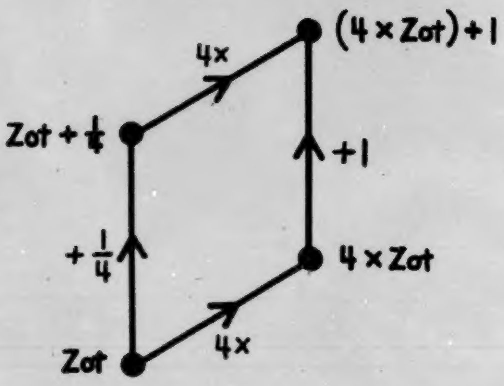

(4 × Zot) + 1
4x
Zot + 1/4
?
+ 1/4
4 × Zot
4x
Zot

Trace the arrow from 4 x Zot to (4 x Zot) + 1.

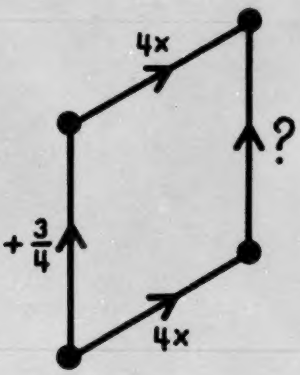
T: What could this arrow be for?

S: +1.

Label the arrow "+1".


(4 × Zot) + 1
4x
Zot + 1/4
+1
+ 1/4
4 × Zot
4x
Zot

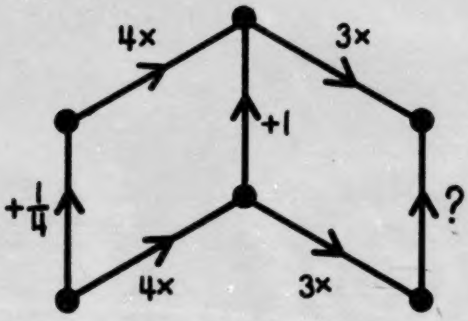
Erase the labels for the dots and for all arrows except the 4 x arrows. Relabel the other arrows as shown below.



T: If I change the function from $+\frac{1}{4}$ to $+\frac{3}{4}$, what could this arrow (point to "?") be for?

S: +3, because $4 \times \frac{3}{4} = 3$.

Repeat this activity several times, each time changing the addition function. Some examples and answers that could be used are $+\frac{7}{4}$ (+7); + 2 (+ 8); $+\frac{3}{2}$ (+6); and $+\frac{1}{8}$ ($+\frac{4}{8}$ or $+\frac{1}{2}$). When most students indicate recognition of the 4 x pattern, extend your arrow picture as shown in the next illustration.

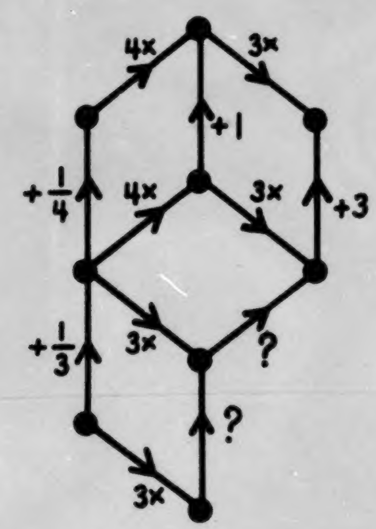


T: What could this arrow be for?

S: +3, because 3 x 1 = 3.

244

Extend your picture again to include five new arrows as shown below.


4x
3x
+1
+$\frac{1}{4}$
4x
3x
+3
3x
?
+$\frac{1}{3}$
?
3x

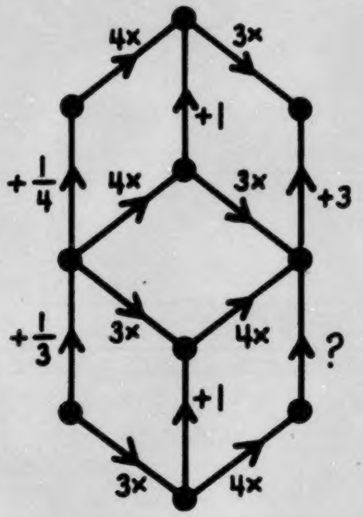
T: What could these arrows be for?

Let the students label these arrows.

S: +1, because $3 \times \frac{1}{3} = 1$.

S: 4 x, because 4 x followed by 3 x is the same as 3 x followed by 4 x.

Complete this part of the picture by adding two additional arrows.

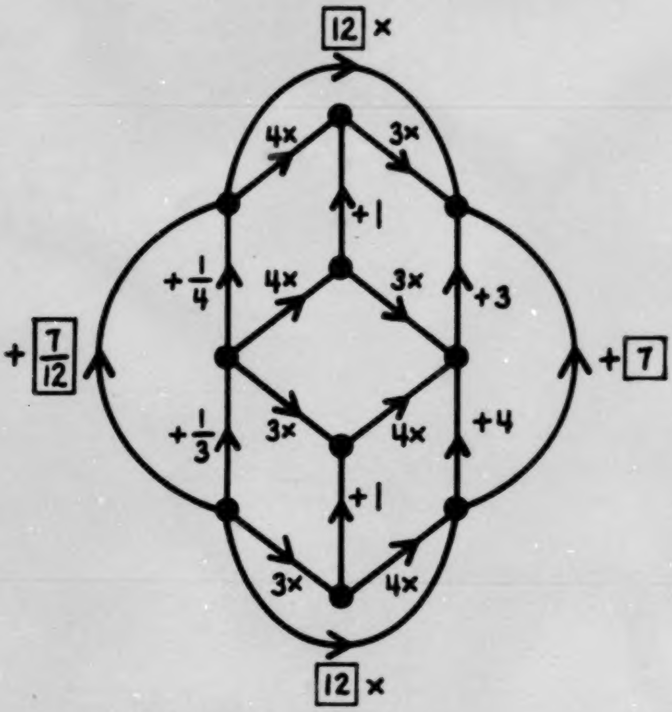

4x
3x
+1
+$\frac{1}{4}$
4x
3x
+3
3x
4x
?
+$\frac{1}{3}$
+1
3x
4x

N 34

T: What could this arrow be for?

S: + 4.

Add four blue arrows as shown in the drawing below. Working from the right side to the left side of the picture, invite students to label the arrows. Answers are in the boxes.



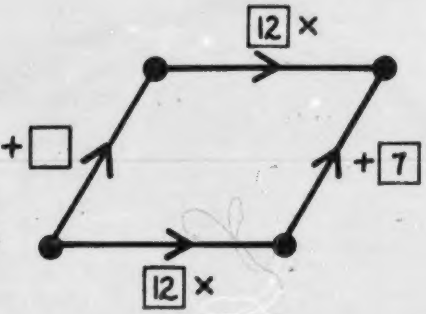
S: + 7, because + 3 followed by + 4 is + 7.

S: 3 x followed by 4 x is 12 x , and also 4 x followed by 3 x is 12 x .

Many students may stumble on labeling the remaining arrow. Give directions as in the following dialogue.

S: The blue arrow is $+\frac{1}{3}$ followed by $+\frac{1}{4}$.

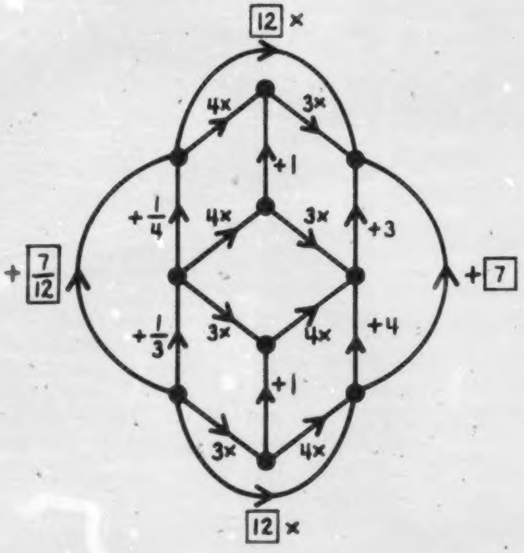
T: Yes, but can you think of another name for this arrow. Let me redraw just the blue arrows in a separate picture for you.

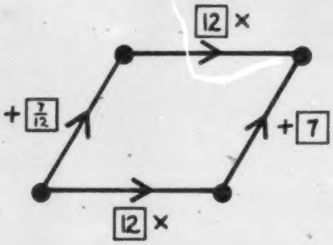


T: This situation is similar to the arrow pictures we used at the beginning of the lesson. How could we fill in the box?

S: $12 \times \frac{7}{12} = 7$, so the left most blue arrow could be for $+\frac{7}{12}$.

Label the remaining arrow "$+ \frac{7}{12}$".


12 ×
4×
3×
+1
$+\frac{1}{4}$
4×
3×
+3
$+\frac{7}{12}$
+ 7
$+\frac{1}{3}$
3×
4×
+4
+1
3×
4×
12 ×


12 ×
$+\frac{7}{12}$
+ 7
12 ×

N 34

T: Do you notice anything interesting about the arrow picture?

Let the students comment freely about the situation.

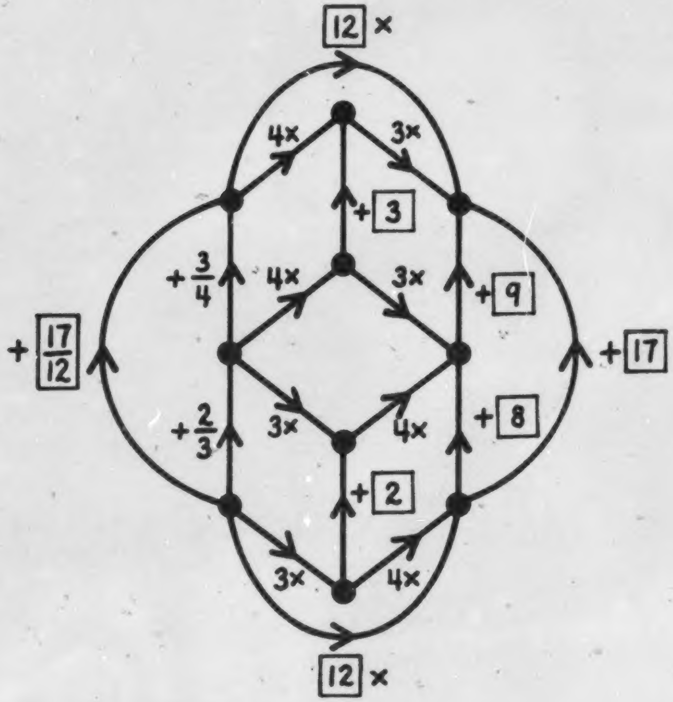
S: Look at the $+\frac{7}{12}$ arrow. $7 = 4 + 3$ and $12 = 4 \times 3$.

S: $\frac{1}{3} + \frac{1}{4} = \frac{4+3}{4 \times 3} = \frac{7}{12}$.

If this last observation is made, summarize the comment by writing an appropriate number story on the board.

$$\frac{1}{3} + \frac{1}{4} = \frac{4+3}{3 \times 4} = \frac{7}{12}$$

If your students enjoy this activity, repeat it by replacing "$+\frac{1}{4}$" and "$+\frac{1}{3}$" with "$+\frac{3}{4}$" and "$+\frac{2}{3}$", respectively. A completed arrow picture is shown below. Answers are in the boxes. Otherwise, proceed to Exercise 2.



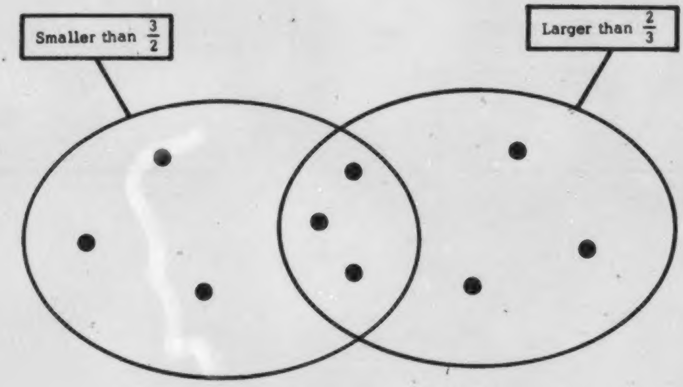
Some students might observe that

$$\frac{3}{4} + \frac{2}{3} = \frac{(3 \times 3) + (2 \times 4)}{4 \times 3} = \frac{17}{12}.$$

Exercise 2

Divide the class into two teams. Draw two strings on the board and assign numbers to the two teams by listing them on two sides of a Team Board as shown below.

Team A	Team B
$3-\frac{3}{2}$	$\frac{1}{4}+\frac{5}{12}$
$\frac{1}{3}\div\frac{1}{2}$	$2-\frac{1}{4}$
$\frac{1}{2}+\frac{1}{3}$	$\frac{3}{2}$
$\frac{2}{5}$	$\frac{1}{3}+\frac{1}{5}$
$\frac{1}{2}+\frac{3}{4}$	$\frac{2}{3}\div\frac{1}{3}$
$\frac{5}{2}\times\frac{4}{5}$	$\frac{7}{2}\times\frac{1}{4}$
$\frac{5}{4}-\frac{3}{8}$	$\frac{1}{2}-\frac{1}{10}$

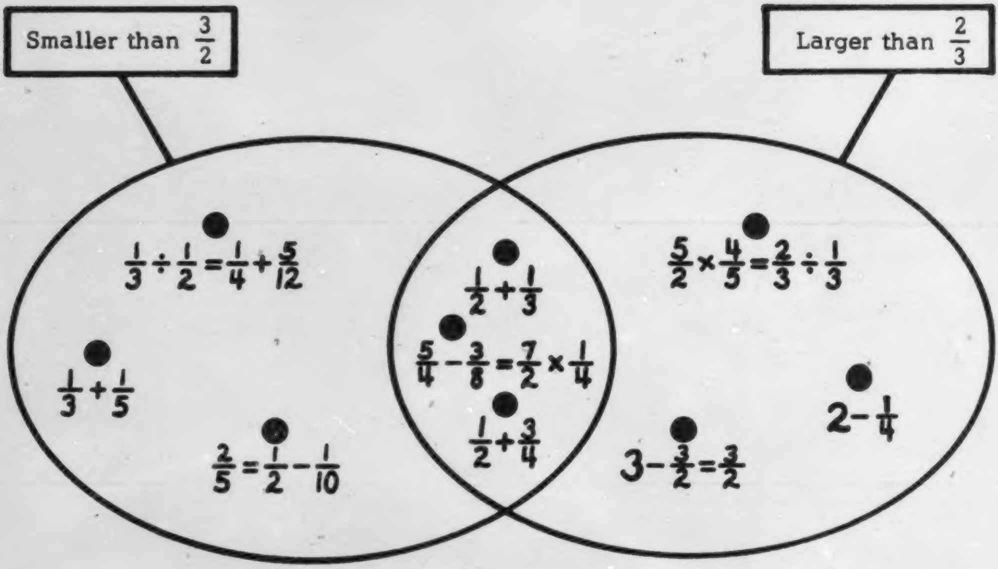


T: We are going to play a game. The teams will take turns trying to label correctly the dots in the string picture. You may only use the labels on your side of the Team Board. The team that correctly uses all its labels first is the winner. By the way, sometimes there are two different names for the same number here, so a dot may get labeled twice.

Alternating teams and among the members of a team, call on students to label the dots. When a label is correct (the number is correctly placed in the string picture), erase it from the Team Board.

N 34

The following picture shows all the dots labeled using the names given to both teams. It is for your reference during the play of the game.


Smaller than $\frac{3}{2}$
Larger than $\frac{2}{3}$
$\frac{1}{3} \div \frac{1}{2} = \frac{1}{4} + \frac{5}{12}$
$\frac{1}{3} + \frac{1}{5}$
$\frac{2}{5} = \frac{1}{2} - \frac{1}{10}$
$\frac{1}{2} + \frac{1}{3}$
$\frac{5}{4} - \frac{3}{8} = \frac{7}{2} \times \frac{1}{4}$
$\frac{1}{2} + \frac{3}{4}$
$\frac{5}{2} \times \frac{4}{5} = \frac{2}{3} \div \frac{1}{3}$
$2 - \frac{1}{4}$
$3 - \frac{3}{2} = \frac{3}{2}$

5.6 Guess My Rule

CAPSULE LESSON SUMMARY

Play a Guess My Rule Game for an operation $*$ where $a * b = (a \div b) - 1 = (a - b) \div b$. Find pairs of numbers satisfying equations such as:

$$\square * \triangle = 3$$

Observe patterns in the possible number pairs.

MATERIALS

Teacher: Chalk
Student: Paper; pencil; Worksheets N26, N27, N28, and N29

DESCRIPTION OF LESSON

Exercise 1

Write this information on the board.

$$\begin{aligned} 8 * 2 &= 3 \\ 20 * 4 &= 4 \\ 20 * 2 &= 9 \\ 10 * 2 &= 4 \\ 40 * 5 &= \square \end{aligned}$$

T: I have a secret rule for $*$. Here are four examples of how to use $*$. With these examples as clues, try to figure out my rule. Which number do you think goes in the box?

NOTE: The rule is $a * b = (a \div b) - 1 = (a - b) \div b$. The number in the box is 7 because $40 * 5 = (40 \div 5) - 1 = (40 - 5) \div 5 = 7$. Do not explain the rule to the class at this time.

Suggest that students whisper their guesses in your ear. Acknowledge, aloud, correct guesses and reject incorrect guesses. For example, "Yes, that's correct" or "No, 8 is not the number I get using my rule." Let a student who guesses correctly tell the class that 7 is in the box, but do not permit the student to give away the rule. If no one guesses correctly, put 7 in the box yourself.

Continue the activity with these examples.

$$15 * 2 = \boxed{6.5}$$
$$36 * 10 = \boxed{2.6}$$
$$8 * 3 = \boxed{\tfrac{5}{3}}$$
$$56 * 8 = \boxed{6}$$
$$10 * 4 = \boxed{1.5}$$
$$9 * 4 = \boxed{1.25}$$

T: Who can explain my rule?

S: Divide the first number by the second number, then subtract one from the result.

S: Subtract the second number from the first number, then divide the result by the second number.

Accept any reasonable and correct explanation your students offer. If you feel it would be helpful, write an equation for the rule on the board.

$$a * b = (a \div b) - 1 = (a - b) \div b$$

Distribute copies of Worksheet N26 to the students and let them work independently for several minutes. Allow students to use whichever formulation of the rule they prefer. Check the problems collectively (an answer key is on page 187.)

Exercise 2

Write this number story on the board.

$$\square * \triangle = 3$$

T: On your paper write pairs of numbers for box and triangle so that when we use the operation * on the numbers the result is 3.

Allow several minutes for independent work. When most students have found a few pairs of numbers, record some of their results on the board. Some possible pairs of numbers are listed below.

$$\square * \triangle = 3$$

8	2
16	4
12	3
32	8
4	1
20	5
40	10

T: Do you notice anything interesting about these pairs of numbers?

Allow a moment for the students to look for patterns.

S: If you know one solution like $\square$ is 8 and $\triangle$ is 2, just multiply both numbers by the same thing (except 0) and you get another solution.

N 38

S: Each number in the first column is four times the corresponding number in the second column. (8 = 4 x 2, 16 = 4 x 4, etc.)

T: Why do you think there is a 4 x pattern?

S: According to the rule for *, (□ ÷ △) - 1 = 3 so □ ÷ △ = 4. That means □ is 4 x △.

Let the students express their ideas no matter how awkward their explanations may be.

Repeat this activity using "□ * △ = 0.5". Number pairs suggested by students in one of our development classes are listed below.

□ * △ = 0.5

3	2
384	256
6	4
12	8
1	$\frac{2}{3}$
1.5	1

T: Can you find any pattern here?

S: Yes, a number for □ is one and a half times the corresponding number for △.

S: □ = 1.5 x △ and 1.5 = 0.5 + 1.

Worksheets N27, N28, and N29 are available for individual work. It may be necessary to call students' attention to the fact that the rule for * changes on the two and three star worksheets.

Name ______________________ **N 26**

$$a * b = (a \div b) - 1$$

Complete.

$25 * 5 = \boxed{4}$

$28 * 4 = \boxed{6}$

$42 * \boxed{7} = 5$

$\boxed{6} * 6 = 0$

$\boxed{81} * 9 = 8$

$19 * 2 = \boxed{8.5}$

$\boxed{0} * 8 = {}^{-}1$

$7 * 5 = \boxed{\frac{2}{5}}$

$\frac{7}{3} * 2 = \boxed{\frac{1}{6}}$

$5 * \boxed{3} = \frac{2}{3}$

$\frac{2}{3} * \frac{2}{7} = \boxed{\frac{4}{3}}$

$6 * 0.1 = \boxed{59}$

$8 * 0.2 = \boxed{39}$

$\frac{1}{5} * \frac{1}{4} = \boxed{{}^{-}\frac{1}{5}}$

Name ______________________ **N 27** *

$$a * b = (a \div b) - 1$$

Complete this table.

□ * △ = 6

35	5
21	3
7	1
70	10
3.5	0.5
2	$\frac{2}{7}$
42	6
0.42	0.06

Other solutions are possible.

Name ______________________ **N 28** **

$$a * b = (a \div b) - 1$$

Complete.

$51 * 17 = \boxed{2}$

$225 * 25 = \boxed{8}$

$4.5 * 0.5 = \boxed{8}$

$2.8 * 0.1 = \boxed{27}$

$2.5 * 2 = \boxed{0.25}$

$\frac{3}{2} * \frac{1}{2} = \boxed{2}$

$\frac{5}{4} * \frac{2}{3} = \boxed{\frac{7}{8}}$

$\frac{2}{3} * \frac{5}{9} = \boxed{\frac{1}{5}}$

$\boxed{91} * 13 = 6$

$3.5 * \boxed{0.5} = 6$

$\boxed{0.8} * 0.1 = 7$

$\frac{9}{4} * \boxed{2} = \frac{1}{8}$

$\boxed{8} * \frac{1}{2} = 15$

$\frac{3}{5} * \boxed{\frac{1}{3}} = \frac{4}{5}$

$\boxed{\frac{8}{3}} * \frac{7}{3} = \frac{1}{7}$

$\frac{2}{5} * \boxed{\frac{1}{5}} = 1$

Name ______________________ **N 29** ***

$$a * b = (a \div b) + \frac{1}{2}$$

Complete.

$256 * 64 = \boxed{4.5}$

$4.8 * 4 = \boxed{1.7}$

$7.5 * 2.5 = \boxed{3.5}$

$12.4 * 0.4 = \boxed{31.5}$

$\frac{2}{3} * \frac{1}{6} = \boxed{4\frac{1}{2}}$

$\frac{3}{4} * \frac{1}{3} = \boxed{\frac{11}{4}}$

$\frac{3}{8} * \frac{3}{5} = \boxed{\frac{9}{8}}$

$\frac{2}{7} * \frac{3}{7} = \boxed{\frac{7}{6}}$

$108 * \boxed{12} = 9.5$

$\boxed{11} * 5 = 2.7$

$17 * \boxed{4} = 4.75$

$\boxed{30} * 4 = 8$

$\frac{6}{5} * \boxed{\frac{4}{5}} = 2$

$\boxed{\frac{7}{8}} * 7 = \frac{5}{8}$

$\frac{1}{3} * \boxed{2} = \frac{2}{3}$

$\boxed{\frac{1}{14}} * \frac{5}{7} = \frac{3}{5}$

6

COMBINATORICS

"Combinatorics" is a term meaning "the process of (systematically) counting the number of possible ways or outcomes in certain situations." In this chapter, several situations are presented in which the binary system provides a useful scheme for counting. These situations can all be seen to relate to one another with an appropriate "translation" of the symbols (0 and 1) of binary code words. In one case a problem of finding the probability that a certain kind of outcome will occur provides additional interest to the combinatorics situation.

CAPSULE LESSON SUMMARIES

Section 6.1:	Briefly review the situation in THE HIDDEN TREASURE storybook. Invent a binary code to describe a given path between two points in a grid. Count the number of paths between two points in a grid. (Worksheet N30)
Section 6.2:	Review the messages from Section 6.1 and recall a scheme for counting all of the routes from H to one of the intersections of the grid. Locate grid points at a certain taxi-distance from H and count the collective routes from H to these grid points. Observe a relationship between taxi-distance and the number of routes. Relate the counting scheme in this lesson to count the ways to put a certain number of pennies into a given number of boxes. (Worksheet N31)
Section 6.3:	Relate problems of finding the number of ways of putting checkers on the Minicomputer to those of putting pennies into boxes (see Section 6.1). Answer a probability question in this context. (Worksheets N32, N33)

N6

6.1 Combinatorics #1

CAPSULE LESSON SUMMARY

Briefly review the situation in THE HIDDEN TREASURE storybook. Invent a binary code to describe a given path between two points in a grid. Count the number of paths between two points in a grid.

MATERIALS

Teacher: World of Numbers Poster #1 for IG-VI
Student: Paper; pencil; Worksheet N30

NOTE: There were three lessons in IG-IV about THE HIDDEN TREASURE. If your class did not cover those lessons or individuals in the class have not heard the story, give the students the storybook to read sometime before teaching this lesson.

DESCRIPTION OF LESSON

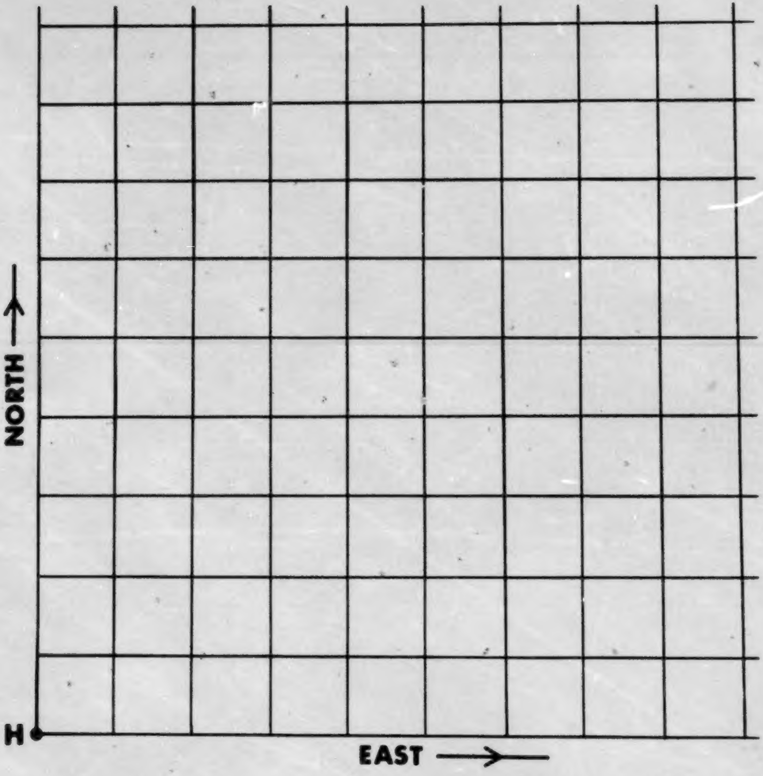
Exercise 1

T: Do you remember the story of THE HIDDEN TREASURE?

Let the students recall as much of the situation as they can, but be sure that the following two details from the story are mentioned.

- Detective Spick lives in a neighborhood of one-way streets running North or East.
- A binary code was used to record the route between two points of the neighborhood.

Distribute individual copies of Worksheet N30 and tape World of Numbers Poster #1 to the board.



T: Spick has a partner, Span. Each time Spick drives to a house in the neighborhood, he leaves a coded message for Span. The message describes the route Spick plans to follow.

One day the message is . . .

Write this binary code word on the board.

10010110001

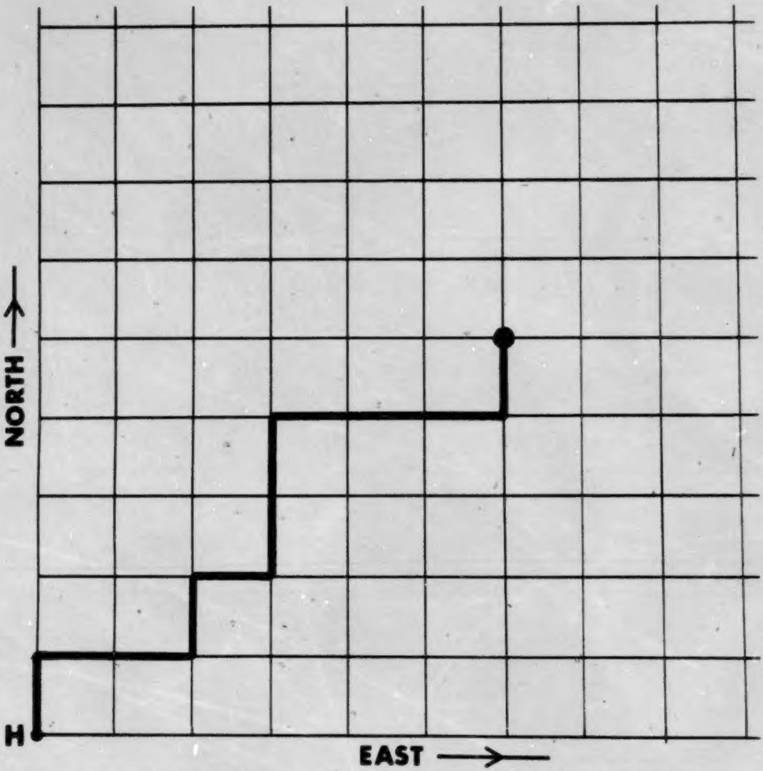
T (pointing to the poster): Can you trace the route that Spick plans to take on this map?

N 6

Let the students respond to your question without commenting yourself. Some students will observe that this question cannot be answered until some agreement concerning "1" and "0" is reached.

T: Let's agree that "1" indicates that Spick travels one block North and that "0" indicates that he travels one block East.

Once this agreement is made, ask the students to trace the path on their worksheet while one student does the same thing at the board.



Write these binary code words on the board.

11100001001   101010010110

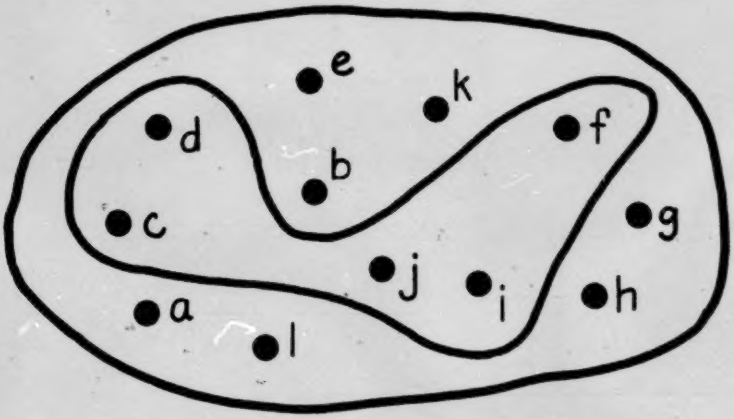
T: One of these messages is for a route that leads to the same house as before. How can we tell without actually tracing the routes?

Allow a few minutes for the students to study the messages. Some students will trace the paths while others will find explanations.

S: For the first message (**10010110001**) there are five "**1**"'s and six "**0**"'s. Any path from H to the house at the end of this path has five "**1**"'s and six "**0**"'s in some order. So **11100001001** describes a path from H to the house; **101010010110** does not.

Ask students to trace the two routes at the board to verify the result.

Draw this string picture on the board as you continue the story.



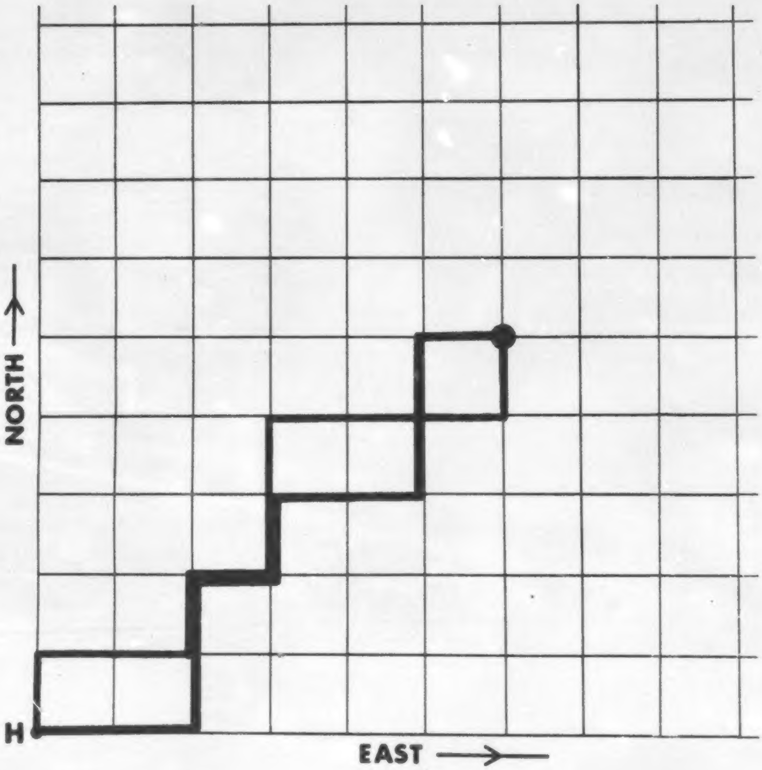
T: On another day, Spick left another kind of coded message in a string picture. It also describes the route he plans to take to a house. On your paper, try to decode the message by writing a binary code word for this picture.

There are two usual responses, **11001011001** and **00110100110**. These two binary code words arise naturally if the dots inside the red string are assigned "**0**" (the former case) or "**1**" (the latter case). To be definite, choose the latter possibility.

N 6

Ask a student to write the binary code word on the board and to trace the path (shown in red).

a	b	c	d	e	f	g	h	i	j	k
0	0	1	1	0	1	0	0	1	1	0

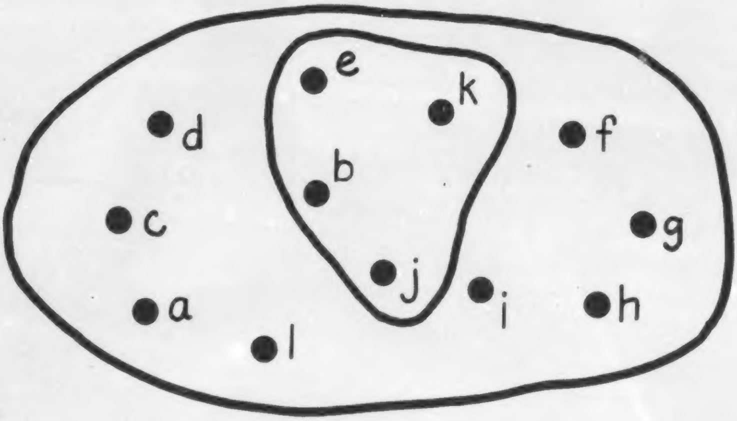


The class should conclude that the route leads to the original house.

Reverse the problem for the students by writing this binary code word on the board . . .

01001000110

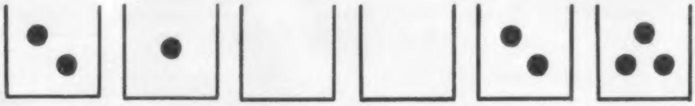
. . . and asking for its string picture message. A string picture on the back of Worksheet N30 is provided for this purpose. When most students have finished, ask for a volunteer to draw the picture on the board.



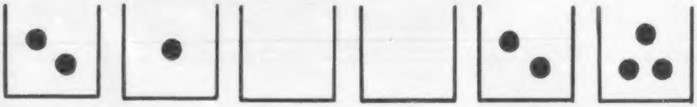
Exercise 2

T: Spick was worried about the secret messages he had been leaving. Maybe they were too easy to decode. So Spick tried to disguise them. He changed his message from a binary code word or a string picture to one using a row of boxes with some pennies in them. Here is a secret message; try to find a binary code word for the message.

Draw this picture of six boxes and eight pennies on the board.



Let the students comment about this situation. Perhaps no one will decode correctly this first message, in which case write the binary code word beneath the picture and ask for observations.

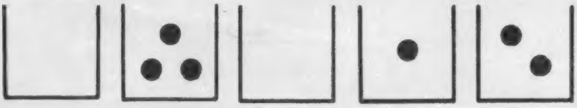


0010111001000

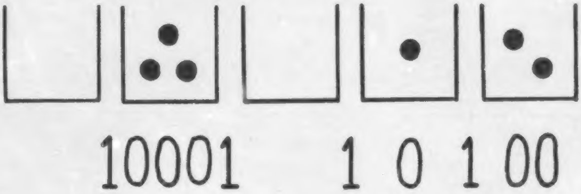
S: There are eight pennies and eight **0**'s.

S: There are six boxes but only five **1**'s.

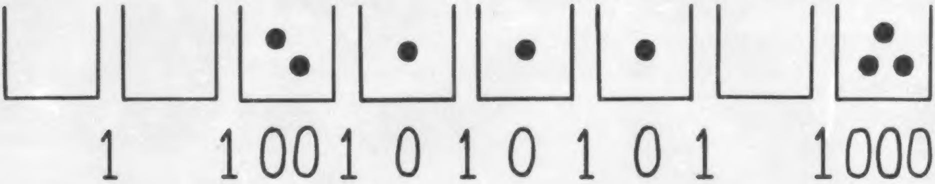
Continue by asking the class to try to decode this message.

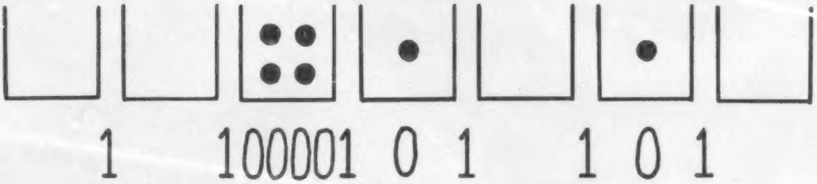


If you think a hint is needed, write the binary code word below the boxes spaced as shown below.



Continue the decoding using the messages below and allowing students to announce the binary code words as they discover the secret.

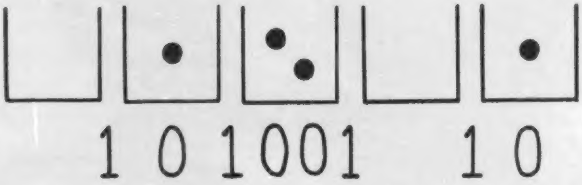


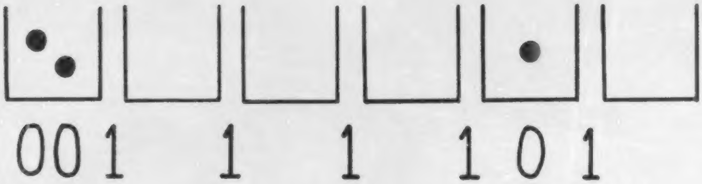


When most of the students understand the code, ask for an explanation.

S: The "**0**"'s are for pennies and the "**1**"'s are for the spaces between the boxes.

To check that everyone understands the code, ask students to decode the messages below.





T: Now I'll write a binary code word and you draw its picture of boxes and pennies.

Write this binary code word on the board and instruct students to show the boxes and pennies message on their papers.

1100100011 1001

If a few students need help getting started, ask the following questions.

T: How many pennies will we need?

S: Seven pennies, because there are seven "0"'s.

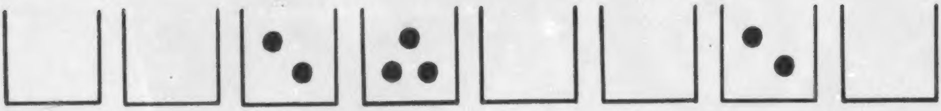
T: How many boxes?

S: There are seven "1"'s, so there must be seven spaces between boxes. That means we must have eight boxes.

N6

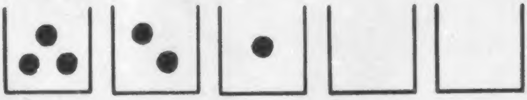
Call on a student to draw the picture on the board.

11001000111001

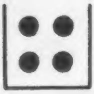


Continue with individual work asking students to draw the boxes and pennies pictures for the following binary code words.

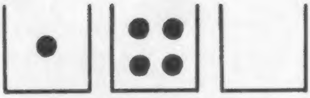
0001001011



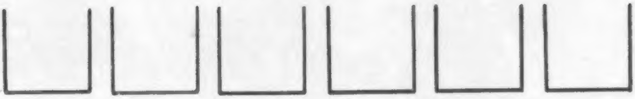
0000



0100001



11111



6.2 Combinatorics #2

CAPSULE LESSON SUMMARY

Review the messages from Lesson N6.1 and recall a scheme for counting all the routes from H to one of the intersections of the grid. Locate grid points at a certain taxi-distance from H and count the collective routes from H to these grid points. Observe a relationship between taxi-distance and the number of routes. Relate the counting scheme in this lesson to counting the ways to put a certain number of pennies into a given number of boxes.

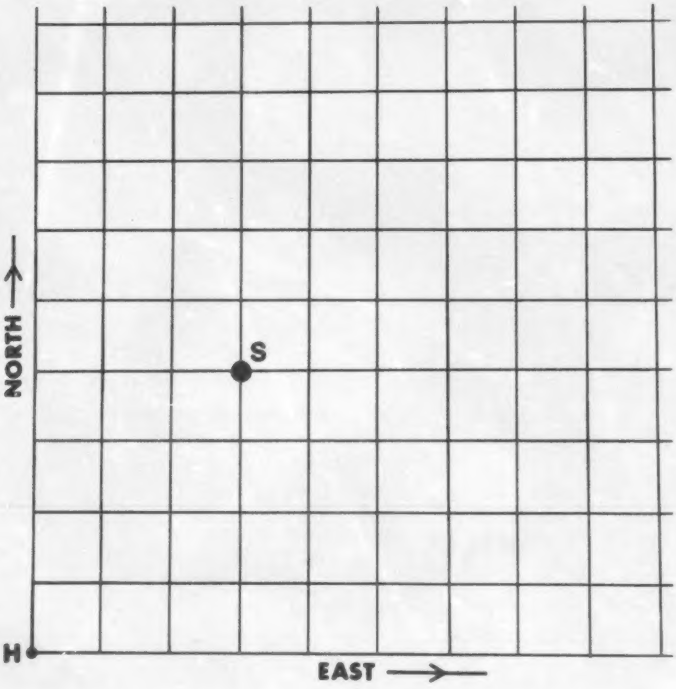
MATERIALS

Teacher:	World of Numbers Posters #1 and #2 for IG-VI; colored pens
Student:	Colored pencils; Worksheet N31

DESCRIPTION OF LESSON

Exercise 1

Review briefly the coded messages in the story of Lesson N6.1. Display World of Numbers Poster #1.



Do not write the letter "S" on the poster. It is here just to facilitate the lesson description.

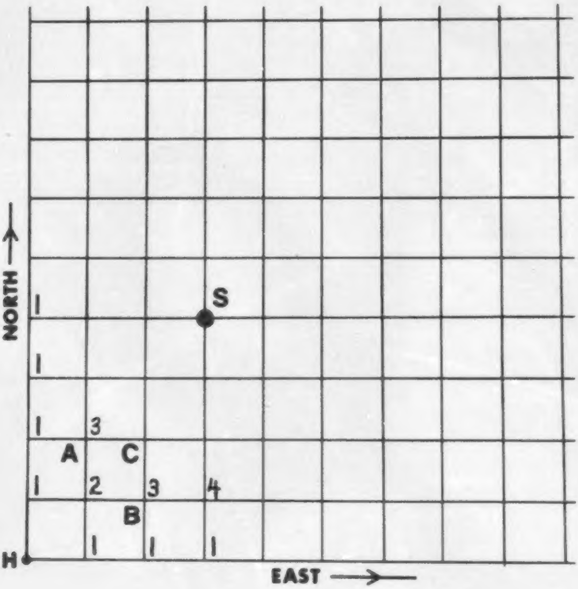
N8

T: In addition to writing secret codes for routes, Spick is also concerned about the number of different routes from his house, H, to another house, for example here (draw a dot at S). How many paths are there that Spick could take from H to this house?

Write the students' estimates on the board.

T: How do you suggest we find this number?

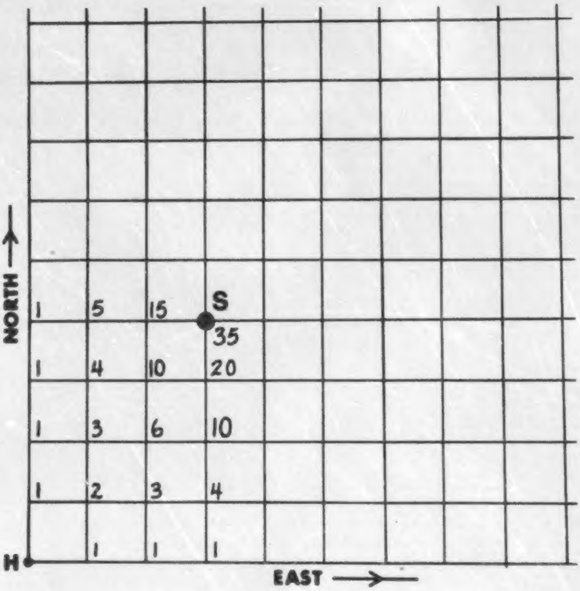
Perhaps some students will remember this kind of problem in THE HIDDEN TREASURE and will suggest a systematic procedure for counting paths. Prompt the students to recall a counting pattern, and as successive corners of the grid are considered, label them with the numbers of paths from H.



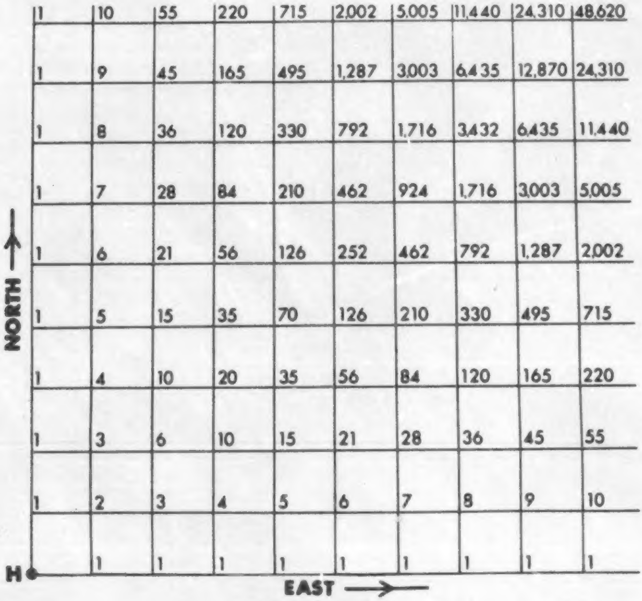
Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

S (pointing to the appropriate intersections): The only way to get here (C) is to travel through this corner (A) or this one (B). Since there are three ways to get to this corner (A) and three ways to get to this corner (B), there must be six ways (3 + 3 = 6) to get here (C).

Label the intersection at C "6". Continue labeling intersections until S is labeled "35".



Call on students to label several other corners of the grid, letting each student have a chance to label at least one corner. Then distribute Worksheet N31 and display World of Numbers Poster #2, checking the work on World of Numbers Poster #1.



1	10	55	220	715	2,002	5,005	11,440	24,310	48,620
1	9	45	165	495	1,287	3,003	6,435	12,870	24,310
1	8	36	120	330	792	1,716	3,432	6,435	11,440
1	7	28	84	210	462	924	1,716	3,003	5,005
1	6	21	56	126	252	462	792	1,287	2,002
1	5	15	35	70	126	210	330	495	715
1	4	10	20	35	56	84	120	165	220
1	3	6	10	15	21	28	36	45	55
1	2	3	4	5	6	7	8	9	10
	1	1	1	1	1	1	1	1	1

N8

Exercise 2

T (pointing to Poster #2): In Spick's neighborhood, each block is 200 m long. One day Spick drove to a secret house 1.6 km away. How many blocks did Spick drive?

S: 8 blocks, because 1600 ÷ 200 = 8.

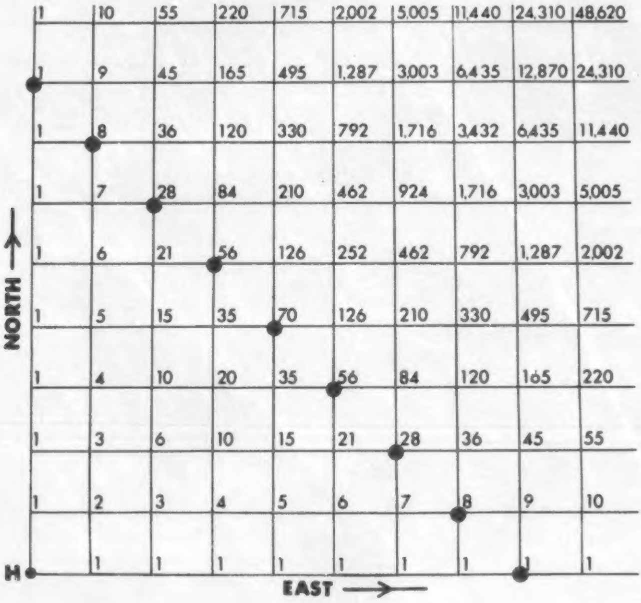
T: So we say that the taxi-distance from Spick's house to the secret house is 8.

Write a statement of this fact on the board.

$$d_T(H, \text{Secret House}) = 8$$

T: On your maps (Worksheet N31), find some places where the secret house could be.

Allow several minutes for individual work before asking volunteers to locate the houses on the poster.



1	10	55	220	715	2,002	5,005	11,440	24,310	48,620
1	9	45	165	495	1,287	3,003	6,435	12,870	24,310
1	8	36	120	330	792	1,716	3,432	6,435	11,440
1	7	28	84	210	462	924	1,716	3,003	5,005
1	6	21	56	126	252	462	792	1,287	2,002
1	5	15	35	70	126	210	330	495	715
1	4	10	20	35	56	84	120	165	220
1	3	6	10	15	21	28	36	45	55
1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1

T: There are nine places where the secret house could be. How many different routes are there altogether from H to a place where the secret house could be, one of these nine dots?

Give the students a few minutes to work independently.

S: 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 = 256.

T: Good. There are 256 paths altogether from H to these nine houses.

On the board, begin a table in which to summarize this information.

Distance (km)	Taxi-Distance (Blocks)	Number of Routes from H
1.6	8	256

Repeat this activity, replacing 1.6 km with 1 km, 1.2 km, and 1.4 km. When finished, the information in your table should be shown below.

Distance (km)	Taxi-Distance (Blocks)	Number of Routes from H
1.6	8	256
1	5	32
1.2	6	64
1.4	7	128

Encourage students to make observations about number patterns in the table. If no one suggests that the numbers in the third column are powers of 2, ask:

T: Look at the numbers in the third column; do you see a way to write them using exponents?

S: $32 = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$.

If this suggestion is not forthcoming, make it yourself. Add a fourth column to the table and in it record as a power of 2 the numbers of routes from H.

Distance (km)	Taxi-Distance (Blocks)	Number of Routes from H	
1.6	8	256	2^8
1	5	32	2^5
1.2	6	64	2^6
1.4	7	128	2^7

T: Do you notice anything interesting in this table?

S: The exponents of the numbers in the fourth column are exactly the same as the corresponding taxi-distances.

T: Yes, let's see if we can explain this fact. Suppose we look again at the houses a taxi-distance of 8 away from H. On your paper write the binary code word for a route from H to any one of these houses.

After a few minutes, ask several students to write their binary code words on the board. For example,

11100010
10101010
11111111
10001100

T: What do you notice about these numbers in the binary code?

S: Each number has eight digits.

T: What is the smallest number using eight digits in binary code?

S: 0.

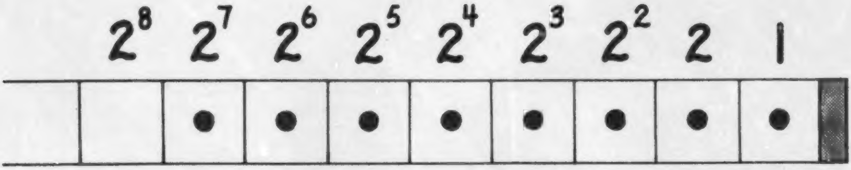
T: What is the largest number using eight digits in binary code?

S: 11111111.

T: Which number is this in decimal writing?

S: 255.

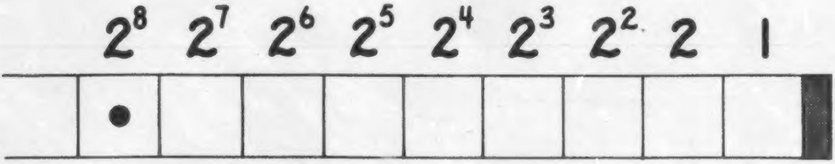
If necessary, draw a binary abacus on the board and put eight checkers on it as shown below.



T (pointing to the abacus): Here is the largest eight digit binary number on the abacus. Do you know an easy way to calculate this number?

S: Put another checker on the ones' board; then make plays.

Ask a student to make plays, or make them yourself.



S: So the number is $2^8 - 1 = 255$.

T: How many binary numbers with eight digits are there?

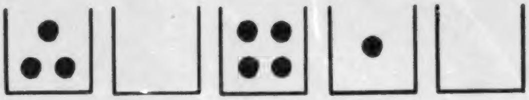
S: From 0 to 255 (inclusively); $256 = 2^8$.

T: So there are 2^8 (or 256) binary numbers with eight digits and there are 2^8 (or 256) routes going from H to one of the houses at a taxi-distance of 8 from H. We could check this for each taxi-distance.

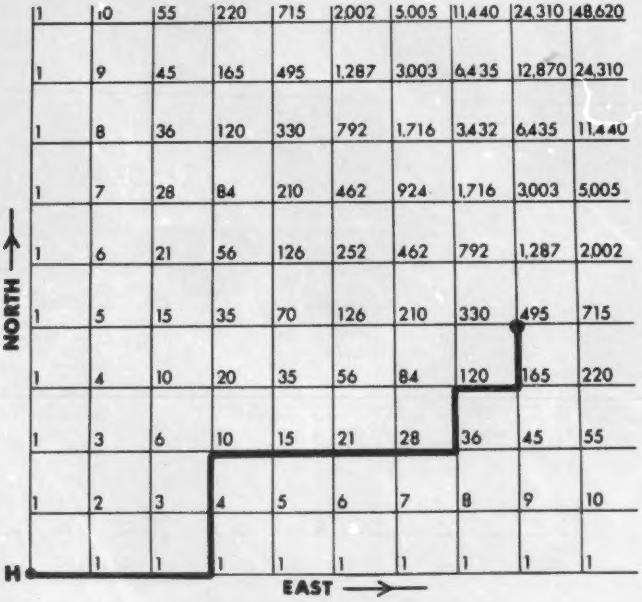
Exercise 3

T: One day Spick left this message for his partner, Span. Can you decode it and locate the corresponding house?

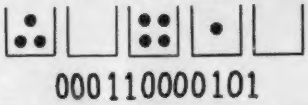
Draw this boxes and pennies picture on the board and display another copy of World of Numbers Poster #2.



Invite a student to write the binary code word corresponding to the message and to locate the house on the grid.



1	10	55	220	715	2,002	5,005	11,440	24,310	48,620
1	9	45	165	495	1,287	3,003	6,435	12,870	24,310
1	8	36	120	330	792	1,716	3,432	6,435	11,440
1	7	28	84	210	462	924	1,716	3,003	5,005
1	6	21	56	126	252	462	792	1,287	2,002
1	5	15	35	70	126	210	330	495	715
1	4	10	20	35	56	84	120	165	220
1	3	6	10	15	21	28	36	45	55
1	2	3	4	5	6	7	8	9	10
	1	1	1	1	1	1	1	1	1



T: On your paper, make up another message using five boxes and eight pennies. Then write the corresponding binary code word and find the house.

Your students may be surprised to discover that each message involving five boxes and eight pennies names a path to the same house, four blocks North and eight blocks East of H. Invite various students to write their binary code words on the board.

111100000000
001001001001
000110100100

T: What do you notice?

S: Each binary code word takes us to the house four blocks North and eight blocks East of H.

T: How can we explain this?

S: Each binary code word has four "1"'s and eight "0"'s; each path must go four blocks North and eight blocks East.

T: How many ways can we put eight pennies into five boxes?

S: 495, the number of paths from H to this house.

Draw a picture of six boxes on the board.

N8

T: How many ways can we put seven pennies into six boxes?

Let the students discuss this situation and lead the discussion toward a solution to the problem.

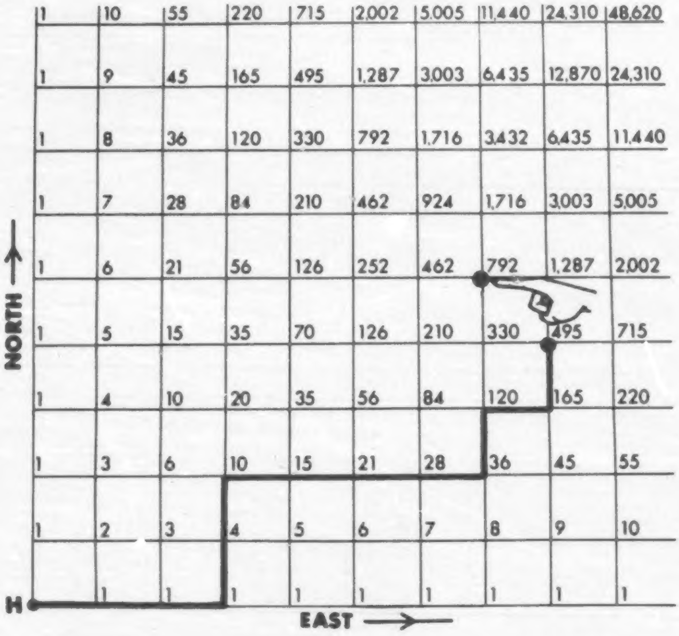
S: Seven pennies mean seven "**0**"'s in each binary code word.

S: Six boxes mean six "**1**"'s in each binary code word.

S: No, the "**1**"'s are for the spaces between the boxes. With six boxes there are five spaces. There can be only five "**1**"'s in each binary code word.

S: So each binary code word will have seven "**0**"'s and five "**1**"'s. That is the same as a path from H to the house that is five blocks North and seven blocks East.

Ask a student to find that house on the map of Spick's neighborhood.



1	10	55	220	715	2,002	5,005	11,440	24,310	48,620
1	9	45	165	495	1,287	3,003	6,435	12,870	24,310
1	8	36	120	330	792	1,716	3,432	6,435	11,440
1	7	28	84	210	462	924	1,716	3,003	5,005
1	6	21	56	126	252	462	792	1,287	2,002
1	5	15	35	70	126	210	330	495	715
1	4	10	20	35	56	84	120	165	220
1	3	6	10	15	21	28	36	45	55
1	2	3	4	5	6	7	8	9	10
	1	1	1	1	1	1	1	1	1

T: How many paths are there from H to this house?

S: 792.

T: How many ways can we put seven pennies into six boxes?

S: 792 ways.

6.3 Combinatorics #3

CAPSULE LESSON SUMMARY

Relate problems of finding the number of ways of putting checkers on the Minicomputer to those of putting pennies into boxes (see Lesson N6.1). Answer a probability question in this context.

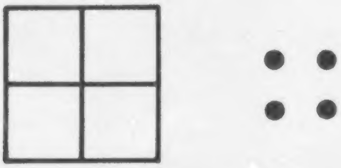
MATERIALS

Teacher: Demonstration Minicomputer Set
Student: Pencil; Worksheets N32 and N33

DESCRIPTION OF LESSON

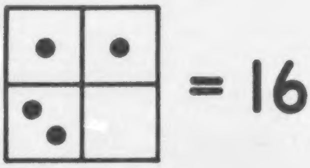
Exercise 1

Display one Minicomputer board and four regular checkers.



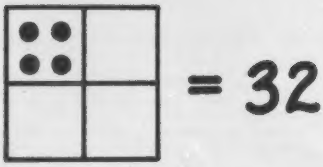
T: Put a number on the ones' board of the Minicomputer using all of these four checkers but no other.

Invite a student to put such a number on the Minicomputer.



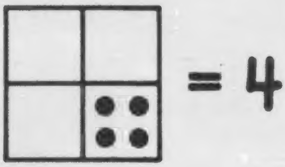
T: What is the largest number we can put on the Minicomputer with exactly these four checkers?

S: 32.



T: . . . and the smallest number?

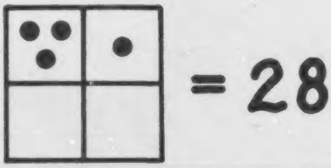
S: 4.



T: Can we put any number between 4 and 32 on the Minicomputer with exactly four regular checkers?

Allow time for the students to think about the question.

S: We cannot put on 31, 30 or 29, because after 32 the next largest number we can put on is 28.



S: Also we cannot put on 27.

There are five numbers between 4 and 32 that cannot be put on the Minicomputer with exactly four positive checkers; 23, 27, 29, 30, and 31. Accept any of these numbers that the students suggest. It is not necessary for the class to find them all at this time.

T: How many ways can these four checkers be put on the ones' board of the Minicomputer?

Again, allow time for the students to contemplate the question.

S: Putting these four checkers on the Minicomputer is like putting four pennies into four boxes. We could use a map of Spick's neighborhood.

If no one suggests this connection between the two problems, present it yourself. Distribute individual copies of Worksheet N32.

T: Yes, it is the same problem stated another way. How can we determine which corner to look at for this problem?

S: The four checkers are like the four pennies. That means there should be four "**0**"'s in a binary code word.

S: Four boxes produce three spaces between boxes, so there should be three "**1**"'s in a binary code word.

S: Each binary code word for the "four pennies into four boxes" problem has four "**0**"'s and three "**1**"'s.

S: We should look at the corner four blocks East and three blocks North of H.

Let the students find the corner point on their worksheets.

S: There are 35 routes from H to the house at this corner, so there are 35 ways to put four pennies into four boxes. There must also be 35 ways to put these four checkars on the ones' board of the Minicomputer.

Encourage students to contribute as much as possible to the class's collective solution of the problem, but be prepared to guide the discussion whenever necessary.

T: If the four checkers are placed randomly on the Minicomputer, what is the probability that the number is between 10 and 20?

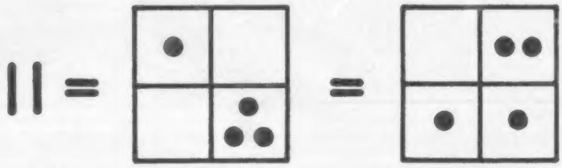
Write some student estimates on the board.

T: How might we solve this problem?

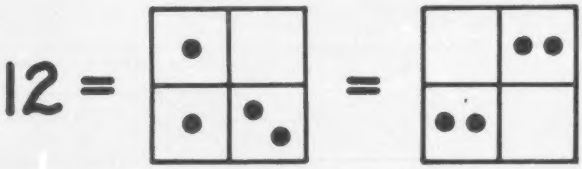
S: We must find out how many ways the numbers between 10 and 20 can be put on the Minicomputer (with exactly four regular checkers).

T: Okay, let's start with 11. How can we put 11 on the Minicomputer with exactly four regular checkers?

Invite a student to put on 11. The class should conclude that there are two ways to do this.



Continue by asking how to put 12 on the Minicomputer with exactly four regular checkers.



N 17

Distribute individual copies of Worksheet N33 and ask the students to work independently on the numbers 13 through 19. While the students are working, draw this table on the board.

Number	Number of ways to be put on the minicomputer
11	2
12	2
13	
14	
15	
16	
17	
18	
19	

Ask various students to help complete the right column of the table and then to sum the numbers in the right column.

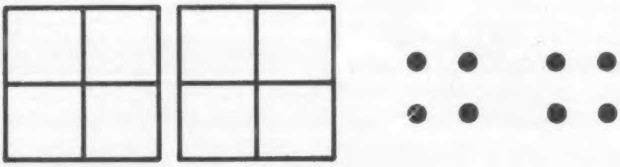
Number	Number of ways to be put on the minicomputer
11	2
12	2
13	2
14	3
15	1
16	2
17	1
18	2
19	1
	16

T: There are 16 ways to put the numbers between 10 and 20 on the Minicomputer with exactly four regular checkers. What is the probability of getting one of these numbers when we randomly place the four checkers on the Minicomputer?

S: $\frac{16}{35}$, because there are 35 ways to put the four checkers on the Minicomputer and 16 of the ways result in a number between 10 and 20.

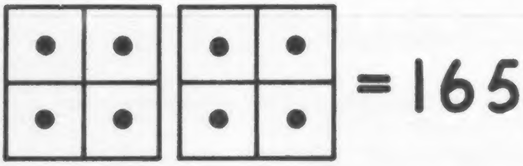
Exercise 2

Display two Minicomputer boards and eight regular checkers.



T: Put any number you wish on this Minicomputer using exactly eight regular checkers.

For example,



Solicit several responses before asking for the largest such number (640) and the smallest such number (8).

T: Are there some numbers between 8 and 640 which we cannot put on this Minicomputer with exactly eight regular checkers?

It is quite easy to argue that the numbers between 600 and 640, and those between 580 and 600 cannot be put on in this way. It is not necessary (nor very easy) to find all such numbers.

T: How many ways are there to put the eight checkers on this Minicomputer?

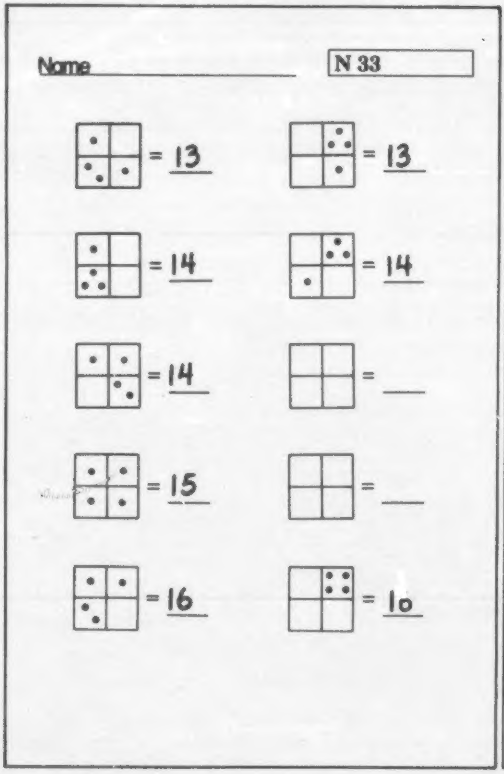
Write some student estimates on the board for future reference. The class should observe the equivalence of this problem to the problem of putting eight pennies into eight boxes. Let the students discuss this situation.

S: There are eight "**0**"'s and seven "**1**"'s in a binary code word for a solution to the "eight pennies and eight boxes" problem.

S: All binary code words with eight "**0**"'s and seven "**1**"'s give ways to put eight pennies into eight boxes.

S: We should look at the corner that is eight blocks East and seven blocks North of H.

The students should consult their worksheets (maps of Spick's neighborhood) and conclude that there are 6,435 routes from H to the house at the appropriate corner and, therefore, 6,435 ways to put eight checkers on the Minicomputer.



Name ____________________ N 33

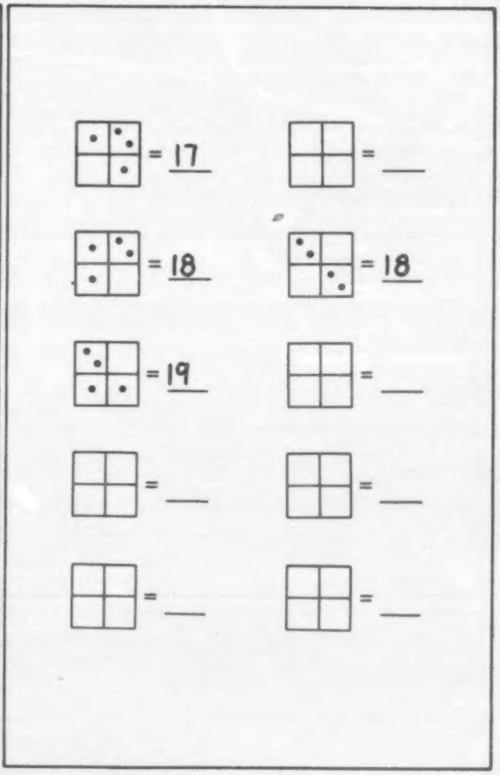
= 13 = 13

= 14 = 14

= 14 = ___

= 15 = ___

= 16 = 10



= 17 = ___

= 18 = 18

= 19 = ___

= ___ = ___

= ___ = ___

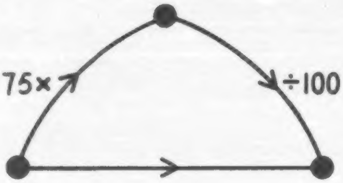
7

PERCENT

The lessons of this chapter introduce the concept of percent as a composite function; for example, $75\% = 75\times \circ \div 100 = \div 100 \circ 75\times$. In this context, the exercises investigate many useful names for certain percent functions (for example, $20\% = \frac{2}{10}\times = \frac{1}{5}\times = 0.2\times$), several interesting patterns, and some helpful properties (for example, a% of b = b% of a). The hand-calculator is employed as a support tool in calculations and as a situation in which to explore patterns involving percent. The lessons all include some applications of percent in the solution of real life problems.

CAPSULE LESSON SUMMARIES

Section 7.1: Find several ways to label a composite arrow such as:



Give a new name to the red arrow: "75%". Use the arrow picture and the various names for 75% to do calculations like the following.

75% of $560 = \square$

75% of $\square = 120$

Progressively build an arrow picture and label the dots or the arrows appropriately for given information. Include composite arrows designated in the form "$\square$%" and fill in the box for each such arrow.

(Worksheets N34, N35, N36, N37)

Section 7.2: Review the notion of a percent function by examining many names for 20%. Observe some patterns. Use many arrow pictures to solve problems related to election results in a small town.
(Worksheets N38, N39)

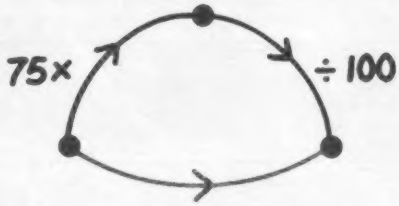
Section 7.3: Review the notion of a percent function by examining many names for 15%. Observe that to calculate "a% of b" you can interchange a and b; a% of b = b% of a. Introduce the $\boxed{\%}$ key on a hand-calculator and use the calculator in solving some problems related to investment.

Section 7.4: Review the notion of a percent function by examining many names for $12\frac{1}{2}\%$. Write fraction names for percent functions. Solve problems related to clothing store discounts.
(Worksheets N40, N41, N42, N43)

7.1 Percent #1

CAPSULE LESSON SUMMARY

Find several ways to label a composite arrow such as:



Give a new name to the red arrow: "75%". Use the arrow picture and the various names for 75% to do calculations like the following.

75% of 560 = ☐

75% of ☐ = 120

Progressively build an arrow picture and label the dots or the arrows appropriately for given information. Include composite arrows designated in the form "☐ %" and fill in the box for each such arrow.

MATERIALS

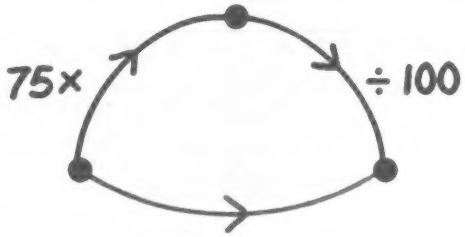
Teacher: Colored chalk

Student : Paper; pencil; Worksheets N34, N35, N36, and N37

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.

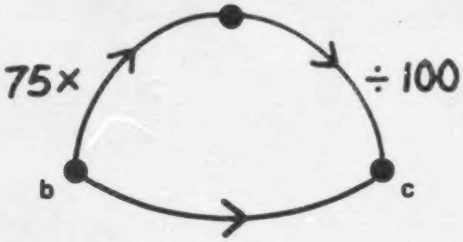


T: What could the red arrow be for?

S: $\frac{75}{100}\times$.

S: $\frac{3}{4}\times$.

As students' suggest different names for the red arrow, list them on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

$$\frac{75}{100}\times \qquad \frac{3}{4}\times \qquad 0.75\times \qquad \div\frac{4}{3}$$

T: Another name for this arrow is "75%" (read: seventy-five percent of).

Write this problem on the board.

$$75\% \text{ of } 560 = \boxed{}$$

T: Calculate "75% of 560". Use the arrow picture to help you.

Allow a few minutes for the students to work on this problem independently.

S (pointing to b): I labeled this dot 560. 75 × 560 = 42,000 and 42,000 ÷ 100 = 420. So 75% of 560 = 420.

S: I calculated "$\frac{3}{4}$ x 560": 560 ÷ 4 = 140 and 140 x 3 = 420.

T: Perhaps "$\frac{3}{4}$ x 560" is an easier calculation than "$\frac{75}{100}$ x 560", but both calculations are correct; they both give us that 75% of 560 = 420.

Instruct the students to solve the following problems, or choose similar problems more appropriate for your class. Answers are in the boxes.

$$75\% \text{ of } 0.48 = \boxed{0.36}$$

$$75\% \text{ of } \boxed{160} = 120$$

$$75\% \text{ of } \boxed{0.52} = 0.39$$

Check the results collectively. Then continue with these problems. Answers are in the boxes.

$$75\% \text{ of } 60 = \boxed{45} \qquad 0.75 \times 120 = \boxed{90}$$

$$75\% \text{ of } 28.8 = \boxed{21.6} \qquad 0.75 \times 4.24 = \boxed{3.18}$$

$$75\% \text{ of } 8.36 = \boxed{6.27}$$

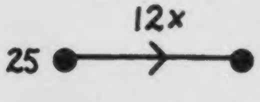
$$75\% \text{ of } 816 = \boxed{612}$$

$$25\% \text{ of } 816 = \boxed{204}$$

Exercise 2

T: I am going to draw an arrow road on the board and you will label the dots or fill in the boxes for the arrows as we proceed.

Start the road with this arrow.

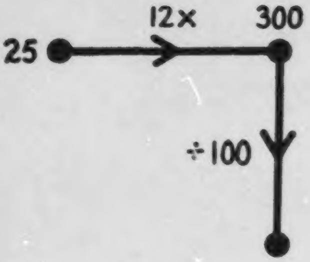


N7

T: Which number is here (point to the ending dot)?

S: 300.

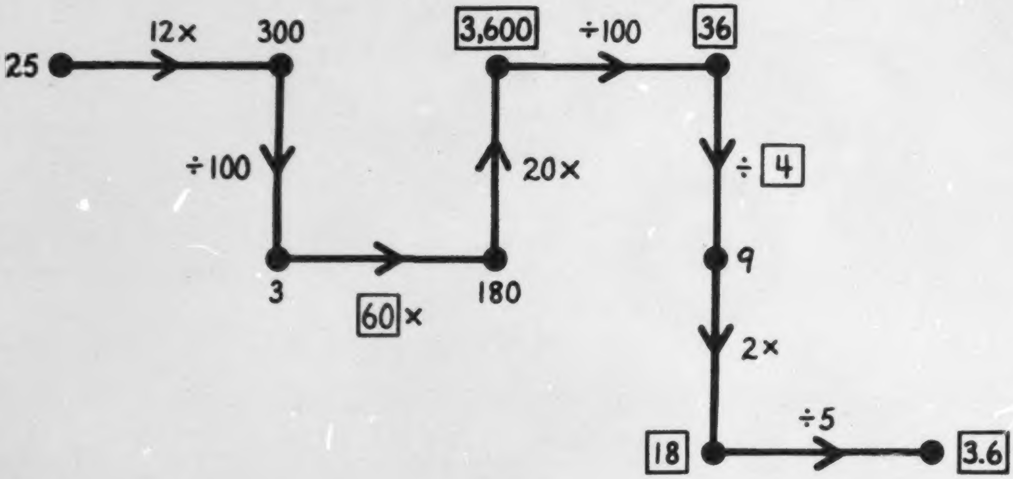
Label the dot "300" and add this arrow to your drawing.


12×
300
25
÷100

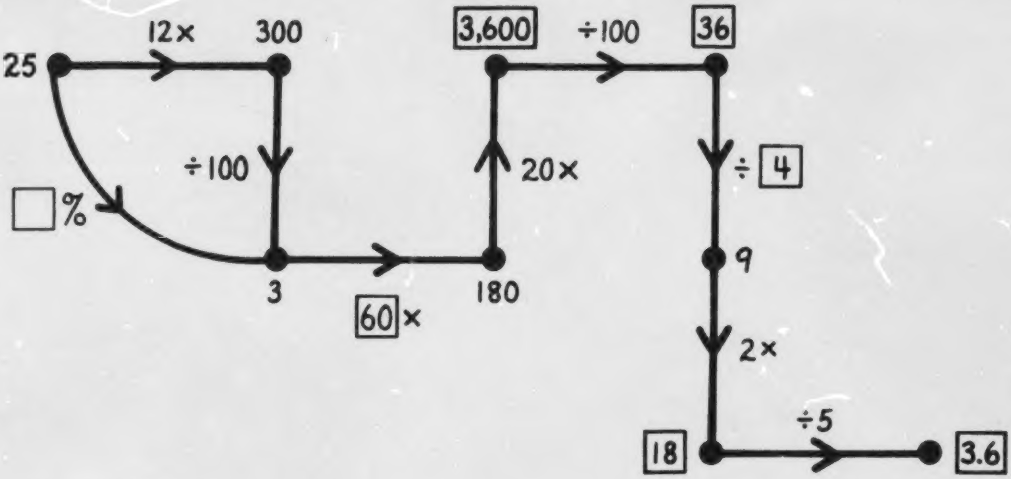
T: Which number is here (point to the ending dot)?

S: 3.

Continue the exercise by producing the arrow road shown below. Correct student responses are in the boxes.


12×
300
25
÷100
3
60 ×
180
20×
3,600
÷100
36
÷ 4
9
2×
18
÷5
3.6

Add this red arrow to your picture.

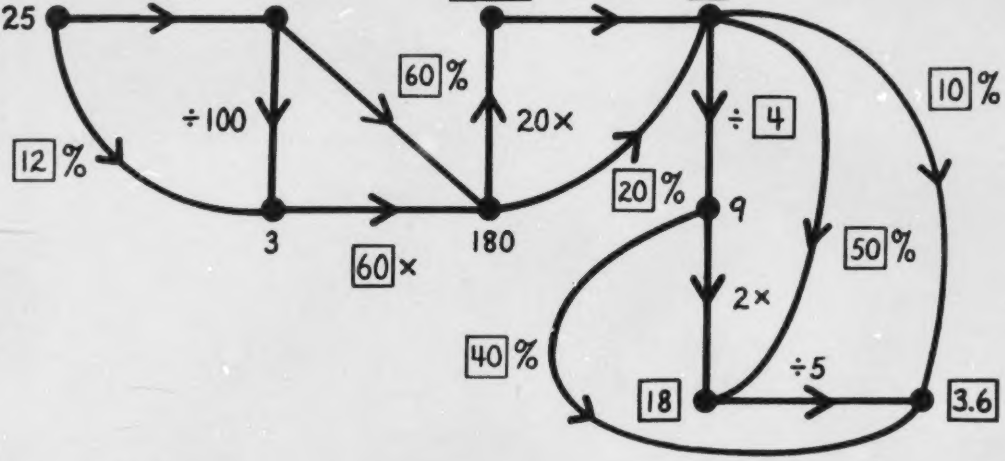

25
12×
300
÷100
3
3,600
÷100
36
20×
÷ 4
9
2×
18
÷5
3.6
60 ×
180
☐ %

T: Fill in this box for the red arrow.

S (tracing the appropriate arrows): 12 × followed by ÷100 is 12%.

Encourage the students to observe that ☐ % = ☐ x ∘ ÷ 100.

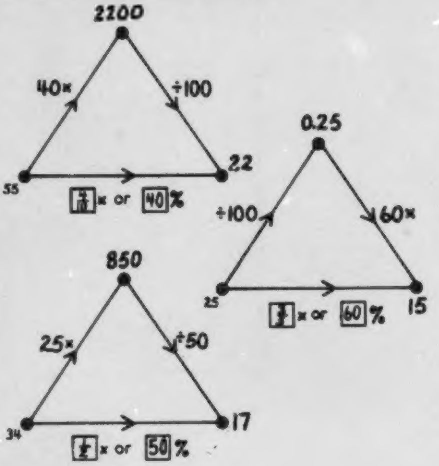
Fill in the box and draw some other red arrows such as in the next illustration. Answers are in the boxes.


25
12×
300
÷100
3
12 %
60 %
3,600
÷100
36
20×
180
60 ×
20 %
÷ 4
9
2×
18
÷5
3.6
10 %
50 %
40 %

Worksheets N34, N35, N36, and N37 are available for individual work.

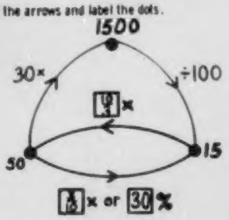
Name ______________________ **N 34** *

Label the dots and fill in the boxes for the arrows.


2200
40×
÷100
55
22
$\boxed{\frac{2}{5}}$ × or $\boxed{40}$ %
0.25
÷100
60×
25
15
$\boxed{\frac{3}{5}}$ × or $\boxed{60}$ %
850
25×
÷50
34
17
$\boxed{\frac{1}{2}}$ × or $\boxed{50}$ %

Name ______________________ **N 35** **

Fill in the boxes for the arrows and label the dots.


1500
30×
÷100
$\boxed{\frac{10}{3}}$ ×
50
15
$\boxed{\frac{3}{10}}$ × or $\boxed{30}$ %

Calculate. (Hint: Use the arrow picture above as an aid.)

30 % of 50 = 15 0.3 × 70 = 21

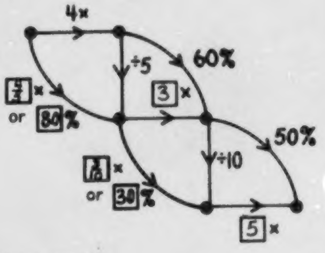
$\frac{3}{10}$ × 20 = 6 $\frac{10}{3}$ × 20 = $\frac{200}{3} = 66\frac{2}{3}$

30 % of 40 = 12

60 % of 40 = 24

Name ______________________ **N 36** ***

Fill in the boxes for the arrows.


4×
÷5
60%
$\boxed{\frac{4}{5}}$ ×
or $\boxed{80}$ %
$\boxed{3}$ ×
50%
÷10
$\boxed{\frac{3}{10}}$ ×
or $\boxed{30}$ %
$\boxed{5}$ ×

Complete these number stories.

80 % of 75 = $\boxed{60}$ 80 % of $\boxed{30}$ = 24

60 % of 75 = $\boxed{45}$ 60 % of $\boxed{40}$ = 24

50 % of 75 = $\boxed{37.5}$ 50 % of $\boxed{48}$ = 24

30 % of 75 = $\boxed{22.5}$ 30 % of $\boxed{80}$ = 24

Name ______________________ **N 37** ****

The Drama club is raising money to buy costumes; their goal is $400. The treasurer announces at the meeting that they now have 65% of the goal.

How much money has the Drama club raised? 260

How much money do they still need to raise? 140

What percent of the goal are they short? 35%

One week later:

The treasurer announces that the Drama club now has raised $340.

What percent of the goal does the Drama club have now? 85%

What percent of the goal are they short? 15%

7.2 Percent #2

CAPSULE LESSON SUMMARY

Review the notion of a percent function by examining many names for 20%. Observe some patterns. Use many arrow pictures to solve problems related to election results in a small town.

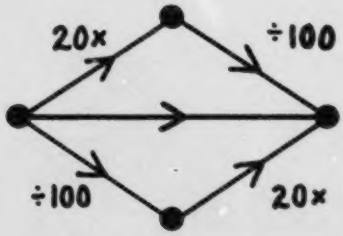
MATERIALS

Teacher: Colored chalk
Student : Paper; colored pencils; Worksheets N38 and N39

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.



T: What could the red arrow be for?

S: $\frac{20}{100}\times$ or 20%.

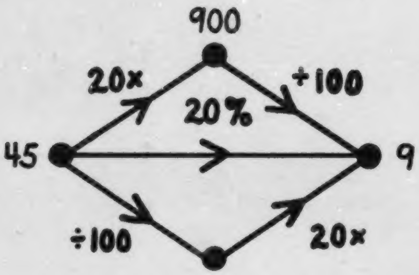
Label the arrow "20%".

T: Calculate "20% of 45". Use the arrow picture to help you.

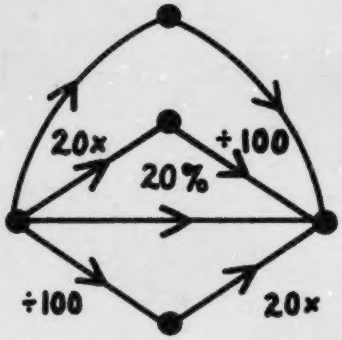
Allow a few minutes for independent work. Then ask a student to explain the calculation at the board.

N 13

S (tracing the appropriate arrows): 20 × 45 = 900 and 900 ÷ 100 = 9, so 20% of 45 = 9.


900
20×
÷100
20%
45
9
÷100
20×

Add these arrows to your picture.

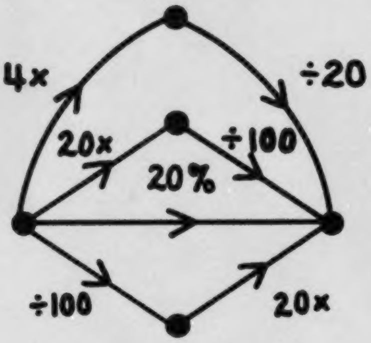

20×
÷100
20%
÷100
20×

T: What could these arrows be for?

S: 4 × followed by ÷20.

NOTE: Many other correct responses might be given here; for example, 5 × followed by ÷25, 10 × followed by ÷50, 100 × followed by ÷500, and so on.

Label the arrows according to your student's response and record the corresponding information to the side of your drawing.



$$20\% = \frac{20}{100}\times = \frac{4}{20}\times$$

Continue in this manner until several other names for 20% have been suggested. Be sure that $\frac{1}{5}\times$, $0.2\times$, and ÷5 are included in your list. Observe that using other names for 20% can sometimes make the calculations simpler.

Write these problems on the board and ask the students to do the calculations mentally.

20% of 75 = ☐

40% of 75 = ☐

60% of 75 = ☐

80% of 75 = ☐

100% of 75 = ☐

N 13

T: What number is 20% of 75?

S: 15, because $\frac{1}{5} \times 75 = 15$.

T: Good. What number is 40% of 75?

S: 30, just double 20% of 75.

T: Yes, and 60% of 75. . . ?

S: 45, $3 \times 15 = 45$.

T: 80% of 75. . . ?

S: 60; $2 \times 30 = 60$.

T: 100% of 75. . . ?

S: 75; 100% = 1 ×. Also, $5 \times 15 = 75$.

Continue this mental arithmetic activity using the problems suggested below. Answers are in the boxes.

120% of 75 = $\boxed{90}$

10% of 75 = $\boxed{7.5}$

5% of 75 = $\boxed{3.75}$

15% of 75 = $\boxed{11.25}$

25% of 75 = $\boxed{18.75}$

2.5% of 75 = $\boxed{1.875}$

Exercise 2

Write this information on the board as you tell the following story.

Population: 800
Eligible voters: 320
Number voting: 240

T: I want to tell you about a little town I know of called "Prairieville." Prairieville has a population of 800. It is election time in Prairieville and the voters must choose a mayor. There are three candidates to choose among: Xandu, Yost, and Zinc. Of the 320 eligible voters in Prairieville only 240 voted in the mayoral election. Here are the results.

Add this information to that which is on the board.

Mayor

Candidate	% of votes	Number of votes
Xandu	60%	
Yost	25%	
Zinc		

T: What percent of the votes did Zinc receive?

S: 15%, because 85% of the votes were for Xandu or Yost.

Add this information to the chart.

T: How many votes did Xandu receive out of the 240 votes cast?

S: We must calculate "60% of 240".

Give the students a few minutes to do this calculation. Encourage those students who need help to use an arrow picture. Check the problem collectively and ask the class to complete the chart.

Mayor

Candidate	% of votes	Number of votes
Xandu	60%	144
Yost	25%	60
Zinc	15%	36

T: How did you calculate the number of votes for Yost?

S: $25\% = \frac{1}{4} \times$ and $\frac{1}{4} \times 240 = 60$.

T: How did you calculate the number of votes for Zinc?

S: 204 people voted for Xandu or Yost. $240 - 204 = 36$, so 36 people voted for Zinc.

S: $\frac{3}{20} \times 240 = 36$.

S: $240 \times 0.15 = 36$.

Let the students make any comments they wish about how to do the calculations.

T: In addition to electing a mayor, the voters had to decide on the issue of whether or not to have a curfew for Prairieville. Here are some of the results.

Draw this chart on the board.

Curfew

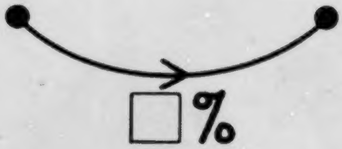
	For	Against
Number of votes		180
% of votes		

T: Again, 240 people voted on this issue. How many people voted for the curfew?

S: 60; 240 - 180 = 60.

Add this information to your chart.

Draw this arrow picture on the board.



T: What percent of the votes cast were for the curfew?
How can we solve this problem?

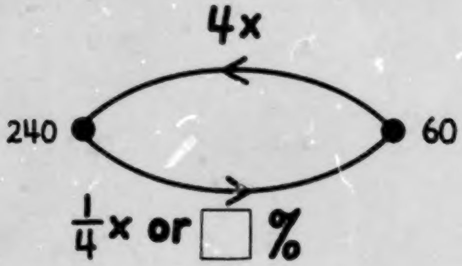
Give the students some time to consider these questions. The following dialogue suggests a possible sequence of steps to solving the problem; your students may have a different method. Follow their suggestions.

S: Put 240 at the start of the arrow and 60 at the end.

S: The return arrow is for 4 × because 4 × 60 = 240.

S: The arrow from 240 to 60 is for $\frac{1}{4}$x.

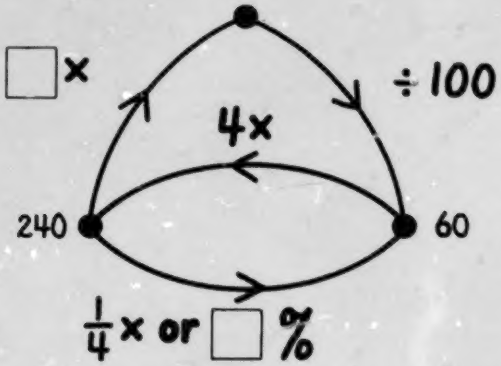
Add this arrow to your picture.


4x
240
60
$\frac{1}{4}$x or ☐ %

T: We still need to fill in the box for the percent function.

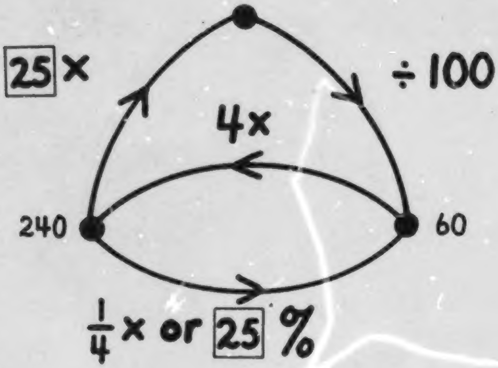
S: We could use a detour; ☐ % is ☐ x followed by ÷100.

Add these arrows to your picture.


☐x
÷100
4x
240
60
$\frac{1}{4}$x or ☐ %

S: $\frac{1}{4} = \frac{25}{100}$, so 25 goes in the boxes.

S: $\frac{1}{4}x = 25\%$, so 25% of 240 is 60.


25 ×
÷ 100
4 ×
240
60
$\frac{1}{4}$ × or 25 %

T: What percent of the votes were against the curfew?

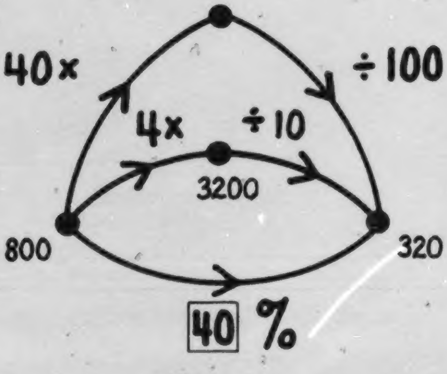
S: 75%.

Curfew

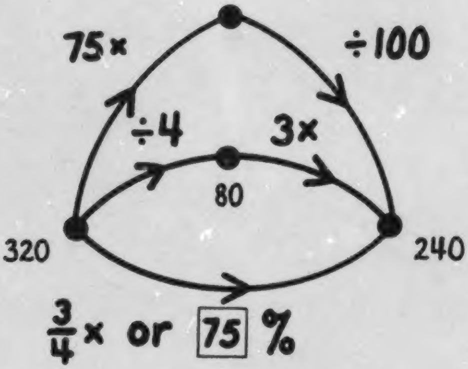
	For	Against
Number of votes	60	180
% of votes	25%	75%

If time permits, ask and discuss these two questions:

- What percent of the population is eligible to vote? (Answer: 40%)


40 ×
÷ 100
4 ×
÷ 10
3200
800
320
40 %

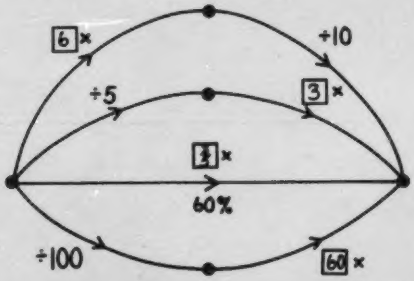
- What percent of the eligible voters voted? (Answer: 75%)


75×
÷100
÷4
3×
80
320
240
$\frac{3}{4}$× or $\boxed{75}$ %

Worksheets N38 and N39 are available for individual work.

Name____________________ N 38 *

Fill in the boxes for the arrows.



Calculate. (Hint: Use the arrow picture above as an aid).

60% of 105 = 63 20% of 105 = 21

30% of 105 = 31.5 10% of 105 = 10.5

Name____________________ N 39 **

Complete this table of Test Results for a 120 question true-false test.

Student	Number Correct	% Correct
A	96	80%
B	78	65%
C	102	85%
D	108	90%
E	66	55%

If 70% or better is a passing grade on this test, who passes? A, C, and D

How many questions must a person get correct to have a passing grade? 84

7.3 Percent #3

CAPSULE LESSON SUMMARY

Review the notion of a percent function by examining many names for 15%. Observe that to calculate "a% of b" you can interchange a and b; a% of b = b% of a. Introduce the $\boxed{\%}$ key on a hand-calculator and use the calculator in solving some problems related to investment.

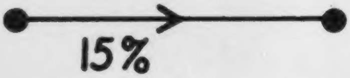
MATERIALS

Teacher: Hand-calculator; colored chalk
Student : Hand-calculator; colored pencils; paper

DESCRIPTION OF LESSON

Exercise 1

Draw this red arrow on the board.



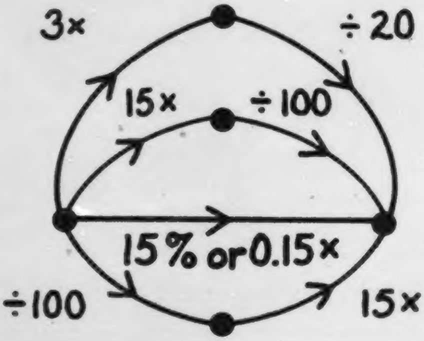
T: How do we calculate 15% of some number?

S: Multiply by 15 and then divide by 100; or divide by 100 and then multiply by 15.

S: Multiply by 0.15.

S: Multiply by 3 and then divide by 20.

Add the appropriate arrows as ways to calculate 15% are suggested.



T: What are some other names for 15%?

S: "$\frac{15}{100}\times$".

S: "$\frac{3}{20}\times$".

There are many correct responses that could be given; for example, "$\frac{6}{40}\times$", "$\frac{60}{400}\times$", "$\frac{600}{4,000}\times$", $\frac{300}{2,000}\times$", and so on. Record the names suggested by the class on the board near the arrow picture.

$$\begin{aligned} 15\% &= \frac{15}{100}\times \\ &= \frac{3}{20}\times \\ &= \frac{6}{40}\times \\ &= 0.15\times \\ &= \frac{30}{200}\times \end{aligned}$$

T: Calculate "15% of 80". Use an arrow picture to help you.

Give the class a few minutes to solve this problem before asking a student to explain the calculation.

S: To calculate "$\frac{3}{20} \times 80$", calculate "80 ÷ 20" and get 4; then calculate "3 × 4" and get 12. So 15% of 80 = 12.

S: $0.15 \times 80 = 12$.

N 25

Write these pairs of problems on the board and ask the students to solve them. Answers are in the boxes.

$$16\% \text{ of } 50 = \boxed{8} \qquad 25\% \text{ of } 12 = \boxed{3} \qquad 28\% \text{ of } 75 = \boxed{21}$$

$$50\% \text{ of } 16 = \boxed{8} \qquad 12\% \text{ of } 25 = \boxed{3} \qquad 75\% \text{ of } 28 = \boxed{21}$$

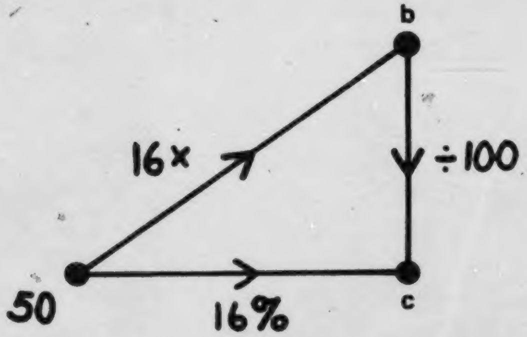
$$24\% \text{ of } 150 = \boxed{36} \qquad 20\% \text{ of } 85 = \boxed{17}$$

$$150\% \text{ of } 24 = \boxed{36} \qquad 85\% \text{ of } 20 = \boxed{17}$$

Even before the work is complete, your students probably will observe that each pair of problems has the same answer. Lead a discussion to explain this observation.

T: Let's use an arrow picture to help us explain what is happening. Suppose we wish to calculate 16% of 50.

Draw this arrow picture on the board.

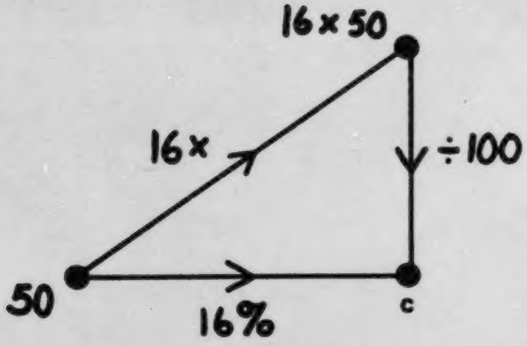


Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (pointing to b): Which number is here?

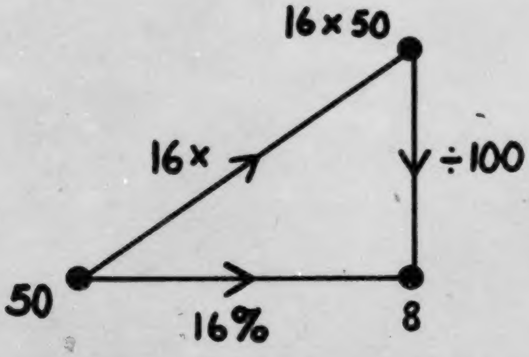
S:	16 × 50 or 800.

Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.


16×50
16×
÷100
50
16%
c

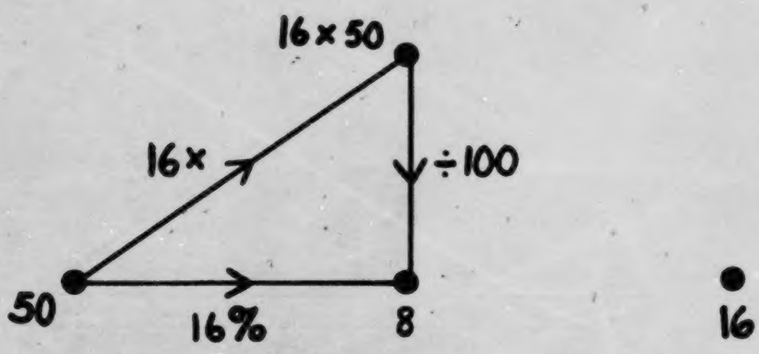
T (pointing to c): And which number is here?

S:	800 ÷ 100 or 8.


16×50
16×
÷100
50
16%
8

T:	Now let's look at an arrow picture for 50% of 16.

Add the dot labeled "16" to your drawing.

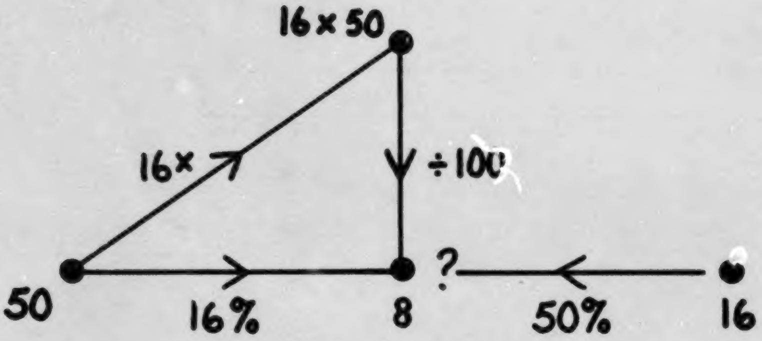

16×50
16×
÷100
50
16%
8
16

N 25

T: We want to draw a 50% arrow starting at 16.
Where do you think this arrow will end?

S: At 8.

Draw an arrow starting at 16 and ending at 8.


16 × 50
16×
÷100
?
50
16%
8
50%
16

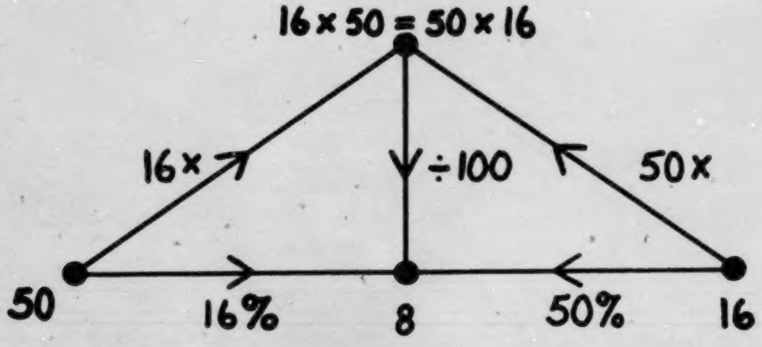
T: Let's check to be sure you are correct.
How do we calculate 50% of 16?

S: Multiply by 50 and then divide by 100.

S: A 50× arrow starting at 16 would end at 16 × 50 because
16 × 50 = 50 × 16.

S: A ÷100 arrow starting at 50 × 16 is already in the picture; it ends at 8.

Draw the 50× arrow and complete the picture.


16 × 50 = 50 × 16
16×
÷100
50×
50
16%
8
50%
16

T: 16% of 50 = 50% of 16. We could draw an arrow picture similar to this one for each pair of problems on the board. In each case, we get the same answer for both problems.

Exercise 2

NOTE: The purpose of this exercise is to explore the use of the $\boxed{\%}$ key found on many hand-calculators. Therefore, the hand-calculators that you provide for this lesson should have a $\boxed{\%}$ key. It is possible to present the investment problem in this exercise without using hand-calculators with a $\boxed{\%}$ key. In that case, computations could be done with pencil and paper.

Distribute a hand-calculator to each student and write these problems on the board.

65% of 84 = □

13% of 120 = □

T: How could we solve these problems using the calculator?

S: To calculate "65% of 84" we could put 84 on the display and then press $\boxed{\times}\boxed{6}\boxed{5}\boxed{\div}\boxed{1}\boxed{0}\boxed{0}\boxed{=}$.

T: Yes, 65× followed by ÷100. Is there a special key that might be used for this calculation?

S: The $\boxed{\%}$ key.

T: Let's see how the $\boxed{\%}$ key works. Put 84 on the display and press $\boxed{\times}\boxed{6}\boxed{5}\boxed{\%}$ (do not press $\boxed{=}$). Which number is on the display?

S: 54.6.

T: So 65% of 84 = 54.6. How would we use the calculator and the $\boxed{\%}$ key to calculate "13% of 120"?

S: Put 120 on the display and press $\boxed{\times}\ \boxed{1}\ \boxed{3}\ \boxed{\%}$. 13% of 120 = 15.6.

Encourage the class to check this result by doing the calculation "(13 × 120) ÷ 100" on the calculator and comparing the results. Be sure students understand how to do percent calculations with their calculators before proceeding.

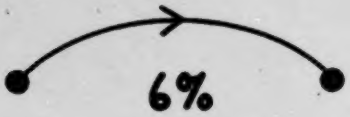
T: Suppose $500 is invested at a 6% yearly interest rate. How much interest is due at the end of one year?

Let the students have a few minutes to consider the situation.

S: We must calculate "6% of 500".

T: Yes, let's describe this situation in an arrow picture.

Draw an arrow and label it "6%".



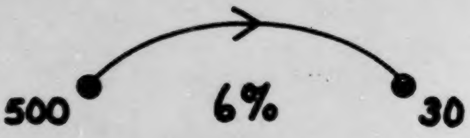
T: Which dot is for 500?

S: The starting number is 500.

Label the dot "500" and ask the students to do the calculation.

S: Put 500 on the display and press $\boxed{\times}\ \boxed{6}\ \boxed{\%}$. 6% of 500 = 30. So $30 in interest is due.

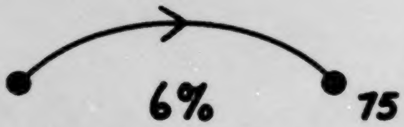
Label the ending dot "30".



Erase the dot labels and pose the next problem.

T: How much money should be invested at 6% yearly interest rate to get $75 of interest in one year? How can we describe this situation in an arrow picture?

S: The ending number is 75 because the interest for one year is $75.

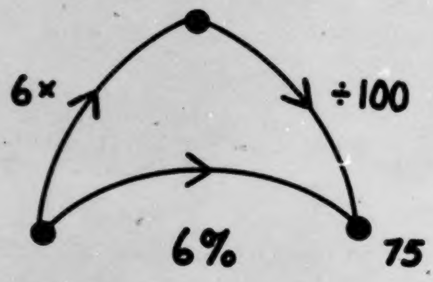


S: We must find the starting number.

Sometimes without thinking a student will calculate "6% of 75" to get the starting number. In this case, point out that it would not make sense to get more interest ($75) than was invested ($4.50).

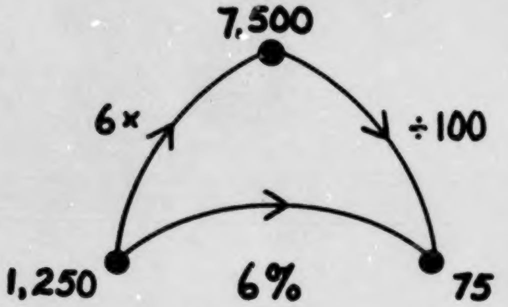
S: We could use a detour; 6% is 6 × followed by ÷100.

If a detour is not suggested by someone in class, do so yourself and add the arrows to your picture.



N 25

Let the students have a few minutes to solve this problem using hand-calculators. Encourage the students to copy the arrow picture and to label the dots. Invite a student to give an explanation of the problem at the board.



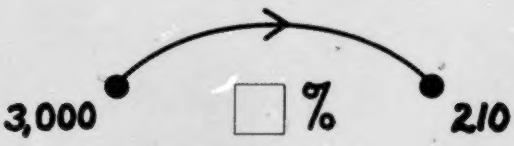
S (tracing "against" the appropriate arrows): 75 × 100 = 7,500 and 7,500 ÷ 6 = 1,250. \$1,250 must be invested at 6% to get \$75 interest.

NOTE: Some students may observe that they can do this calculation on the calculator by putting 75 on the display and pressing $\boxed{\div}$ $\boxed{6}$ $\boxed{\%}$.

T: Very good. Suppose \$3,000 is invested and \$210 is paid for one year's interest. What is the yearly interest rate? How can we picture this situation?

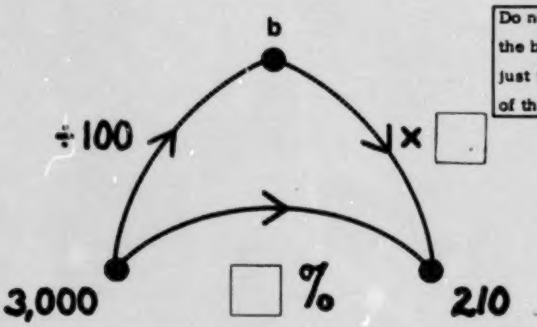
S: The red arrow is for the interest rate. It starts at 3,000 and ends at 210.

Label the arrow picture appropriately.



T: How can we find the number that goes in the box?

Some students may use estimation or trial and error. Others may suggest taking a detour. For example,



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (pointing to b): Which number is here?

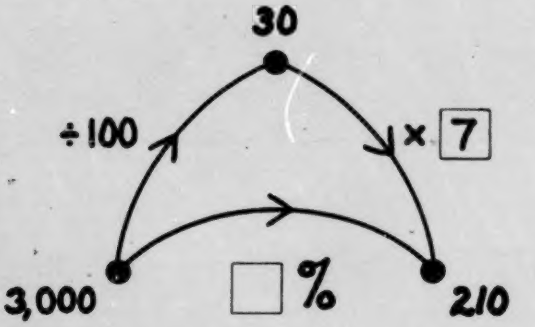
S: 30, because 3,000 ÷ 100 = 30.

Label b "30".

T (tracing the remaining blue arrow): This arrow is for times which number?

S: 7, because 30 × 7 = 210.

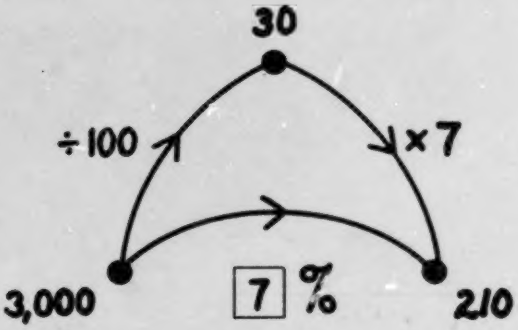
Label the arrow "×7".



T: What is the red arrow for?

S: 7%.

Label the red arrow and check that 7% of 3,000 = 210.



Exercise 3

NOTE: This is an optional exercise that could be done on another day if you are short on time, or it might serve as a challenging activity for stronger students.

Problem A:

Invest $5,000 at a yearly interest rate of 8%. At the end of each year, collect just the interest from the account and leave $5,000 in the account. How long before you have collected $5,000 in interest (or doubled your money)?

Problem B:

Invest $5,000 at a yearly interest rate of 8%. Let the interest accumulate in the account so that the next year interest is figured on a new balance. How many years is it before the initial $5,000 has doubled to $10,000? Is it the same result as in Problem A?

Let the students explore these situations with their hand-calculators.

S (referring to Problem A): 8% of 5,000 = 400, so $400 is the interest collected the first year.

S: $400 is collected each year.

S: Altogether, $800 is collected by the end of the second year; $1,200 by the end of the third year; $1,600 by the end of the fourth year; and so on.

S: It will take 12.5 years because 12.5 × 400 = 5,000.

T: Yes, it takes 12.5 years to double the $5,000. Think about Problem B. How many years do you think it will take to double the $5,000?

Record the estimates on the board for future reference.

T: At the end of the first year, how much money has accumulated?

S: $5,400; the same as the first problem. $5,000 to start and $400 interest.

Record this information in a table.

Year	Starting Balance	Interest at 8% Yearly	Ending Balance
1	$5,000.00	$400.00	$5,400.00

T: Good. What about the second year . . . ?

S: The new starting balance is $5,400 because we leave the interest in the account.

S: 8% of 5,400 = 432. So the interest is $432.

S: The ending balance is $5,832.

Add this data to your table.

Year	Starting Balance	Interest at 8% Yearly	Ending Balance
1	$5,000.00	$400.00	$5,400.00
2	$5,400.00	$432.00	$5,832.00

Continue this discussion as long as you feel necessary. Let the students work independently on this problem while you walk around and give individual attention to those who need it. Check the problem collectively by completing the table. You should conclude that it takes between 9 and 10 years to double your money. A complete table is provided below.

Year	Starting Balance	Interest at 8% Yearly	Ending Balance
1	$5,000.00	$400.00	$5,400.00
2	$5,400.00	$432.00	$5,832.00
3	$5,832.00	$466.56	$6,298.56
4	$6,298.56	$503.88	$6,802.44
5	$6,802.44	$544.20	$7,346.64
6	$7,346.64	$587.73	$7,934.37
7	$7,934.37	$634.75	$8,569.12
8	$8,569.12	$685.53	$9,254.65
9	$9,254.65	$740.37	$9,995.02
10	$9,995.02	$799.60	$10,794.62

7.4 Percent #4

CAPSULE LESSON SUMMARY

Review the notion of a percent function by examining many names for $12\frac{1}{2}\%$. Write fraction names for percent functions. Solve problems related to clothing store discounts.

MATERIALS

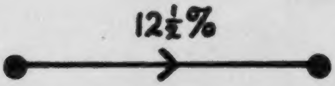
Teacher: Hand-calculator; colored chalk

Student : Hand-calculator; paper; pencil; Worksheets N40, N41, N42, and N43

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow on the board.



T: How can we calculate $12\frac{1}{2}\%$ of some number?

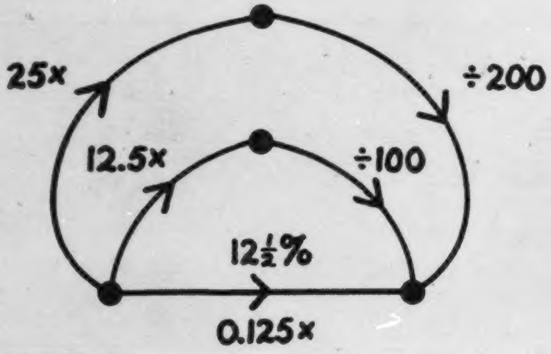
S: Multiply by 12.5 and then divide by 100.

S: Multiply by 25 and then divide by 200.

S: Multiply by 0.125.

N 31

Add the appropriate arrows as ways to calculate $12\frac{1}{2}\%$ are suggested.



T: What are some other names for $12\frac{1}{2}\%$?

S: "$\frac{12.5}{100}x$".

S: "$\frac{1}{8}x$".

S: "$\frac{6\frac{1}{4}}{50}x$".

There are many other correct responses that might be given. Record the suggestions of the class on the board near the arrow picture.

$$\begin{aligned} 12\tfrac{1}{2}\% &= \frac{12.5}{100}x \\ &= 0.125x \\ &= \frac{1}{8}x \\ &= \frac{6\frac{1}{4}}{50}x \\ &= \div 8 \end{aligned}$$

T: Calculate "$12\frac{1}{2}\%$ of 72". Use an arrow picture to help you.

Give the class a few minutes to solve this problem independently before asking a student to explain the calculation.

S: $\frac{1}{8} \times 72 = 9$, so $12\frac{1}{2}\%$ of $72 = 9$.

S: $0.125 \times 72 = 9$.

Exercise 2

Distribute Worksheet N 40 and allow several minutes for individual work. Observe how the students are doing and offer assistance to those that need it. Check the worksheet collectively after most of the students have finished. Answers to this worksheet can be found on page 256.

Exercise 3

Distribute Worksheet N41 and provide each student with a hand-calculator. Give the class a few minutes to read over the worksheet. If you prefer, students could read the worksheet aloud.

T: Suppose the sweater is purchased at store A, how can we calculate the sale price?

Let the students have a minute to consider your question.

S: We must calculate "25% of $34.50".

Of course this is only a first step in calculating the sale price. Let the students use their hand-calculators to calculate "25% of 34.50".

S: 25% of 34.50 is 8.625. So 25% of $34.50 is about $8.63.

T: Is $8.63 the sale price of the sweater?

S: No. $8.63 is the amount that must be subtracted from $34.50 to get the sale price.

S: 34.50 - 8.63 = 25.87.

Record this information on the board.

Item	Discounted Price at store
Sweater	A: $ 25.87 B: C:

Lead a similar discussion to calculate the discounted price of the sweater at stores B and C.

Item	Discounted Price at store
Sweater	A: $ 25.87 B: $ 23.00 C: $ 24.15

T: Which store has the best discounted price for the sweater?

S: Store B.

Instruct students to record this information in the appropriate box on the worksheet.

T: If the cost of the sweater is $23.00, how much sales tax is paid on the purchase?

S: $5\frac{1}{4}$% of 23 is 1.2075. So the sales tax is $1.21.

Add the sales tax information to your table and instruct students to do the same on their worksheets.

Item	Discounted Price at store	5¼% Sales tax
Sweater	A: $ 25.87 B: $ 23.00 C: $ 24.15	$ 1.21

T: What is the total cost of the purchase?

S: $23.00 + 1.21 = 24.21. The total cost is $24.21.

Add the information to your table while the students do so on their worksheets.

Item	Discounted Price at store	5¼% Sales tax	Best Total cost
Sweater	A: $ 25.87 B: $ 23.00 C: $ 24.15	$ 1.21	$24.21

Let the students work independently on the remainder of this worksheet. Walk around and observe their progress. Feel free to stop the class and discuss any of the items that cause the class difficulty.

Worksheets N42 and N43 are available for the faster students or for homework.

Name ____________________ **N 40** *

Write a fraction in each box to get a true number story.

$25\% = \boxed{\frac{1}{4}} \times$ $40\% = \boxed{\frac{2}{5}} \times$
$12\frac{1}{2}\% = \boxed{\frac{1}{8}} \times$ $5\% = \boxed{\frac{1}{20}} \times$
$6\frac{1}{4}\% = \boxed{\frac{1}{16}} \times$ $2\frac{1}{2}\% = \boxed{\frac{1}{40}} \times$
$50\% = \boxed{\frac{1}{2}} \times$ $31\frac{1}{4}\% = \boxed{\frac{5}{16}} \times$
$10\% = \boxed{\frac{1}{10}} \times$ $30\% = \boxed{\frac{3}{10}} \times$
$20\% = \boxed{\frac{1}{5}} \times$ $150\% = \boxed{\frac{3}{2}} \times$

Complete.

$\boxed{60}\,\% = \frac{3}{5} \times$ $\boxed{35}\,\% = \frac{7}{20} \times$
$\boxed{80}\,\% = \frac{4}{5} \times$ $\boxed{75}\,\% = \frac{3}{4} \times$
$\boxed{70}\,\% = \frac{7}{10} \times$ $\boxed{125}\,\% = \frac{5}{4} \times$
$\boxed{62\frac{1}{2}}\,\% = \frac{5}{8} \times$ $\boxed{150}\,\% = \frac{3}{2} \times$
$\boxed{15}\,\% = \frac{3}{20} \times$ $\boxed{300}\,\% = 3 \times$

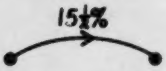
Name ____________________ **N 41** **

A	B	C
Spring Sale! 25% off all winter wear Additional 10% off on items originally priced $50 or more	Spring Sale! $\frac{1}{3}$ off all winter clothing Make room for Spring!	Spring Sale! 30% off every winter clothing item in stock

Three nearby stores are having winter clearance sales. Some items are available at all three stores; some only at two of the three stores. Fill in this table to show where to pay the least amount for individual items.

Item	Original Price	Available At	Best Discounted Price (at which store)	$5\frac{1}{4}$% Sales Tax	Best Total Cost
Sweater	$34.50	A, B, & C	$23 (B)	$1.21	$24.21
Boots	$19.75	A & C	$13.83 (C)	$0.73	$14.56
Coat	$78	A, B, & C	$50.70 (A)	$2.66	$53.36
Jacket	$62	A & B	$40.30 (A)	$2.12	$42.42
Suit	$210.90	B & C	$140.60 (B)	$7.38	$147.98

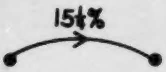
Name ____________________ **N 42** ***



Tom has $36.50 and wants to buy an anniversary present for his parents. He considers a silver serving tray costing $32. However, Tom must remember that there is a $15\frac{1}{2}$% combined luxury and sales tax.

How much tax is there on the $32 silver tray? $4.96

Does Tom have enough money to buy the silver tray? No
If yes, how much change would he receive? ______
If no, how much more money does he need? $0.46



Tom also considers a large Gardenia plant costing $34. On the plant there is only a $5\frac{1}{4}$% sales tax.

How much tax is there on the Gardenia plant? $1.79

Does Tom have enough money to buy the Gardenia plant? Yes
If yes, how much change would he receive? $0.71
If no, how much more money does he need? ______

Name ____________________ **N 43** ****

1. The Midland Savings & Loan company pays $6\frac{1}{3}$% yearly interest on regular savings. How much interest would be earned in one year by a savings deposit of $1,125? $71.25

2. The Midland Savings & Loan company charges $1\frac{3}{8}$% monthly interest on loans. How much interest would a customer pay monthly on a loan of $1,000? $13.75
What equivalent yearly rate of interest is the customer paying? $16\frac{1}{2}$%

TIRE SALE!
All prices
reduced 15%

3. Joyce saved $12 by buying two tires at the tire sale. What was the original price of the two tires? $80
How much did Joyce pay for the tires? $68

4. You can buy a set of luggage for $160 cash. If you buy it on the installment plan, you must make 12 monthly payments of $15.25 each. What is the total cost to buy the luggage on the installment plan? $183
What percent over the cash price is the installment plan price? $14\frac{3}{8}$%

8

DECIMALS AND DIVISION

The lessons in this chapter reinforce and extend the experiences of Part V of the CSMP Intermediate Grades curriculum relative to situations involving decimals and the standard division algorithm. Some of the activities include decimals on the Minicomputer and in the Minicomputer Golf game, in arrow roads with restrictions on the arrows permitted to appear, and in estimation techniques where a rule for placing the decimal point in a product is explored. Problems related to sharing an amount of money equally among some specified number of people present the standard division algorithm with decimals. A multiplication technique is employed to transform a division problem into an easier problem or perhaps one that uses the results of other related calculations.

CAPSULE LESSON SUMMARIES

Section 8.1:	Write decimal names for fractions and vice versa. Using the hand-calculator, discover a rule for placing the decimal point in the product of two decimal numbers. (Worksheets N44, N45)
Section 8.2:	Use multiplication to transform a division problem such as $3.5\overline{)463.75}$ to an easier problem such as $35\overline{)4637.5}$ or $7\overline{)927.5}$.
Section 8.3:	Put some numbers on the Minicomputer using a ⑦-checker, and then using only a ③-checker and a ⑨-checker. Label arrows in an arrow picture where the arrows are for +, −, ×, or ÷ some decimal number. Build arrow roads between numbers where the arrows must be for +, −, ×, or ÷ some decimal number (that can be displayed on a hand-calculator) between 0 and 1.

Section 8.4: Perform some division calculations using the results of other related calculations. Use multiplication to transform a division problem such as

$$7.5\overline{)271.8}$$

to an easier problem such as

$$15\overline{)543.6} \qquad \text{or} \qquad 75\overline{)2718}.$$

Section 8.5: Put some numbers on the Minicomputer using a ③-checker, a ④-checker, and a ⑤-checker. Put numbers on the Minicomputer using exactly two of the checkers ②, ③, . . . , ⑨. Moving exactly one checker, change a number on the Minicomputer by a specified amount. Play Minicomputer Golf.

8.1 Decimals #1

CAPSULE LESSON SUMMARY

Write decimal names for fractions and vice versa. Using the hand-calculator, discover a rule for placing the decimal point in the product of two decimal numbers.

MATERIALS

Teacher: Hand-calculator

Student: Hand-calculator; paper; pencil; Worksheets N44 and N45.

DESCRIPTION OF LESSON

Exercise 1

Write this information on the board.

$$\frac{6}{10} = \boxed{}$$

T: Let's express $\frac{6}{10}$ in decimal writing.

Invite a student to fill in the box.

$$\frac{6}{10} = \boxed{0.6}$$

Continue this activity with other numbers. Alternate between giving the decimal name and the fractional name and then asking for the other name. Answers for the problems at the top of the next page are in the boxes.

$$\frac{6}{10} = \boxed{0.6} \qquad \boxed{\frac{31}{100}} = 0.31$$

$$\frac{32}{100} = \boxed{0.32} \qquad \frac{75}{10} = \boxed{7.5}$$

$$\boxed{\frac{9}{100}} = 0.09 \qquad \boxed{\frac{108}{1{,}000}} = 0.108$$

$$\boxed{\frac{17}{10}} = 1.7 \qquad \frac{105}{10} = \boxed{10.5}$$

$$\frac{205}{100} = \boxed{2.05}$$

Although there are many fractional names for decimals, encourage your students to use fractions with denominators 10, 100, 1000, etc.

T: Do you notice anything interesting about these problems?

Encourage students to observe a relationship between the decimal name and the number of zeros in the denominator of the corresponding fraction.

Exercise 2

Write these problems on the board and ask the students to complete the calculations.

$$62 \times 0.2 = \boxed{}$$

$$0.8 \times 0.6 = \boxed{}$$

$$3.2 \times 0.8 = \boxed{}$$

$$\begin{array}{r} 2.06 \\ \times\ \ 6.4 \\ \hline \end{array} \qquad \begin{array}{r} 0.34 \\ \times 0.02 \\ \hline \end{array}$$

As students finish, distribute hand-calculators and ask them to check their work using the hand-calculator. Many students will probably comment that their work was correct except for the decimal point. Invite students to write the correct solutions to these problems on the board.

$$62 \times 0.2 = \boxed{12.4}$$

$$0.8 \times 0.6 = \boxed{0.48}$$

$$3.2 \times 0.8 = \boxed{2.56}$$

$$\begin{array}{r} 2.06 \\ \times \quad 6.4 \\ \hline 13.184 \end{array} \qquad \begin{array}{r} 0.34 \\ \times 0.02 \\ \hline 0.0068 \end{array}$$

T: Look at the calculations on the board. Do these calculations suggest any rules we could use to correctly place decimal points in products?

Let the class consider this situation for several minutes.

S: Look at 2.06 × 6.4. Count the digits to the right of the decimal points in the numbers being multiplied. Together there are three digits to the right of the decimal points in these numbers. Also, there are three digits to the right of the decimal point in the answer (13.184).

T: Very good. Does this pattern work for each of the other problems?

When the class finds a workable rule like the one described above, verify that it does work for each example by counting the digits to the right of the respective decimal points. Ask the students to turn off their hand-calculators.

N11

Write this calculation on the board.

$$3.72 \times 1.6 = \boxed{}$$

T: Can you predict how many digits will be to the right of the decimal point in the answer?

S: Three; 2 + 1 = 3.

T: Good. About how much is the answer to this problem?

S: More than 3 because 3 x 1 = 3 and less than 8 because 4 x 2 = 8.

S: 3.72 is almost 4 and 1.6 is close to $1\frac{1}{2}$ and $4 \times 1\frac{1}{2} = 6$, so the answer will be about 6.

Write the estimates on the board and then ask the students to solve the problem using their hand-calculators.

$$3.72 \times 1.6 = \boxed{5.952}$$

Solve a few more problems such as those suggested below. For each one, ask students to predict the number of digits to the right of the decimal point in the answer and to estimate the answer.

$$1.28 \times 3.1 = \boxed{3.968}$$

$$0.03 \times 0.2 = \boxed{0.006}$$

$$0.026 \times 0.003 = \boxed{0.000078}$$

$$\begin{array}{r} 3.14 \\ \times\ 3.14 \\ \hline \boxed{9.8596} \end{array} \qquad \begin{array}{r} 2.07 \\ \times\ 1.2 \\ \hline \boxed{2.484} \end{array}$$

$$\begin{array}{r} 44 \\ \times\ 0.2 \\ \hline \boxed{8.8} \end{array}$$

Write this calculation on the board.

$$1.7 \times 0.6 = \boxed{}$$

T: Let's calculate "1.7 x 0.6" using fractions. Give a fraction for 1.7.

S: "$\frac{17}{10}$".

T: . . . a fraction for 0.6.

S: "$\frac{6}{10}$".

Write this information on the board.

$$1.7 \times 0.6 = \frac{17}{10} \times \frac{6}{10}$$

T: What number is $\frac{17}{10} \times \frac{6}{10}$?

S: $\frac{102}{100}$.

$$1.7 \times 0.6 = \frac{17}{10} \times \frac{6}{10} = \frac{102}{100}$$

T: Can you give a decimal name for $\frac{102}{100}$?

S: "1.02".

$$1.7 \times 0.6 = \frac{17}{10} \times \frac{6}{10} = \frac{102}{100} = 1.02$$

N 11

T: Look at "$\frac{17}{10} \times \frac{6}{10}$". Which calculations did we do?

S: "17 x 6" and "10 x 10".

T: Yes. What number is 17 x 6?

S: 102.

Lead the students to see that multiplying decimal numbers is exactly like multiplying whole numbers except for the problem of correctly placing the decimal point in the answer.

Exercise 3

Write these problems on the board.

$$2.44 \times 0.35 = \boxed{\quad\quad\quad}$$

$$126 \times 0.45 = \boxed{\quad\quad\quad}$$

$$3.60 \times 0.40 = \boxed{\quad\quad\quad}$$

$$3.7 \times 0.30 = \boxed{\quad\quad\quad}$$

$$100 \times 5.7 = \boxed{\quad\quad\quad}$$

T: Let's predict the number of digits to the right of the decimal points in some of these products. How many are there in this first product?

S: There should be four digits to the right of the decimal point in the product.

Record this information on the board next to the problem. Do this for the first three problems.

④ $2.44 \times 0.35 = \boxed{\quad\quad}$

② $126 \times 0.45 = \boxed{\quad\quad}$

④ $3.60 \times 0.40 = \boxed{\quad\quad}$

$3.7 \times 0.30 = \boxed{\quad\quad}$

$100 \times 5.7 = \boxed{\quad\quad}$

T: Use your hand-calculators to do the first three calculations.

Record the products on the board.

④ $2.44 \times 0.35 = \boxed{0.854}$

② $126 \times 0.45 = \boxed{56.7}$

④ $3.60 \times 0.40 = \boxed{1.44}$

$3.7 \times 0.30 = \boxed{\quad\quad}$

$100 \times 5.7 = \boxed{\quad\quad}$

T: What do you notice? Does our rule about counting digits to the right of the decimal points still work?

S: If we do these calculations ignoring the decimal points (i.e. with whole numbers), each of the products ends in at least one zero. The calculator drops the ending zeros when they are to the right of the decimal point.

S: 0.854 = 0.8540.

S: 1.44 = 1.4400

T: What do you think happens in the last two problems?

Let students comment on the expected results and then find the products in the two remaining problems.

$$3.7 \times 0.30 = \boxed{1.11}$$
$$100 \times 5.7 = \boxed{570}$$

Worksheets N44 and N45 are available for independent work.

Name ____________________ **N 44** *

Fill in the boxes.

$\frac{5}{10} = \boxed{0.5}$ $0.8 = \boxed{\frac{8}{10}}$

$1.6 = \boxed{\frac{16}{10}}$ $\frac{26}{100} = \boxed{0.26}$

$\frac{235}{100} = \boxed{235}$

Calculate.

$$1.6 \times 0.8 = \frac{16}{10} \times \frac{8}{10} = \frac{128}{100} = 1.28$$

$$2.35 \times 0.5 = \frac{235}{100} \times \frac{5}{10} = \frac{1175}{1000} = 1.175$$

$$0.26 \times 0.8 = \frac{26}{100} \times \frac{8}{10} = \frac{208}{1000} = 0.208$$

Name ____________________ **N 45** **

All of these calculations are correct, except that a decimal point is missing in the answer.

Place a decimal point in each answer to get a true number story.

$1.26 \times 0.3 = .378$

$0.09 \times 15.64 = 1.4076$

$16.5 \times 33 = 544.5$

$5.78 \times 0.692 = 3.99976$

$1.11 \times 1.1 = 1.221$

$1{,}780 \times 4.2 = 7476.$

$$\begin{array}{r} 5.2 \\ \times\ 4.6 \\ \hline 23.92 \end{array}$$

$$\begin{array}{r} 3.68 \\ \times\ 0.104 \\ \hline 0.38272 \end{array}$$

$$\begin{array}{r} 2.485 \\ \times\ 0.38 \\ \hline 0.9443 \end{array}$$

$$\begin{array}{r} 25.6389 \\ \times\ 0.012 \\ \hline 0.3076668 \end{array}$$

8.2 Division with Decimals #1

CAPSULE LESSON SUMMARY

Use multiplication to transform a division problem such as

$$3.5\overline{)463.75}$$

to an easier problem such as

$$35\overline{)4637.5} \quad \text{or} \quad 7\overline{)927.5}.$$

MATERIALS

Teacher: None

Student: Paper; pencil

DESCRIPTION OF LESSON

Begin the lesson with a brief mental arithmetic activity involving division problems such as the ones suggested here. Feel free to adjust the problems to suit the numerical abilities of your students.

15 ÷ 5 (3)
100 ÷ 5 (20)
115 ÷ 5 (23)
1,150 ÷ 5 (230)

60 ÷ 4 (15)
600 ÷ 4 (150)
660 ÷ 4 (165)
6,600 ÷ 4 (1,650)

0.48 ÷ 3 (0.16)
12 ÷ 3 (4)
12.48 ÷ 3 (4.16)
124.8 ÷ 30 (4.16)
1248 ÷ 300 (4.16)

848 ÷ 4 (212)
1,216 ÷ 4 (304)

963 ÷ 3 (321)
6.96 ÷ 3 (2.32)

627 ÷ 3 (209)
15.75 ÷ 5 (3.15)

Exercise 1

Write this number story on the board.

$$\boxed{256 \div 4 = 64}$$

T: Use this information to help find the answers to these problems.

Write these problems on the board, and ask the students to solve them.

$$512 \div 8 = \boxed{}$$
$$1{,}024 \div 16 = \boxed{}$$
$$128 \div 2 = \boxed{}$$

The students should comment that each problem has the same answer — 64. Add two more problems to your list.

$$512 \div 8 = \boxed{64}$$
$$1{,}024 \div 16 = \boxed{64}$$
$$128 \div 2 = \boxed{64}$$
$$1{,}280 \div 20 = \boxed{}$$
$$768 \div 12 = \boxed{}$$

T: Do you think the answers to these two problems will also be 64? Why?

S: Yes, because when you multiply both numbers in a division problem by the same number, the answer doesn't change.

T: Does that always work? Let's check with some simple division problems.
What number is 24 ÷ 4?

S: 6.

T: Use the "multiplying" method to give me another problem with 6 as the answer.

S: "48 ÷ 8 = ?" I multiplied both numbers by 2.

S: "72 ÷ 12 = ?" I multiplied both numbers by 3.

S: "120 ÷ 20 = ?" I multiplied both numbers by 5.

S: "12 ÷ 2 = ?" I multiplied both numbers by $\frac{1}{2}$.

Refer back to the original statement on the board, "256 ÷ 4 = 64", and the list of related facts. Add these problems to the list, give the students a couple minutes to study them, and then solve them collectively. Answers are in the boxes.

$$25{,}600 \div 400 = \boxed{64}$$

$$25.6 \div 0.4 = \boxed{64}$$

$$2.56 \div 0.04 = \boxed{64}$$

T: As long as both numbers in a division problem are multiplied by the same number, the answer is the same.

Let's look at some other problems related to this fact (point to "256 ÷ 4 = 64").

Write these problems on the board and ask students to solve them. Answers are in the boxes.

$$256 \div 8 = \boxed{32} \qquad 25.6 \div 8 = \boxed{3.2}$$

$$256 \div 16 = \boxed{16} \qquad 25.6 \div 4 = \boxed{6.4}$$

Let the class comment about this set of problems.

Exercise 2

Write this problem on the board.

$$3.5\overline{)463.75}$$

T: Can you think of another problem with the same answer, but that might be easier to solve?

S: Multiply each number by 10.

S: Or multiply each number by 2.

Write the problem both ways on the board. If both ways are not suggested by the students, suggest them yourself.

$$\begin{array}{ccc} & \text{(multiply by 10)} & \text{(multiply by 2)} \\ 3.5\overline{)463.75} & 35\overline{)4637.5} & 7\overline{)927.5} \end{array}$$

T: Let's solve this problem (point to "$35\overline{)4637.5}$"). How could we begin?

S: 100 × 35 = 3,500 and 200 × 35 = 7,000. So start with 100.

Write this information on the board.

$$\begin{array}{r|l} 35\overline{)4637.5} & \\ -3500.0 & 100 \\ \hline 1137.5 & \end{array}$$

N 20

Complete this problem and then solve the problem "$7\overline{)927.5}$" collectively. A solution for each problem is shown below.

$$\begin{array}{r|r}
132.5 & \\
35\overline{)4637.5} & \\
-3500.0 & 100 \\ \hline
1137.5 & \\
-1050.0 & 30 \\ \hline
87.5 & \\
-70.0 & 2 \\ \hline
17.5 & \\
-17.5 & 0.5 \\ \hline
0 & 132.5
\end{array}
\qquad\qquad
\begin{array}{r|r}
132.5 & \\
7\overline{)927.5} & \\
-700.0 & 100 \\ \hline
227.5 & \\
-140.0 & 20 \\ \hline
87.5 & \\
-70.0 & 10 \\ \hline
17.5 & \\
-14.0 & 2 \\ \hline
3.5 & \\
-3.5 & 0.5 \\ \hline
0 & 132.5
\end{array}$$

T: What number is 463.75 ÷ 3.5?

S: 132.5.

$$\begin{array}{r}
132.5 \\
3.5\overline{)463.75}
\end{array}$$

T (pointing to "3.5"): Whenever this number (divisor) is a (non-integer) decimal, it is usually easier to find another problem with the same answer using multiplication to get a whole number divisor.

Write a set of problems on the board (see the next page) and ask the students to copy and solve them independently. Give assistance to those students who need help getting started.

For the problems, answers are in boxes and suggested multiplication functions are circled (of course any multiplication function except 0× can be used, but the suggested ones are convenient because they yield whole number divisors).

$$\begin{array}{r} \boxed{64.5} \\ 2.5\overline{)161.25} \end{array} \qquad \begin{array}{c} 10\times \\ 4\times \\ 2\times \end{array} \qquad\qquad \begin{array}{r} \boxed{13.7} \\ 0.6\overline{)8.22} \end{array} \qquad \begin{array}{c} 10\times \\ 5\times \end{array}$$

$$\begin{array}{r} \boxed{28.75} \\ 1.4\overline{)40.25} \end{array} \qquad \begin{array}{c} 10\times \\ 5\times \end{array} \qquad\qquad \begin{array}{r} \boxed{20.9} \\ 3.7\overline{)77.33} \end{array} \qquad 10\times$$

Faster students should try solving these problems.

$$\begin{array}{r} \boxed{231.5} \\ 7.06\overline{)1{,}634.39} \end{array} \qquad 100\times \qquad\qquad \begin{array}{r} \boxed{12.4} \\ 4.15\overline{)51.46} \end{array} \qquad 100\times$$

8.3 Decimals #2

CAPSULE LESSON SUMMARY

Put some numbers on the Minicomputer using a ⑦ -checker, and then using only a ③ -checker and a ⑨ -checker. Label arrows in an arrow picture where the arrows are for +, -, ×, or ÷ some decimal number. Build arrow roads between numbers where the arrows must be for +, -, ×, or ÷ some decimal number (that can be displayed on a hand-calculator) between 0 and 1.

MATERIALS

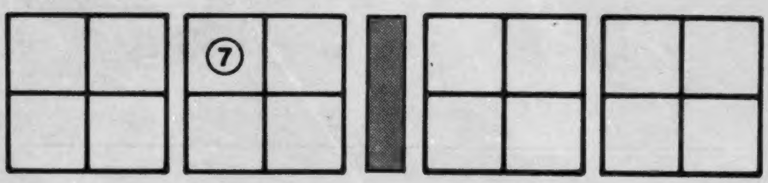
Teacher: Hand-calculator; Minicomputer set; multiplication checkers; colored chalk

Student : Hand-calculator; paper; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Display four Minicomputer boards and a ⑦ -checker.



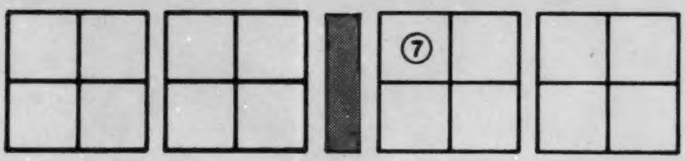
T: Which number is on the Minicomputer?

S: 56, because 7 × 8 = 56.

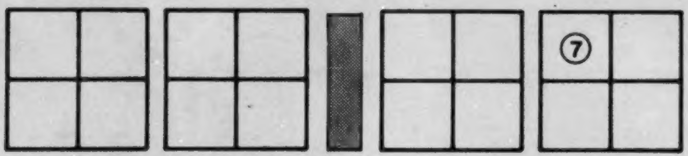
Record the calculation on the board.

$$7 \times 8 = 56$$

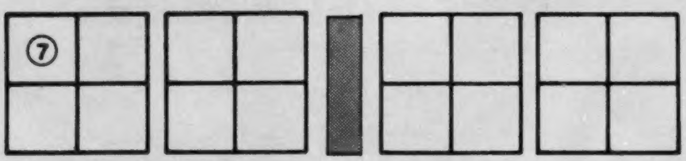
Continue by moving the ⑦ -checker to the squares indicated below. In each case, record the appropriate calculation on the board.



$= 7 \times 0.8 = 5.6$



$= 7 \times 0.08 = 0.56$



$= 7 \times 80 = 560$

Write these multiplication problems next to those already on the board and ask the class to solve them. Answers are in the boxes.

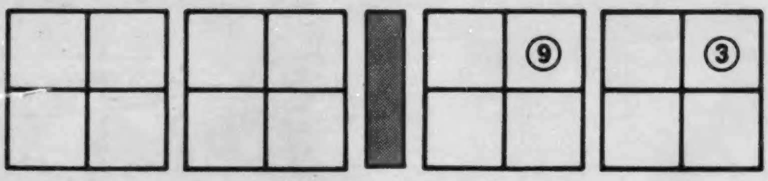
$$0.7 \times 8 = \boxed{5.6}$$

$$0.7 \times 0.8 = \boxed{0.56}$$

$$0.7 \times 0.08 = \boxed{0.056}$$

$$0.7 \times 80 = \boxed{56}$$

Display this configuration on the Minicomputer.



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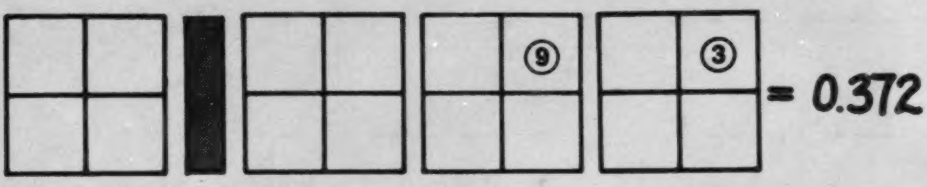
T: Which number is on the Minicomputer?

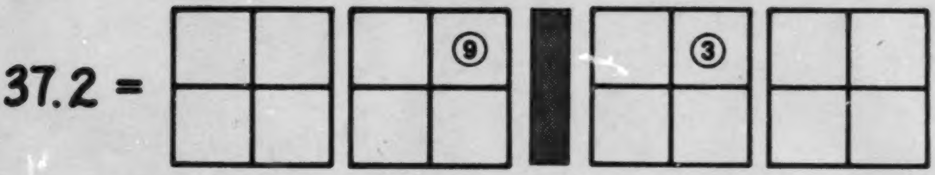
S: 3.72 (9 × 0.4 + 3 × 0.04 = 3.6 + 0.12 = 3.72).

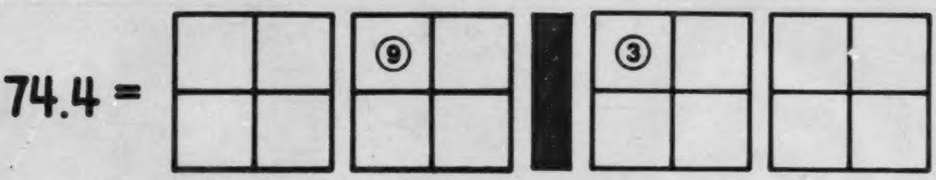
T: Who can put 0.372 on the Minicomputer using only the ⑨ -checker and the ③ -checker?

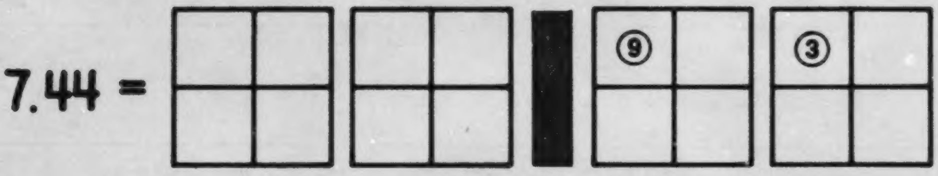
Invite a student to move the checkers (and probably a board) resulting in this configuration.

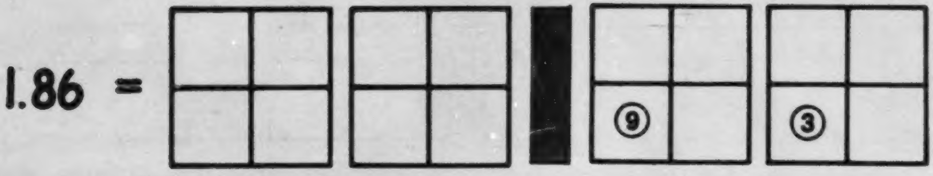


Continue this activity by asking various students to put the following numbers on the Minicomputer using just the ⑨ -checker and the ③ -checker.



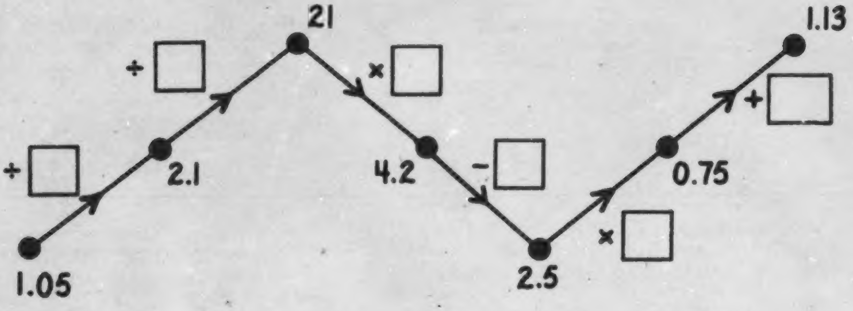






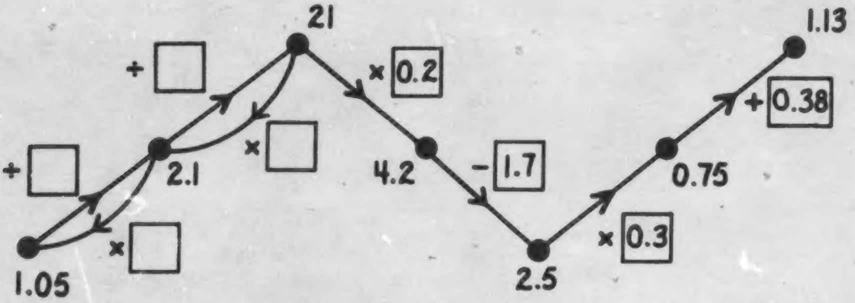
Exercise 2

Draw this arrow picture on the board.


21
1.13
÷
÷
×
−
×
+
1.05
2.1
4.2
0.75
2.5

Ask students to fill in the boxes for the arrows allowing individual students to choose which arrow to label.

Perhaps the class will need a hint for labeling the division arrows. In that case, draw a blue return arrow from each of the division arrows.


21
1.13
÷
× 0.2
+ 0.38
×
÷
2.1
4.2
− 1.7
0.75
×
× 0.3
1.05
2.5

T (tracing the blue arrow starting at 21 and ending at 2.1): What could this arrow be for; times what number?

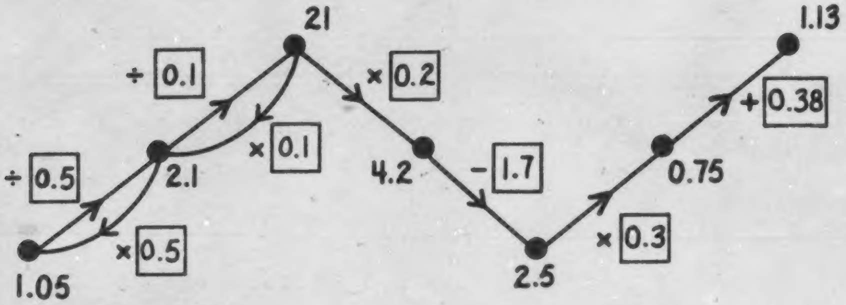
S: ×0.1.

Fill in the box for that blue arrow.

T: Yes. So what could the red arrow from 2.1 to 21 be for?

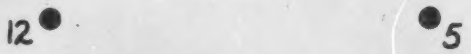
S: +0.1.

In a similar manner, lead the students to see that the other blue arrow is for ×0.5 and its return red arrow is for +0.5.



Exercise 3

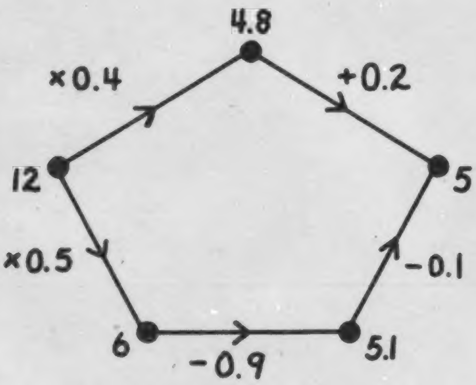
Distribute a hand-calculator to each student and write this information on the board.



T: You must build an arrow road from 12 to 5 with arrows for +, -, ×, or ÷ some decimal number between 0 and 1 that you can display on your hand-calculator.

Allow several minutes for independent work on this problem. Observe the students' work, offering assistance to those that have difficulty getting started.

When most students have finished, ask some volunteers to put their solutions on the board. Two possible solutions are illustrated here.



Continue this activity by asking the students to build other arrow roads. Some starting and ending numbers are suggested below.

*	● 10	● 6	● 18	● 6	● 20	● 15
* *	● 2	● 20	● 8	● 3	● 5	● ⁻5
* * *	● 15	● 25	● 95	● 7	● 1.5	● 9.8

Encourage the faster students to find a shortest road for each problem. Be sure to allow a few minutes at the end of class to put some solutions on the board. One of the many solutions for each problem is shown in the illustration on the next page.

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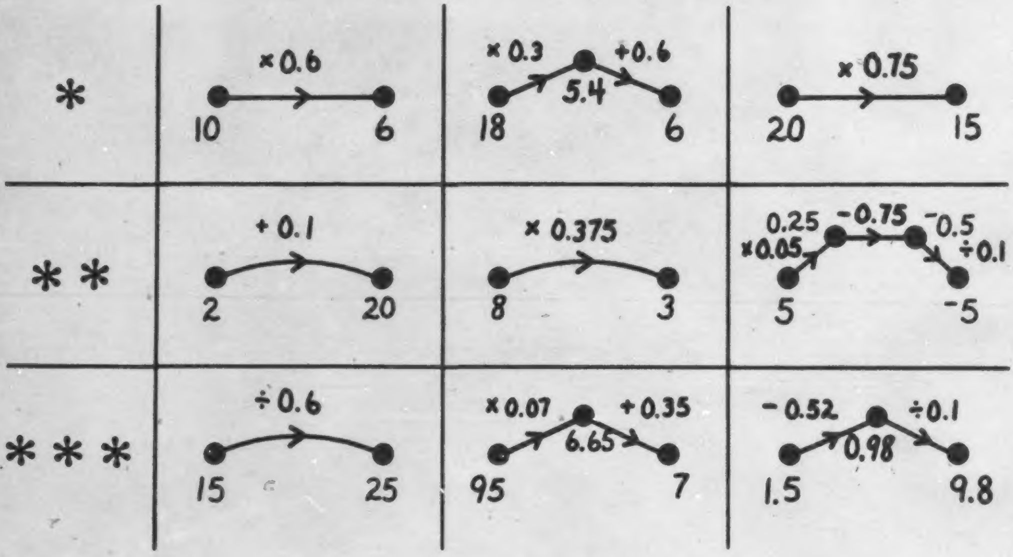
In each case the road is a shortest road.


*
×0.6
10
6
×0.3
+0.6
5.4
18
6
×0.75
20
15
* *
+ 0.1
2
20
× 0.375
8
3
0.25
−0.75
⁻0.5
×0.05
÷0.1
5
⁻5
* * *
÷0.6
15
25
×0.07
+0.35
6.65
95
7
−0.52
÷0.1
0.98
1.5
9.8

8.4 Division with Decimals #2

CAPSULE LESSON SUMMARY

Perform some division calculations using the results of other related calculations. Use multiplication to transform a division problem such as

$$7.5\overline{)271.8}$$

to an easier problem such as

$$15\overline{)543.6} \quad \text{or} \quad 75\overline{)2718}.$$

MATERIALS

Teacher: None
Student : Paper; pencil

DESCRIPTION OF LESSON

Begin the lesson with a brief mental arithmetic activity with division calculations such as those suggested here. Feel free to adjust the problems to suit the numerical abilities of your students.

60 ÷ 1 (60)
60 ÷ 2 (30)
60 ÷ 3 (20)
60 ÷ 4 (15)
60 ÷ 5 (12)

6 ÷ 0.1 (60)
6 ÷ 0.2 (30)
6 ÷ 0.3 (20)
6 ÷ 0.4 (15)
6 ÷ 0.5 (12)

56 ÷ 8 (7)
560 ÷ 8 (70)
112 ÷ 8 (14)
672 ÷ 8 (84)

749 ÷ 7 (107)
848 ÷ 8 (106)
1,236 ÷ 6 (206)
816 ÷ 4 (204)

Exercise 1

Write this number story on the board and call the class's attention to it.

$$\boxed{357 \div 7 = 51}$$

T: Use this information to help find the answers to the problems I write on the board.

Write these problems on the board and instruct the students to solve them. Answers are in the boxes.

$$3570 \div \boxed{70} = 51$$

$$\boxed{3570} \div 7 = 510$$

$$714 \div 14 = \boxed{51}$$

$$1428 \div \boxed{14} = 102$$

$$35.7 \div \boxed{0.7} = 51$$

$$\boxed{3.57} \div 0.07 = 51$$

Check the problems collectively asking various students to explain their solutions to a particular problem.

Exercise 2

Write this problem on the board.

$$7.5\overline{)271.8}$$

T: Can you think of another division problem with the same answer but that might be easier to solve?

S: Multiply each number by 2 and get a problem where you divide by 15.

S: Multiply each number by 10 and get a problem where you divide by 75.

T: Let's use the doubling method first. What number is 2 × 7.5?

S: 15.

T: . . . and 2 × 271.8?

S: 543.6.

Write the resulting problem next to the original problem.

$$7.5\overline{)271.8} \qquad\qquad 15\overline{)543.6}$$

T: What is a shortcut for multiplying by 10?

S: Move the decimal point one place to the right.

T: Yes. What number is 10 × 7.5?

S: 75.

T: . . . and 10 × 271.8?

S: 2718.

T: Rather than copy the problem again, I will indicate multiplying each number by 10 this way.

$$7\underset{\curvearrowright}{.5}\overline{)271\underset{\curvearrowright}{.8}} \qquad\qquad 15\overline{)543.6}$$

Instruct the students to solve one of the problems (of their choice) independently or, if you prefer, solve both problems collectively. Ask volunteers to explain how they are solving the problems. The students should find that indeed the problems do have the same answer and that 271.8 ÷ 7.5 = 36.24.

$$\begin{array}{r|l}
\multicolumn{1}{r}{36.24} & \\
7.\underset{\curvearrowright}{5}\,\overline{)\,271.\underset{\curvearrowright}{8}} & \\
-2250 & 30 \\ \hline
468 & \\
-450 & 6 \\ \hline
18 & \\
-15 & 0.2 \\ \hline
3 & \\
-3 & 0.04 \\ \hline
0 & 36.24
\end{array}
\qquad\qquad
\begin{array}{r|l}
\multicolumn{1}{r}{36.24} & \\
15\,\overline{)\,543.6} & \\
-450.0 & 30 \\ \hline
93.6 & \\
-90.6 & 6 \\ \hline
3.6 & \\
-3.0 & 0.2 \\ \hline
0.6 & \\
-0.6 & 0.04 \\ \hline
0 & 36.24
\end{array}$$

Write these problems on the board and ask the students to copy and solve them independently. Give help to those students who have difficulty getting started. Answers are in boxes. Use of any of the circled multiplication functions will yield a problem with a whole number divisor.

$$\boxed{*} \qquad\qquad \boxed{**} \qquad\qquad \boxed{***}$$

$$\begin{array}{r}\boxed{23.6}\\ 0.8\,\overline{)\,18.88}\end{array}\ \begin{matrix}5\times\\10\times\end{matrix}
\qquad
\begin{array}{r}\boxed{16.8}\\ 1.25\,\overline{)\,21}\end{array}\ \begin{matrix}4\times\\100\times\end{matrix}
\qquad
\begin{array}{r}\boxed{109.09}\\ 3.14\,\overline{)\,342.5426}\end{array}\ \begin{matrix}100\times\end{matrix}$$

$$\begin{array}{r}\boxed{60.2}\\ 3.5\,\overline{)\,210.7}\end{array}\ \begin{matrix}2\times\\10\times\\4\times\end{matrix}
\qquad
\begin{array}{r}\boxed{1.2}\\ 2.5\,\overline{)\,3}\end{array}\ \begin{matrix}2\times\\4\times\\10\times\end{matrix}
\qquad
\begin{array}{r}\boxed{26.\overset{*}{6}}\\ 0.3\,\overline{)\,8}\end{array}\ \begin{matrix}10\times\end{matrix}$$

8.5 Minicomputer Golf with Decimals

CAPSULE LESSON SUMMARY

Put some numbers on the Minicomputer using a ③ -checker, a ④ -checker, and a ⑤ -checker. Put numbers on the Minicomputer using exactly two of the checkers ② , ③ , . . . , ⑨ . Moving exactly one checker, change a number on the Minicomputer by a specified amount. Play Minicomputer Golf.

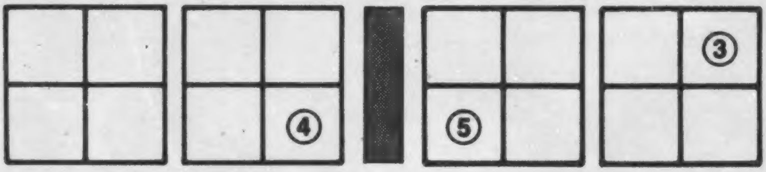
MATERIALS

Teacher:	Minicomputer set; multiplication checkers; colored chalk
Student:	Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

Display four Minicomputer boards and this configuration of checkers.



T: Which number is on the Minicomputer?

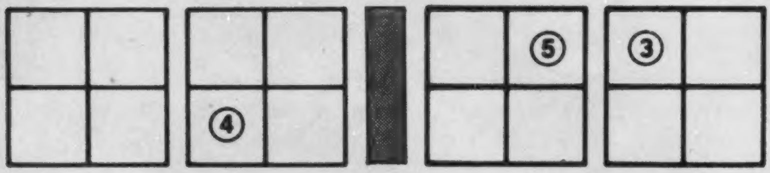
S: 5.12.

T: Who can put 10.24 on the Minicomputer using just a ③ -checker, a ④ -checker, and a ⑤ -checker?

S: Move each checker to the square of next higher value; that doubles the number.

N 36

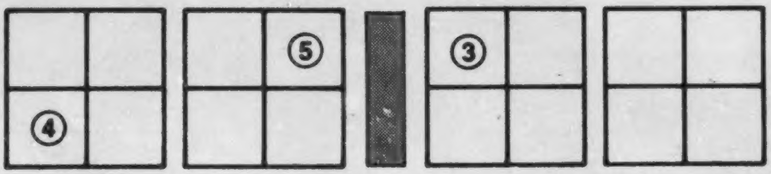
Invite a student to move the checkers to this configuration.



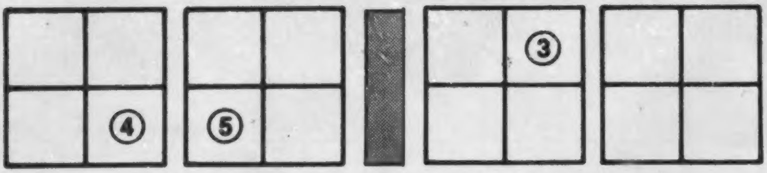
= 10.24

Continue this activity by asking different students to put the numbers suggested here on the Minicomputer using only a ③ -checker, a ④ -checker, and a ⑤ -checker.

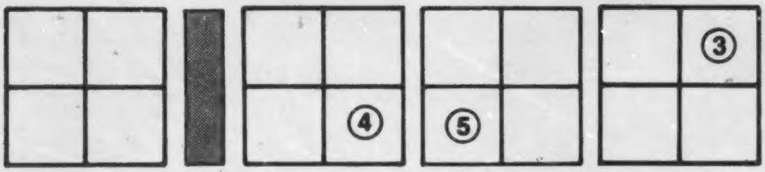
102.4 =



51.2 =

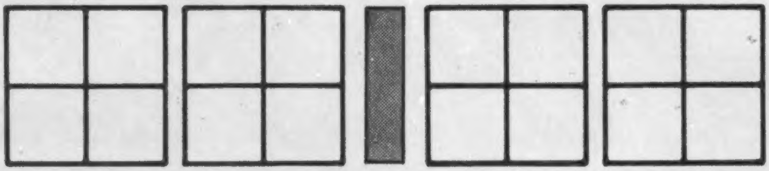


0.512 =



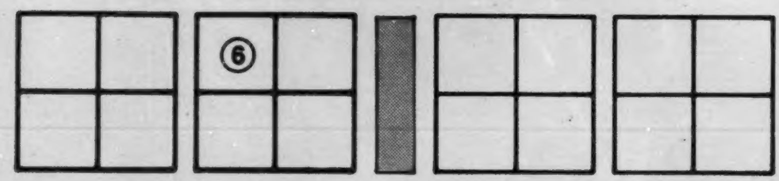
Display four Minicomputer boards and your set of multiplication checkers.

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



T: Can we put 49.4 on the Minicomputer using exactly two of these checkers?

Invite a student to place one of the checkers. Suppose a ⑥ -checker is placed first.



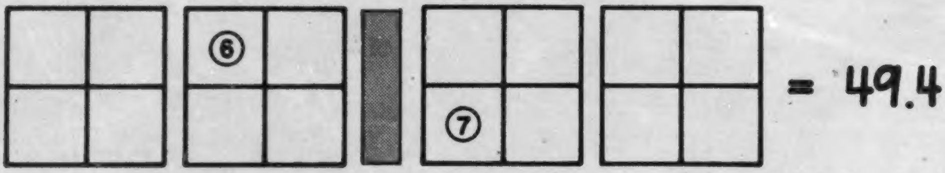
T: Which number is on the Minicomputer?

S: 48, because 6 × 8 = 48.

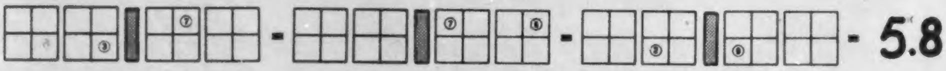
T: How much larger is 49.4 than 48?

S: 1.4.

T: Who can place the second checker?

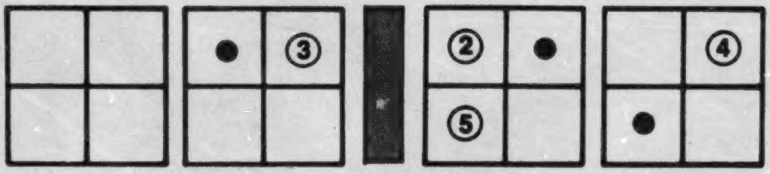


Repeat this activity with 5.8. Several solutions are possible but only insist on the class finding one of them.



Exercise 2

Display four Minicomputer boards with this configuration of checkers.



T: I will do something to this number by moving some checkers. Each time you must tell me if we get a larger number, a smaller number, or the same number. Also, when we get a larger or a smaller number, how much larger or smaller is it?

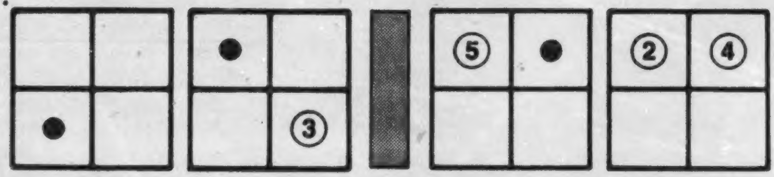
Move the ③ -checker from the 4-square to the 1-square.

S: 9 smaller, because 12 - 9 = 3.

Continue in this manner making the following moves:

- Move the ⑤ -checker from the 0.2-square to the 0.8-square. (3 larger; 1 + 3 = 4)
- Move the regular checker from the 0.02-square to the 20-square. (19.98 larger; 0.02 + 19.98 = 20)
- Move the ② -checker from the 0.8-square to the 0.08-square. (1.44 smaller; 1.6 - 1.44 = 0.16)

At this point the configuration on the Minicomputer should be as below. If your configuration of checkers does not agree, move any incorrectly placed checkers.



T: By moving a checker from the square it's on to another square, can you make the number 21 larger?

S: Move the ③-checker from the 1-square to the 8-square.

Make the move as the student suggests and continue the activity by asking similar questions. Feel free to adjust the level of difficulty of these questions to the numerical abilities of your class.

T: Who can make the number

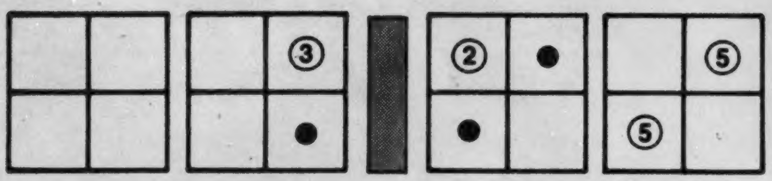
. . . 0.3 smaller? (Move the regular checker from the 0.4-square to the 0.1-square.)

. . . 1 larger? (Move the ⑤-checker from the 0.8-square to the 1-square.)

. . . 0.12 smaller? (Move the ②-checker from the 0.06-square to the 0.02-square or move the ④-checker from the 0.04-square to the 0.01-square.)

Exercise 3

Display four Minicomputer boards and this configuration of checkers.



N 36

T: Which number is on the Minicomputer?

Give the students a minute to study the configuration and compute.

S: 15.5

T: Yes. We are going to play a game of Minicomputer Golf using this number as the starting number.

Divide your class into two groups. One group will be the Red Team and the other will be the Blue Team.

Write this information on the board.

Goal : 88.8

Red Team →

Blue Team →

T: In this game the starting number is 15.5 and the goal is 88.8. Students from the Red Team and from the Blue Team play in turn. Each time, you may move one checker from the square it's on to any other square. Your move either increases or decreases the number on the Minicomputer. The first team to reach the goal is the winner. Who would like to play first for the Red Team?

Suppose the first player for the Red Team moves the regular checker from the 1-square to the 40-square. (We are describing an actual game played in one of our development classes.)

T: Is the number on the Minicomputer larger or smaller?

S: Larger.

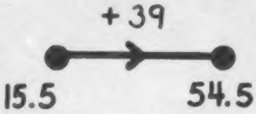
T: How much larger?

S: 39 larger.

T: Which number is on the Minicomputer now?

S: 54.5, because 15.5 + 39 = 54.5.

Draw this red arrow on the board.



Now a player from Team B takes a turn. Suppose a ③ -checker is moved from the 4-square to the 10-square.

T: Is the number larger or smaller?

S: Larger.

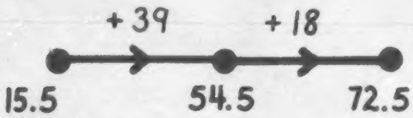
T: How much larger?

S: 18 larger, because 12 + 18 = 30.

T: Which number is on the Minicomputer now?

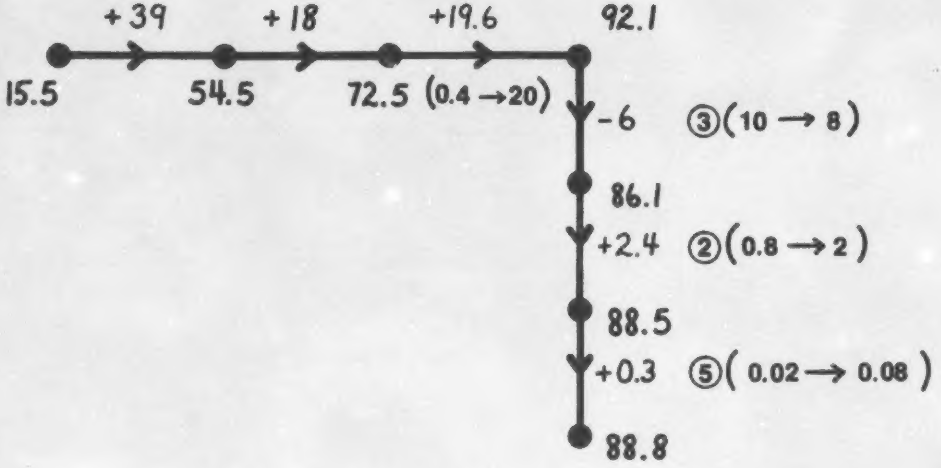
S: 72.5, because 54.5 + 18 = 72.5.

Add this blue arrow to the picture.



N 36

Follow similar steps for the next players, alternately from the Red Team and from the Blue Team. The results of our game are described in this red-blue road. The Blue Team won.


+39
+18
+19.6
92.1
15.5
54.5
72.5
(0.4 → 20)
-6
③(10 → 8)
86.1
+2.4
②(0.8 → 2)
88.5
+0.3
⑤(0.02 → 0.08)
88.8

NOTE: In this picture, "(0.4 ⟶ 20)" indicates that a regular checker was moved from the 0.4-square to the 20-square, while "③(10 ⟶ 8)" indicates that a ③ -checker was moved from the 10-square to the 8-square.

9

CARTESIAN GRAPHS

The lessons of this chapter provide experience in graphing relations in a two dimensional coordinate system. The resulting graphs are called "Cartesian graphs" because one commonly refers to the two dimensional coordinate system as the "Cartesian Plane."[†] Graphing several kinds, similar and different, of relations provides opportunities to make observations which describe the effect on the graph of changing a relation by modifying its algebraic expression and which distinguish Cartesian graphs of functions from other relations.

The last two lessons of the chapter investigate a linear programming problem in which cost must be minimized on a commodity subject to certain restrictions and requirements. The solution makes use of the Cartesian graphs for several linear functions.

CAPSULE LESSON SUMMARIES

Section 9.1:	Use a Guess My Rule activity to explore a squaring relation: $a \longrightarrow a^2$. Graph this relation and other similar relations, such as $a \longrightarrow a^2 + 3$, on the same set of axes. (Worksheet N46)
Section 9.2:	Play the Guess My Rule game using a rounding operation as the secret operation. That is, the rule for the game is: a Ⓡ b is the whole number multiple of b closest to a.
Section 9.3:	Review the rounding operation introduced in Section 9.2. Label dots in an arrow picture for the Ⓡ 4 function. Graph the Ⓡ 4 function. (Worksheet N47)

[†] "Cartesian" is a name honoring the French mathematician René Descartes (1596-1650) for his unification of algebra and geometry in the creation of analytic geometry.

Section 9.4: Introduce a story posing a linear programming problem in which cost must be minimized while certain requirements are fulfilled. Investigate solutions which satisfy the requirements and examine their costs. Look at cost lines on a grid. This preliminary work is background for Section 9.5 in which the problem is solved. (Worksheet N48)

Section 9.5: Review the story and problem of Section 9.4. Examine each of the minimum daily requirements (MDR) for the various nutrients and find corresponding lines on the grid. Locate a region for mixtures all of which satisfy the MDR. Use the fact that the cost lines are parallel to locate the least expensive mixture satisfying the MDR.

9.1 Cartesian Graphs #1

CAPSULE LESSON SUMMARY

Use a Guess My Rule activity to explore a squaring relation: $a \longrightarrow a^2$. Graph this relation and other similar relations, such as $a \longrightarrow a^2 + 3$, on the same set of axes.

MATERIALS

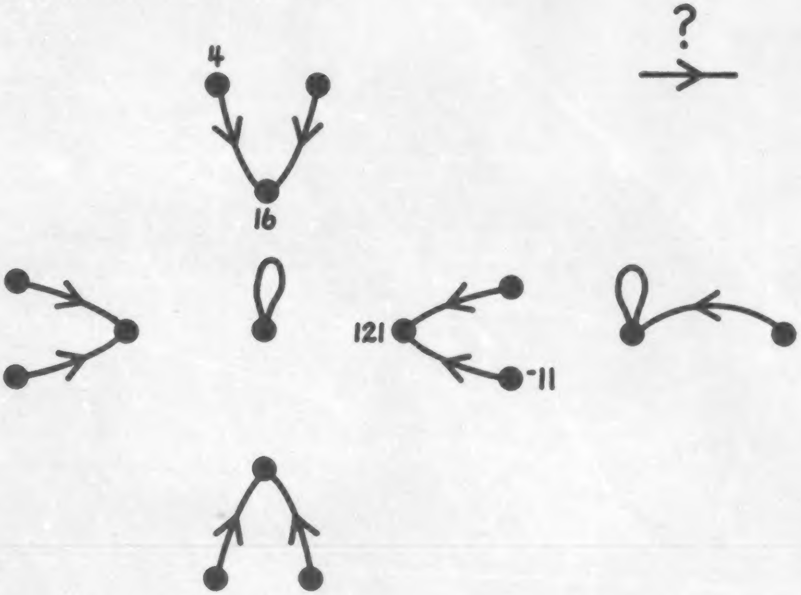
Teacher: Colored chalk; grid board (optional)
Student: Paper; colored pencils; Worksheet N46

NOTE: If a grid board is available, you may prefer to use it rather than to draw a grid on the chalkboard as described in the lesson.

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.



T: I have a secret rule for red arrows. This picture gives you some hints; try to decide my rule for the red arrows.

N 12

NOTE: The rule for red arrows is:

$$\xrightarrow{\text{you are my square}}$$

That is, the ending number of a red arrow is the square of the starting number. Do not announce the rule to the class at this time.

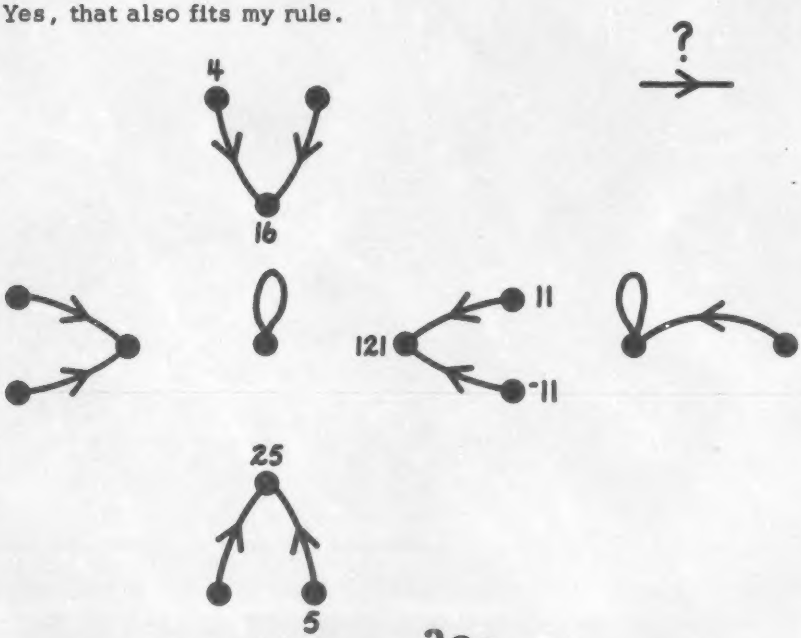
Allow a minute for the class to study the picture and then invite students to label dots of their choice. When a student labels a dot according to the rule, announce that is is correct. Erase labels which do not follow the rule with the explanation that the label is not consistent with your rule. For example,

S: The starting number for this other arrow ending at 121 is 11.

T: Yes, that fits my rule for the red arrows.

S (pointing to the red arrow at the bottom of the picture): Here the starting number could be 5 and the ending number 25.

T: Yes, that also fits my rule.



Continue this "Guess My Rule" game until it is clear that many students know the rule. Ask a student to explain the rule to the class.

S: The ending number is the square of the starting number.

S: Multiply the starting number by itself to get the ending number.

S: A red arrow is for "you are my square."

Be prepared for a suggestion that the relation is "is the square of." Point out the error by asking:

T (tracing the arrow starting at 5 and ending at 25): Is it correct to say "5 is the square of 25"?

S: No, 25 is the square of 5.

S: 5 can say to 25 "you are my square."

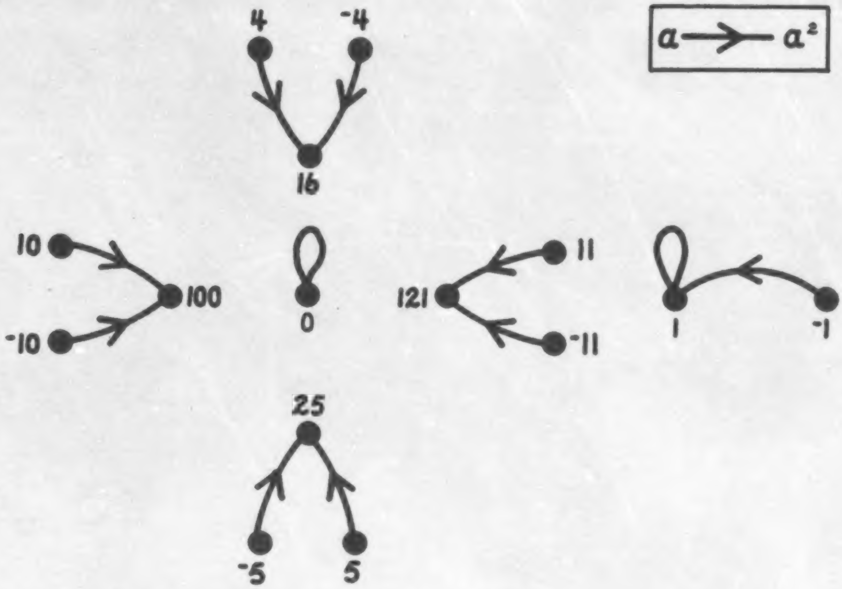
Your class may not be comfortable with squaring negative numbers or with the fact that a negative number times a negative number is a positive number. Since the focus of this lesson is drawing Cartesian graphs for relations, simply announce that for any whole number n the square of n and ⁻n are the same. You could also use a hand-calculator to explore this idea, but do not spend too much time on it.

Once the rule is established, write it on the board.

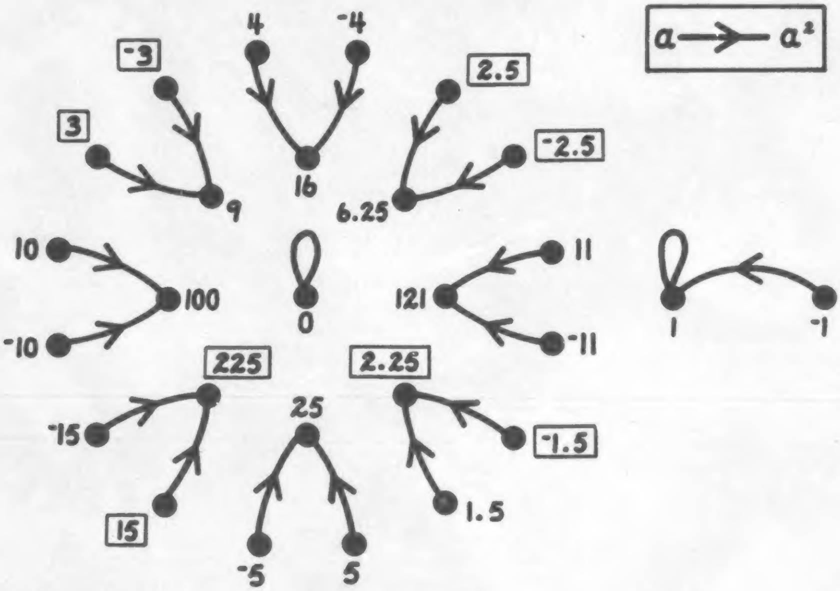
$$a \longrightarrow a^2$$

N 12

Continue to label the dots until your picture looks similar to the one below.


4
-4
16
a → a²
10
100
-10
0
11
121
-11
1
-1
25
-5
5

Add more arrows to the picture as shown below and ask the students to copy them and label the dots. Check the work collectively. Answers are in the boxes.

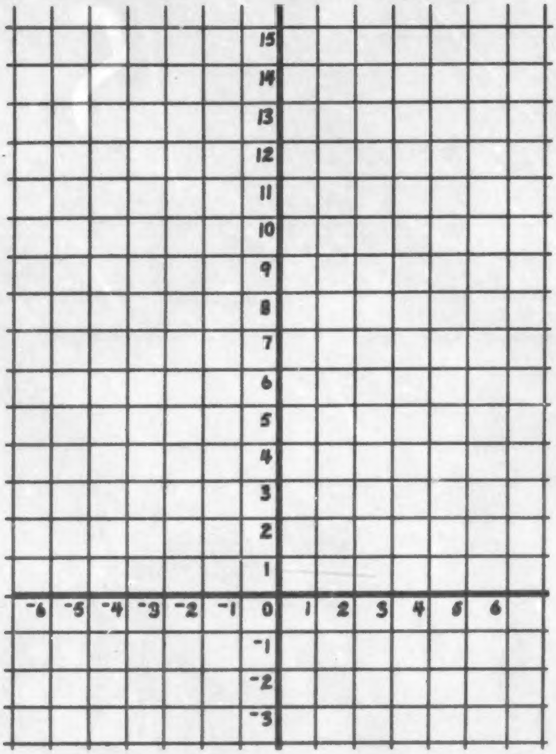

a → a²
4
-4
16
-3
3
9
2.5
-2.5
6.25
10
100
-10
0
11
121
-11
1
-1
225
-15
15
2.25
-1.5
1.5
25
-5
5

T: Is this relation a function?

S: Yes; at most one arrow starts from each dot.

Exercise 2

Draw this grid on the board.

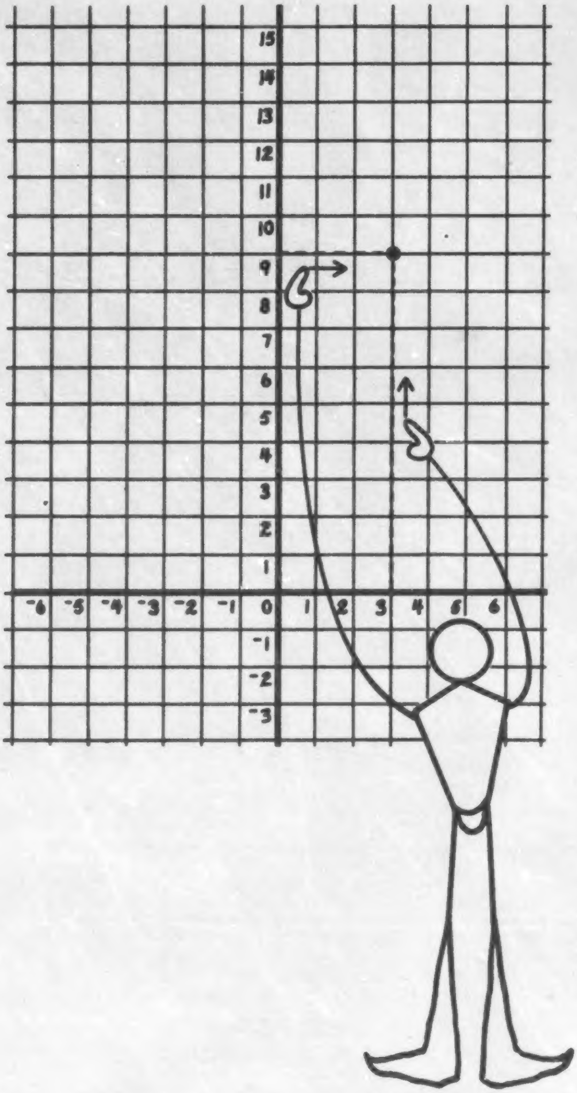


T: Let's draw the Cartesian Graph for the squaring function. How do we locate the point for this arrow starting at 3 and ending at 9 (trace the appropriate arrow)?

S: Find the starting number going across from 0 (on the horizontal axis) and the ending number going up from 0 (on the vertical axis). Then see where those grid lines meet.

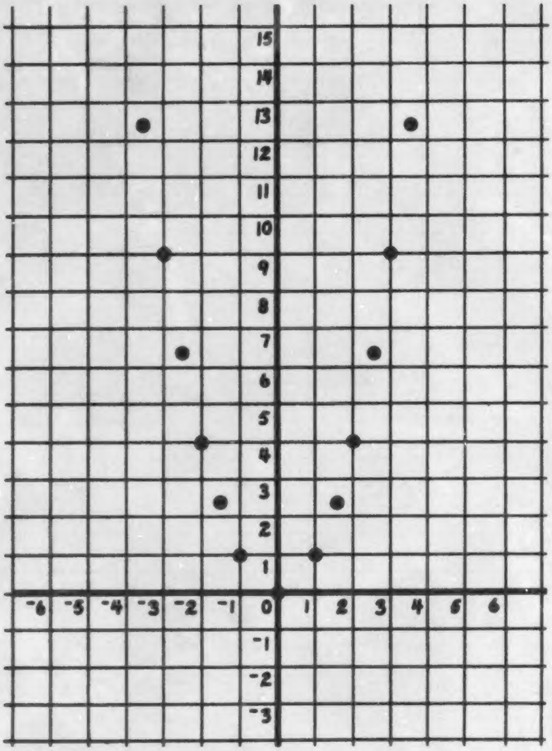
N 12

Invite a student to locate the point.



Ask students to locate points for two or three more arrows collectively; for example, (0, 0) and (⁻2.5, 6.25). Distribute individual copies of Worksheet N46 and instruct the students to locate on the grid a point for each arrow, if possible within the limits of the axes, in the arrow picture on the board. Then encourage students to find points for arrows in the squaring relation that are not in the arrow picture.

After several minutes of independent work, complete the picture at the board.



T: Did we draw a dot for every arrow of the squaring function?

S: No, there are many other arrows that could be drawn; for example, an arrow starting at $\frac{1}{2}$ and ending at $\frac{1}{4}$.

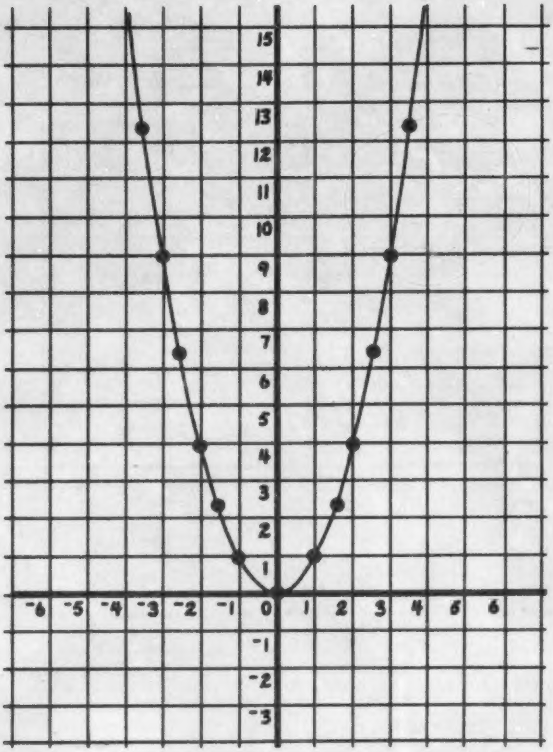
S: There is an endless number of points.

T: Where are there points for other arrows in the squaring relation?

S: Along this curve.

N 12

Complete the curve as shown below.



Exercise 3

Write these ordered pairs on the board.

(2,7)   (7,52)   (3,☐)

(-4,19)   (5,28)

T: Can you guess my rule for these ordered pairs?

Which number could go in the box?

NOTE: The rule is: the second component is three more than the square of the first component.

$$a \rightarrow a^2 + 3$$

The number in the box is 12 because $12 = 3^2 + 3$. Do not announce the rule to the class at this time.

Suggest that students whisper their guesses in your ear. Acknowledge, aloud, correct guesses and reject incorrect ones. For example, "Yes, that's correct." or "No, my rule does not allow 6 to be in the box." Let a student who guesses correctly tell the class that 12 is in the box, but do not give away the rule.

Continue this activity with several more ordered pairs (answers are in the boxes).

$$(^-3, \boxed{12}) \qquad (10, \boxed{103})$$

$$(12, \boxed{147}) \qquad (1.2, \boxed{4.44})$$

T: Who can explain my rule?

S: It is similar to the squaring function except 3 is added to the square of the first number to get the second number.

S: Start with a number, square it, then add 3 to the result.

Summarize the explanations by writing this information on the board.

$$a \rightarrow a^2 + 3$$

N 12

Ask the students to copy and solve these problems independently.

$$(\square, 67)$$
$$(1.5, \square)$$
$$(\square, 3)$$
$$(4, \square)$$
$$(\square, 7)$$

Check the problems collectively (answers follow) and ask for an explanation of each answer.

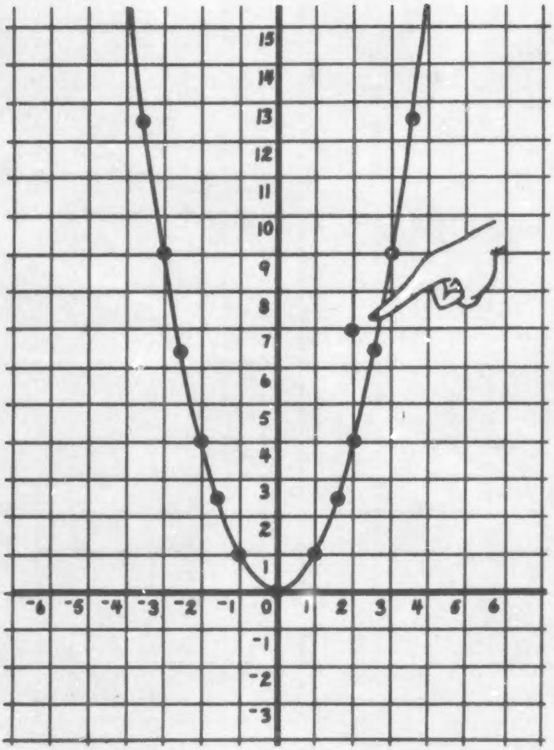
S: 8 is in the box of the first problem, because (8 × 8) + 3 = 67.

S: ¯8 also could be in the box, because the square of ¯8 is 64 and 64 + 3 = 67.

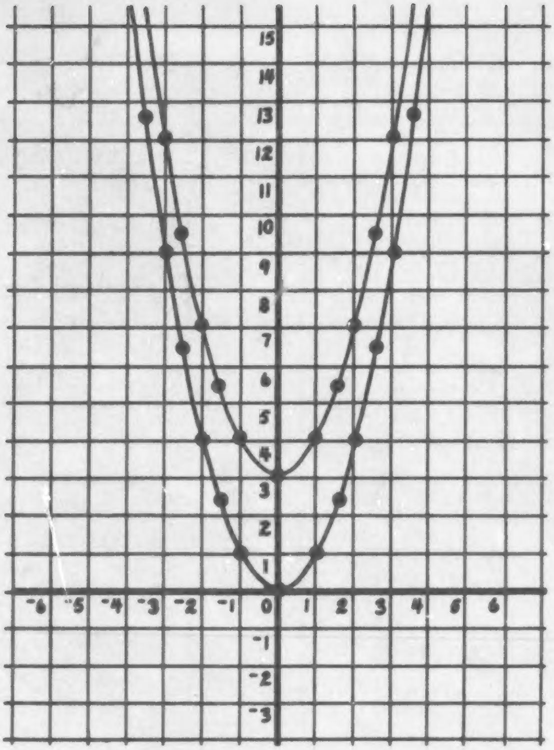
$$(\boxed{8}, 67) \text{ or } (\boxed{^-8}, 67)$$
$$(1.5, \boxed{5.25})$$
$$(\boxed{0}, 3)$$
$$(4, \boxed{19})$$
$$(\boxed{2}, 7) \text{ or } (\boxed{^-2}, 7)$$

T: Let's draw the Cartesian Graph for the blue relation. We can use the same grid as we did for the squaring function. Locate the point for the ordered pair (2, 7) on the grid.

Invite a student to locate the point.



Instruct the class to continue locating points on their grids (Worksheet N46) for the ordered pairs on the board. Encourage some students to find points for ordered pairs in the blue relation that have not been discussed. When most students have finished, complete the picture at the board.



T: Do you notice anything interesting about these graphs?

S: They look the same.

S: Each point on the blue curve is just three units higher than a corresponding point on the red curve.

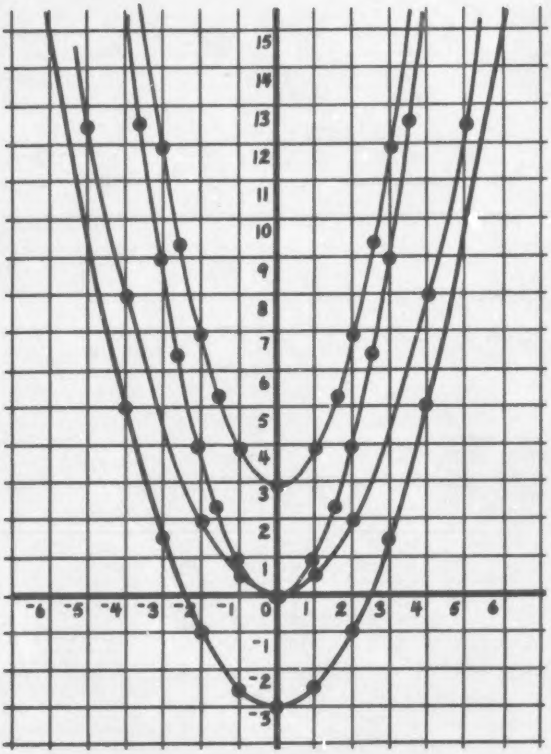
Exercise 4 (Optional)

Write these rules for relations on the board and ask students to draw their Cartesian Graphs on Worksheet N46.

$$a \rightarrow \frac{1}{2} \times a^2$$

$$a \rightarrow \left(\frac{1}{2} \times a^2\right) - 3$$

Observe the student's work and be ready to give help to those having difficulty. When most of the class has finished, let them comment about the graphs.



9.2 Rounding Operation #1

CAPSULE LESSON SUMMARY

Play the Guess My Rule game using a rounding operation as the secret operation. That is, the rule for the game is: a Ⓡ b is the whole number multiple of b closest to a.

MATERIALS

Teacher: None
Student: Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

Write this information on the board.

$$26 * 10 = 30$$
$$12 * 5 = 10$$
$$5 * 12 = 0$$
$$18 * 6 = 18$$
$$33 * 7 = \boxed{}$$

T: I have a secret rule for *; here are four clues. Can you guess my rule? Which number do you think goes in the box?

NOTE: The rule is: a * b is the whole number multiple of b closest to a. The number in the box is 35 because 35 is the multiple of 7 closest to 33. Do not tell the class your rule at this time.

Suggest that students whisper their guesses in your ear. Acknowledge, aloud, correct guesses and reject incorrect ones. For example, "Yes, that's correct" or "No, I do not get 30 using my rule." Let a student who guesses correctly tell the class that 35 is in the box, but do not give away the rule. If no one guesses correctly, put "35" in the box yourself.

Continue the activity with several more examples (answers are in the boxes).

$$16 * 3 = \boxed{15}$$

$$24 * 8 = \boxed{24}$$

$$21 * 1 = \boxed{21}$$

$$24 * 10 = \boxed{20}$$

If you find it necessary to give the class a hint about the rule, try asking:

T: What do you notice about the numbers in the second and third columns (from the left)?

Let the students have a moment to study the situation.

S: The result is always a multiple of the second number.

T: Yes, and the rule decides which multiple of the second number to take as the result. Who can explain my rule?

S: You always take the multiple of the second number closest to the first number.

S: 26 * 10 = 30, because 30 is the nearest multiple of 10 to 26.

Discuss one or two other examples and be sure the class understands the rule for *.

T: This operation is called a "rounding operation." Rather then use "*" to denote this operation we use " Ⓡ " (read: circle R) to suggest the idea of rounding.

N 21

Write the rounding problems below on the board and instruct the students to solve them on their own. Answers are in the boxes.

$$24 \circledR 20 = \boxed{20}$$

$$4 \circledR 12 = \boxed{0}$$

$$125 \circledR 100 = \boxed{100}$$

$$695 \circledR 100 = \boxed{700}$$

$$1{,}568 \circledR 1{,}000 = \boxed{2{,}000}$$

After a few minutes, check the work collectively.

T (pointing to the last problem): 1,568 rounded to the nearest 1,000 is 2,000.

Read a few more of the problems so that the students will begin to become familiar with such expressions as "... rounded to the nearest 10 is ..." or "... rounded to the nearest 100 is ...".

Write this problem on the board and let the class discuss it.

$$15 \circledR 6 = \boxed{}$$

S: The problem is to round 15 to the nearest 6.

S: Multiples of 6 are 0, 6, 12, 18, and so on; 15 is in the middle between 12 and 18.

S: 15 is 3 away from 12 and 3 away from 18. There are two nearest multiples of 6 in this case.

T: In order to have just one solution to problems such as these, let's agree to always take the greater of the two nearest multiples. We say that we "round up" in cases like this. So 15 ® 6 = 18.

$$15 \circledR 6 = \boxed{18}$$

Exercise 2

Write these problems on the board and instruct the students to find solutions whenever possible. Answers are in boxes. Discuss the last three problems collectively.

$$8.6 \circledR 4 = \boxed{8}$$

$$\hat{3} \circledR 5 = \boxed{\hat{5}}$$

$$73 \circledR 7 = \boxed{70}$$

$$25 \circledR 10 = \boxed{30}$$

$$\widehat{10} \circledR 6 = \boxed{\widehat{12}}$$

$$6{,}213 \circledR 100 = \boxed{6{,}200}$$

$$7.5 \circledR 4 = \boxed{8}$$

$$\boxed{\quad} \circledR 6 = 36$$

$$16 \circledR \boxed{\quad} = 15$$

$$14 \circledR \boxed{\quad} = 9$$

T: Which number rounded to the nearest 6 is 36?

S: 34, because 34 is closer to 36 than to 30.

S: Also 36, because the nearest multiple of 6 to 36 is 36.

List solutions on the board as they are suggested. If 33 is not mentioned, ask:

T: Which is the smallest number that could go in the box?

S: 33; 32 ® 6 = 30 and 33 ® 6 = 36, because we agreed to take the greater multiple when the first number is exactly in the middle between two multiples.

Add "33" to your list of solutions.

33, 34, 35, 36, 37, 38

T: Are there other numbers that could go in the box?

Give the students a few minutes to think about your question.

S: Those are all of the whole numbers that could go in the box.

S: 34.5, because 34.5 is closer to 36 than to 30.

S: 37.26.

S: 33.12486.

T: Yes, these are many other numbers that could go in the box. Is there a largest number that could go in the box?

S: 38.9.

S: 38.99.

S: 38.9999.

S: There is no largest number. We can always put one more "9" at the end.

If students conjecture that 39 is the greatest number that could go in the box, just remind them that 39 Ⓡ 6 = 42.

T: What about "16 Ⓡ □ = 15"; which numbers could go in the box?

S: 15 must be a multiple of the number in the box.

S: 3 or 5 or 15.

If students suggest that a number like 2.5 could go in the box, discuss why this is correct. Counting by 2.5 you get . . . 10, 12.5, 15, 17.5, 20, ... So 15 is the nearest multiple of 2.5 to 16. However, do not force such a solution at this time. In Exercise 3 you will discuss such situations.

T: Can you find a solution for this last problem, "$14 \circledR \square = 9$"?

S: I tried 3 and 9, but they don't work.

Let the class discuss this problem for a few minutes. They should conclude that there is no solution.

Exercise 3

Erase the board and write this problem.

$$2.6 \circledR \frac{1}{2} = \boxed{}$$

T: Think of a calculator with $\frac{1}{2}$ (0.5) on the display. Suppose we press $\boxed{+}$ $\boxed{=}$ $\boxed{=}$ and so on. Which numbers would appear on the display?

S: 0.5 and 1 and 1.5 and so on. The calculator would count by $\frac{1}{2}$'s.

T: Which of the numbers that would appear on the display is closest to 2.6?

S: 2.5, because $2.6 - 2.5 = 0.1$ and $3 - 2.6 = 0.4$.

T (writing "2.5" on the box): 2.6 rounded to the nearest $\frac{1}{2}$ is 2.5.

$$2.6 \circledR \frac{1}{2} = \boxed{2.5}$$

N 21

Write these problems on the board and ask the students to solve them. Answers are in boxes.

$$4.7 \circledR \tfrac{1}{2} = \boxed{4.5}$$

$$8.67 \circledR 1 = \boxed{9}$$

$$7.342 \circledR 0.01 = \boxed{7.34}$$

$$3.68 \circledR 0.1 = \boxed{3.7}$$

$$\tfrac{1}{3} \circledR \tfrac{1}{10} = \boxed{\tfrac{3}{10}=0.3}$$

Check the problems collectively, using a hand-calculator discussion as necessary for the class' understanding.

9.3 Rounding Operation #2

CAPSULE LESSON SUMMARY

Review the rounding operation introduced in Lesson N9.2. Label dots in an arrow picture for the Ⓡ 4 function. Graph the Ⓡ 4 function.

MATERIALS

Teacher: Colored chalk; Grid Board (Optional)
Student : Colored pencils; Worksheet N47

DESCRIPTION OF LESSON

Exercise 1

Write this number story on the board.

$$17 \circledR 4 = \boxed{}$$

T: Do you remember the rounding operation?

S: For the problem on the board, we must find the multiple of 4 closest to 17.

S: 17 Ⓡ 4 = 16.

Write "16" in the box and ask the students to solve a second problem.

$$\begin{aligned} 17 \circledR 4 &= \boxed{16} \\ 27 \circledR 6 &= \boxed{} \end{aligned}$$

S: 27 is exactly in the middle between 24 and 30: it is 3 away from 24 and 3 away from 30.

S: We agreed to take the greater multiple in such a case, so $27 \circledR 6 = 30$.

Complete the number story with 30 in the box and instruct the students to work independently on these problems. After a few minutes, check their solutions collectively. Answers are in the boxes.

$$36 \circledR 10 = \boxed{40}$$
$$3 \circledR 10 = \boxed{0}$$
$$129 \circledR 100 = \boxed{100}$$
$$350 \circledR 100 = \boxed{400}$$
$$1{,}186 \circledR 100 = \boxed{1{,}200}$$

$$2{,}657 \circledR 1{,}000 = \boxed{3{,}000}$$
$$10{,}986 \circledR 1{,}000 = \boxed{11{,}000}$$
$$8.36 \circledR 0.1 = \boxed{8.4}$$
$$14.52 \circledR 0.1 = \boxed{14.5}$$
$$3.093 \circledR 0.01 = \boxed{3.09}$$

T: Suppose we have a rounding problem with a sum of two numbers. Do you think it would matter whether we first add the numbers and then do the rounding or first round each number and then add the results? Let me give you an example of what I mean.

Write these number stories on the board.

$$(5+3) \circledR 4 \qquad (5 \circledR 4) + (3 \circledR 4)$$

T (pointing to "$(5 + 3) \circledR 4$"): The parentheses in this problem tell us to first add "5 + 3" and then to round the sum to the nearest multiple of 4. In the other problem, the parentheses tell us first to round each number, 5 and 3, to the nearest multiple of 4 and then to add the results. Do both problems and see what happens.

Let the students work independently for a minute or two. Then call on a volunteer to solve the problems at the board.

$$(5+3) \circledR 4 = 8 \qquad (5 \circledR 4) + (3 \circledR 4) = 8$$

S: They are equal.

T: Do you think it always happens this way?

Encourage students to comment about this situation. Then ask the class to carry this experiment further by working two more pairs of problems. Answers are in the boxes.

$$(15+3)\circledR 7 = \boxed{21} \qquad (15\circledR 7)+(3\circledR 7)=\boxed{14}$$

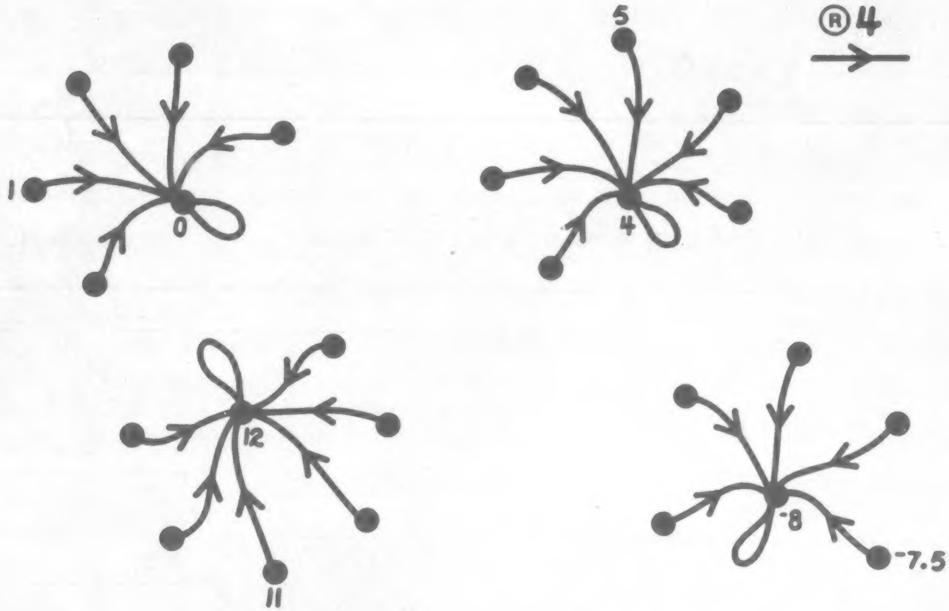
$$(16+9)\circledR 4 = \boxed{24} \qquad (16\circledR 4)+(9\circledR 4)=\boxed{24}$$

S: Sometimes it does make a difference.

Encourage students to make observations about when there is a difference.

Exercise 2

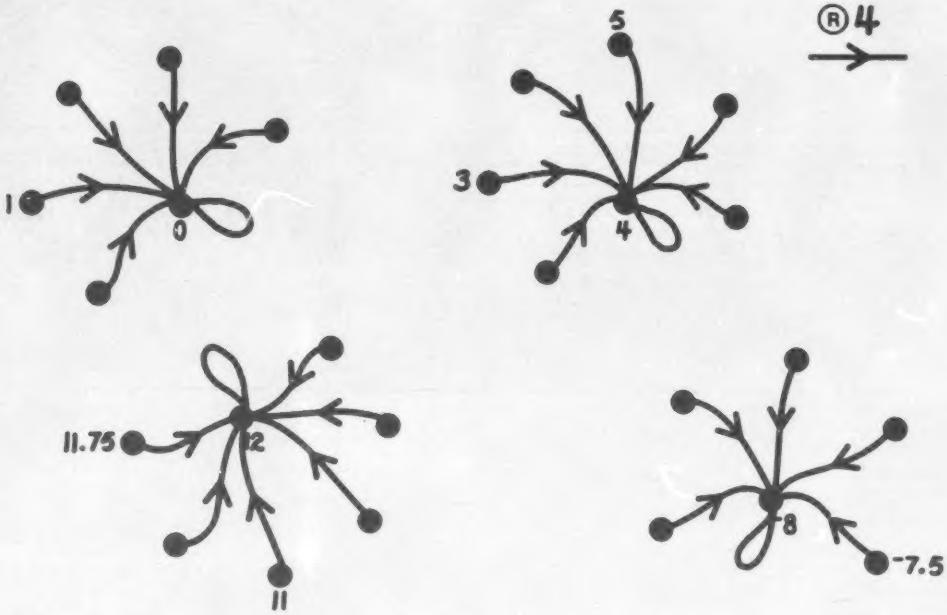
Draw this arrow picture on the board.



N 33

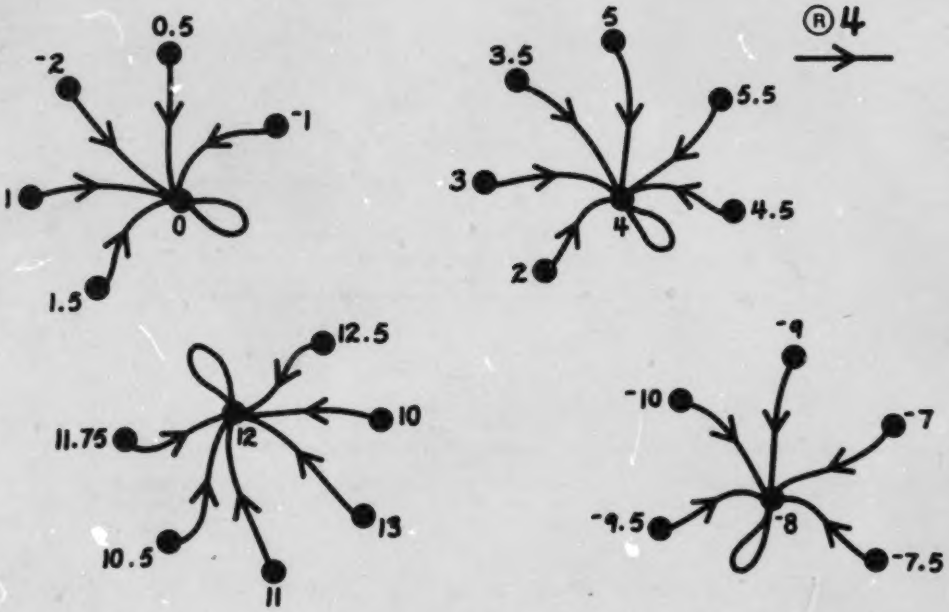
T: I have drawn part of an arrow picture for the Ⓡ4 function. Label a dot in this picture.

Invite a couple students to label dots of their choice.



T (tracing the arrow starting at 3 and ending at 4): This dot could be for 3 because 3 Ⓡ 4 = 4. On your paper, copy this picture and label the other dots.

If you prefer, the dots could be labeled collectively at the board. If you choose to let the students work independently, then periodically ask various students to label dots at the board. Upon completion of this activity, your arrow picture will look similar to the one on the next page.


Ⓡ4
0.5
-2
-1
1
0
1.5
5
3.5
5.5
3
4
4.5
2
12.5
10
11.75
12
10.5
13
11
-9
-10
-7
-9.5
-8
-7.5

T (pointing to the piece of arrow picture centered around 0): Could I draw more red arrows in this piece of the picture?

S: Yes, there could be many more arrows in each piece.

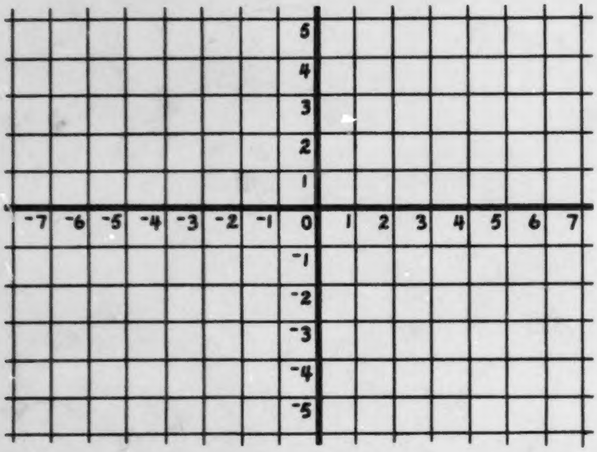
T: Think about the ending number for all of the arrows in one piece. What can we say about these "center" numbers of the pieces?

S: There will be a separate piece for each multiple of 4 and the ending number for all of the arrows of a piece will be a multiple of 4.

N 33

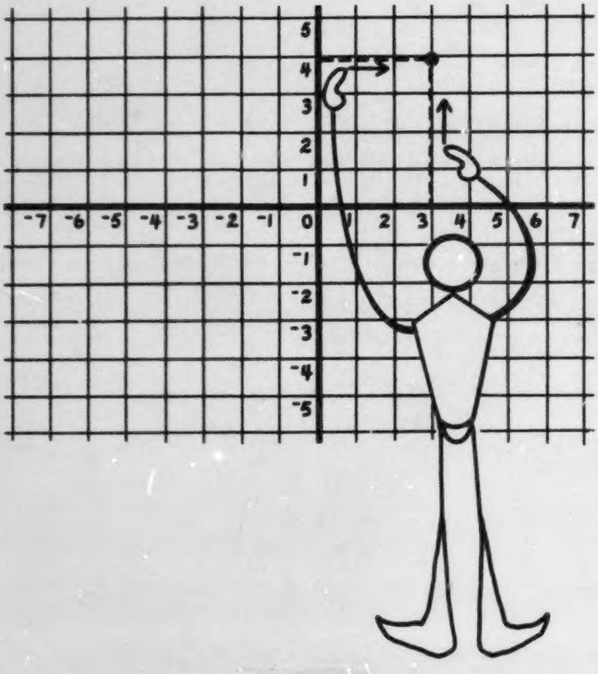
Exercise 3

Draw this grid close to the arrow picture.

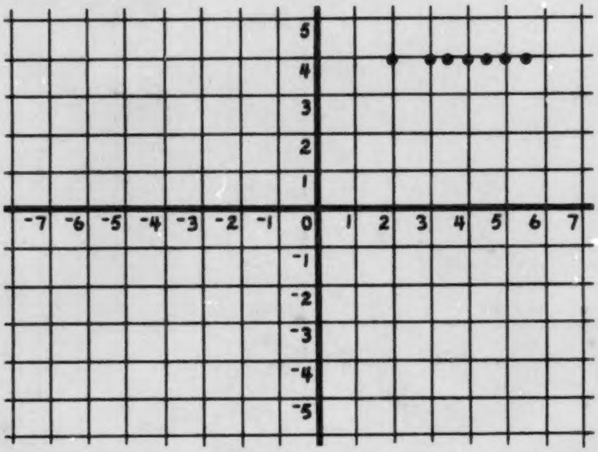

5
4
3
2
1
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7
-1
-2
-3
-4
-5

T: Let's draw the Cartesian Graph for the Ⓡ4 function. How do we locate the point for the arrow that starts at 3 and ends at 4 (trace this arrow)?

Invite a student to locate the point.


5
4
3
2
1
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7
-1
-2
-3
-4
-5

Ask the students to locate other points for arrows in the piece centered at 4; all arrows ending at 4.



T: We mentioned previously that there are many arrows, in fact infinitely many, that could be drawn in each piece. How can we reflect this fact in our graph of the Ⓡ 4 function?

S: We must fill in the gaps between 2 and 6.

T: What about at 6?

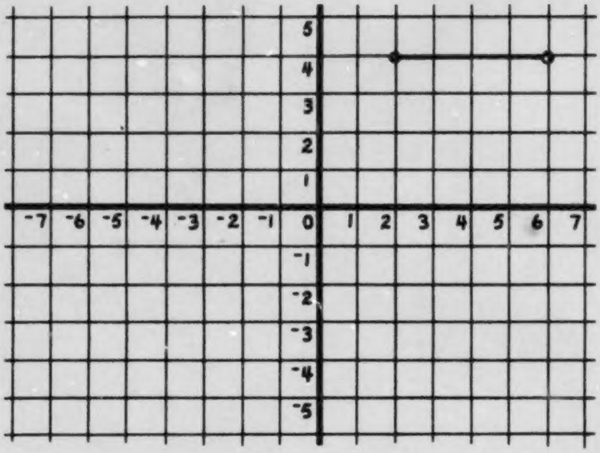
S: We can put a dot at 5.9, 5.99, 5.999, and so on, but not at 6 because 6 Ⓡ 4 = 8.

T: What should we do about 6?

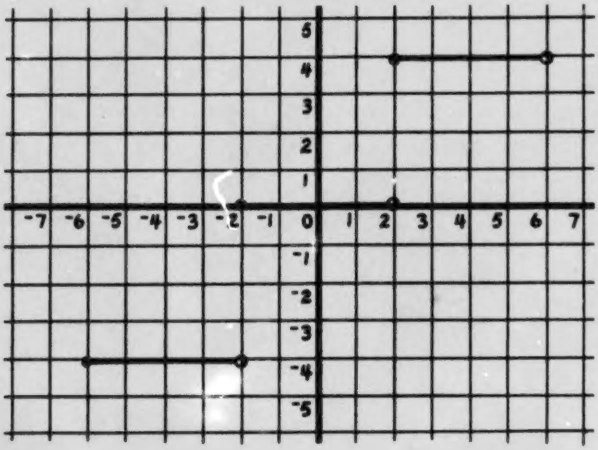
S: Put an open circle at 6 to show that it is left out.

N 33

Complete this part of the graph based on the class's comments.



Distribute individual copies of Worksheet N47 and instruct the students to locate on their grids a point for each remaining arrow on the board. Encourage some students to fill in the graph for the pieces of the picture on the board and to think about how it goes on. When many students are finished, complete the graph at the board.



T: Do you notice anything interesting about our graph?

S: It looks like stair-steps.

Let the students comment about the graph and discuss how it would go on were the grid bigger.

N 35

9.4 Linear Programming #1

CAPSULE LESSON SUMMARY

Introduce a story posing a linear programming problem in which cost must be minimized while certain requirements are fulfilled. Investigate solutions which satisfy the requirements and examine their costs. Look at cost lines on a grid. This preliminary work is background for Lesson N9.5 in which the problem is solved.

MATERIALS

Teacher: World of Numbers Poster #3 for IG-VI
Student: Worksheet N48; pencil

DESCRIPTION OF LESSON

Present the following situation to the class.

T: Dr. Dantzig is a veterinarian. She was hired by the Wolfe Kennel to help provide good healthy diets for the dogs. The Wolfe Kennel specializes in Great Danes and today Dr. Dantzig is concerned about what diet to recommend for a Great Dane.

The kennel stocks two kinds of dog food: Brand X and Brand Y. Here is what Dr. Dantzig has found out about one scoop of each kind of dog food.

Draw this table on the board.

	Units of Carbohydrates	Units of Vitamins	Units of Protein
Brand X	1	4	5
Brand Y	3	4	1

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T: This table tells the number of units of carbohydrates, of vitamins, and of protein supplied per scoop of dog food. How many units of each nutrient does two scoops of Brand X provide?

S: 2 units of carbohydrates, 8 units of vitamins, and 10 units of protein.

T: What about three scoops of Brand Y?

S: 9 units of carbohydrates, 12 units of vitamins, and 3 units of protein.

T: How many units of each nutrient does three scoops of Brand X and five scoops of Brand Y mixed together provide?

Allow a minute for students to make the necessary calculations.

S: Three scoops of Brand X provide 3 units of carbohydrates and five scoops of Brand Y provide 15 units of carbohydrates. Altogether that's 18 units of carbohydrates.

S: Of vitamins there are 12 units from three scoops of Brand X and 20 units from five scoops of Brand Y, so altogether that's 32 units of vitamins.

S: Of protein there are 15 units from three scoops of Brand X and 5 units from five scoops of Brand Y, so altogether that's 20 units of protein.

T: Being a veterinarian, Dr. Dantzig knows that Great Danes require a minimum of 8 units of carbohydrates, 20 units of vitamins, and 7 units of protein each day to remain healthy. These amounts determine the minimum daily requirement (MDR) for Great Danes.

N 35

Add this information to your table.

	Units of Carbohydrates	Units of Vitamins	Units of Protein
Brand X	1	4	5
Brand Y	3	4	1
MDR	8	20	7

T: Dr. Dantzig must recommend a diet that at least meets the minimum daily requirement for Great Danes. Would three scoops of Brand X and five scoops of Brand Y meet the MDR for all of the nutrients?

S: Yes, it would provide more than the MDR for all three.

T: Could you meet the MDR with only Brand X? How many scoops would it take?

S: 8 scoops of Brand X would provide 8 units of carbohydrates, 32 units of vitamins, and 40 units of protein. That is exactly the MDR for carbohydrates and more than the MDR for vitamins and protein.

T: Could you meet the MDR with only Brand Y? How many scoops would it take?

S: 7 scoops of Brand Y would provide 21 units of carbohydrates, 28 units of vitamins, and 7 units of protein. That is more than the MDR for carbohydrates and vitamins and exactly the MDR for protein.

T: Let's record this information using ordered pairs of numbers. The first number is for the scoops of Brand X and the second number is for the scoops of Brand Y.

Record this information on the board.

Meets the MDR

(3,5)
(8,0)
(0,7)

T: Can you think of some other mixtures of dog food to meet the MDR for the Great Danes?

Allow a couple minutes for students to find such mixtures. Collectively check each suggested mixture.

S: Five scoops of Brand X and one scoop of Brand Y together provide 8 units of carbohydrates, 24 units of vitamins, and 26 units of protein.

Record this information on the board.

Meets the MDR

(3,5)
(8,0)
(0,7)
(5,1)

As different mixtures are suggested, record them on the board. For example,

Meets the MDR

(3,5)
(8,0)
(0,7)
(5,1)
(3,3)
(4,2)
(2,4)
(3,4)

T: In addition to concern for a Great Dane's nutritional requirements, the kennel owner is also concerned about the cost of feeding the animal. He tells Dr. Dantzig that one scoop of Brand X costs about forty cents and that one scoop of Brand Y costs about thirty cents.

Add the cost information to your table.

	Units of Carbohydrates	Units of Vitamins	Units of Protein	Cost
Brand X	1	4	5	$0.40
Brand Y	3	4	1	$0.30
MDR	8	20	7	▨

T (pointing to "(8,0)"): How much would eight scoops of Brand X cost?

S: \$3.20, because 8 × 0.40 = 3.20.

Record the cost on the board, in the manner shown below, as you repeat:

T: The cost of eight scoops of Brand X is \$3.20.

$$c(8,0) = \$3.20$$

T: What is the cost of seven scoops of Brand Y?

S: 7 × 0.30 = 2.10; \$2.10 is the cost.

$$c(0,7) = \$2.10$$

T: What is the cost of three scoops of Brand X and five scoops of Brand Y?

S: $(3 \times 0.40) + (5 \times 0.30) = 2.70$; \$2.70.

Record this cost on the board and ask the students to calculate the cost of each remaining mixture. Check the work collectively.

$c(3,5) = \$2.70$

$c(8,0) = \$3.20$

$c(0,7) = \$2.10$

$c(5,1) = \$2.30$

$c(3,3) = \$2.10$

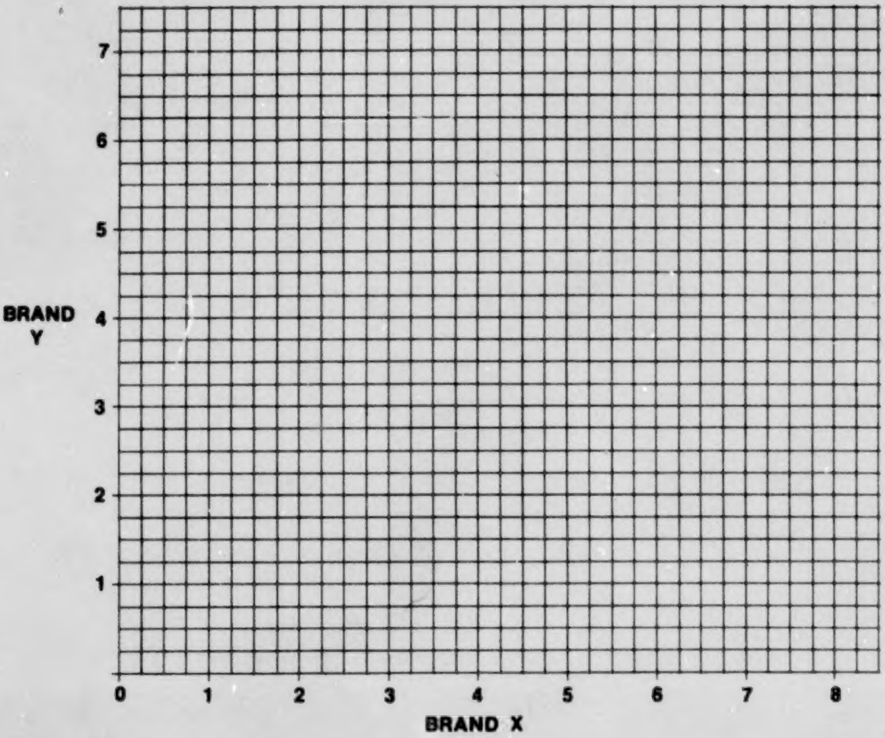
$c(4,2) = \$2.20$

$c(2,4) = \$2.00$

$c(3,4) = \$2.40$

T: Dr. Dantzig must find the least expensive diet for a Great Dane which meets the minimum daily requirement for carbohydrates, for vitamins, and for protein. She decides to look at the cost for various mixtures of dog food.

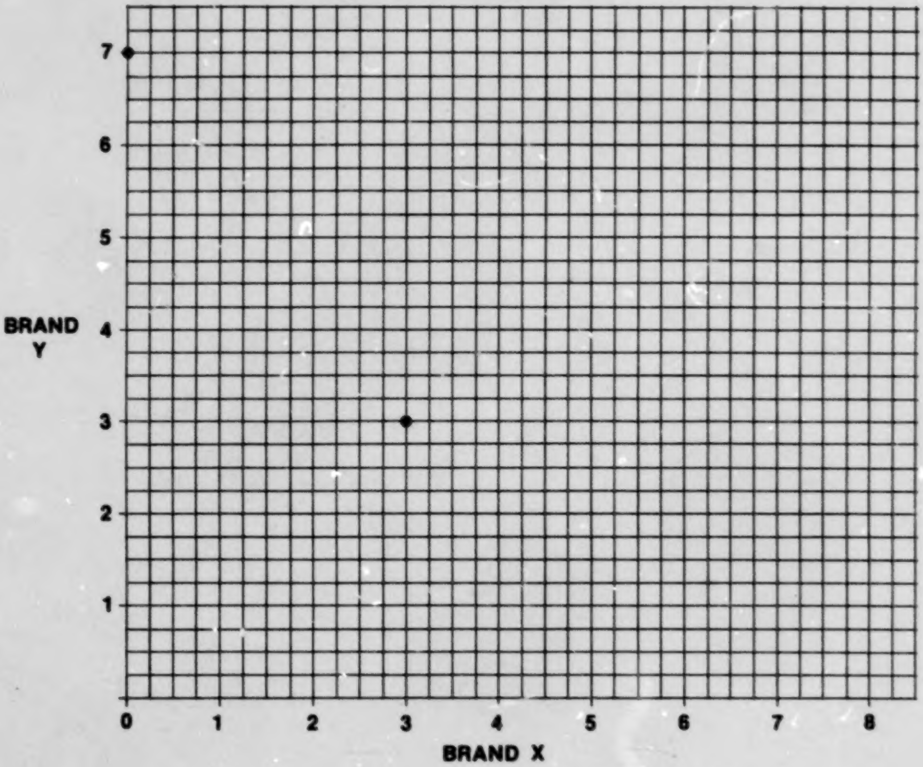
Distribute copies of Worksheet N48 and display World of Numbers Poster #3.



Modify the dialogue in this section to fit your data.

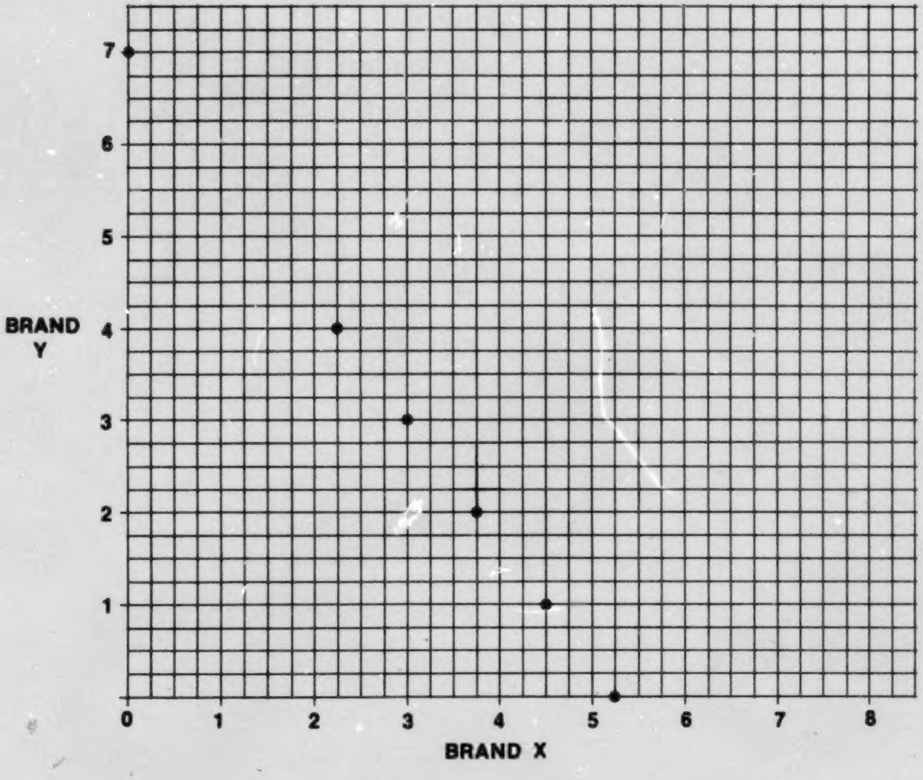
T: Notice that there are two mixtures that cost $2.10. Locate on the grid the point for (0, 7) and for (3, 3).

Invite a student to locate these points and you draw the dots in black. Try to work as carefully as you can on this poster as it will be needed again in Lesson N9.5.



T: Find some other mixtures that cost $2.10 and locate the points for those mixtures on your grid.

Let the students work either collectively or independently on this problem. Call on students to draw dots for points they find on the grid at the front of the room. Suppose your graph looks like this one.

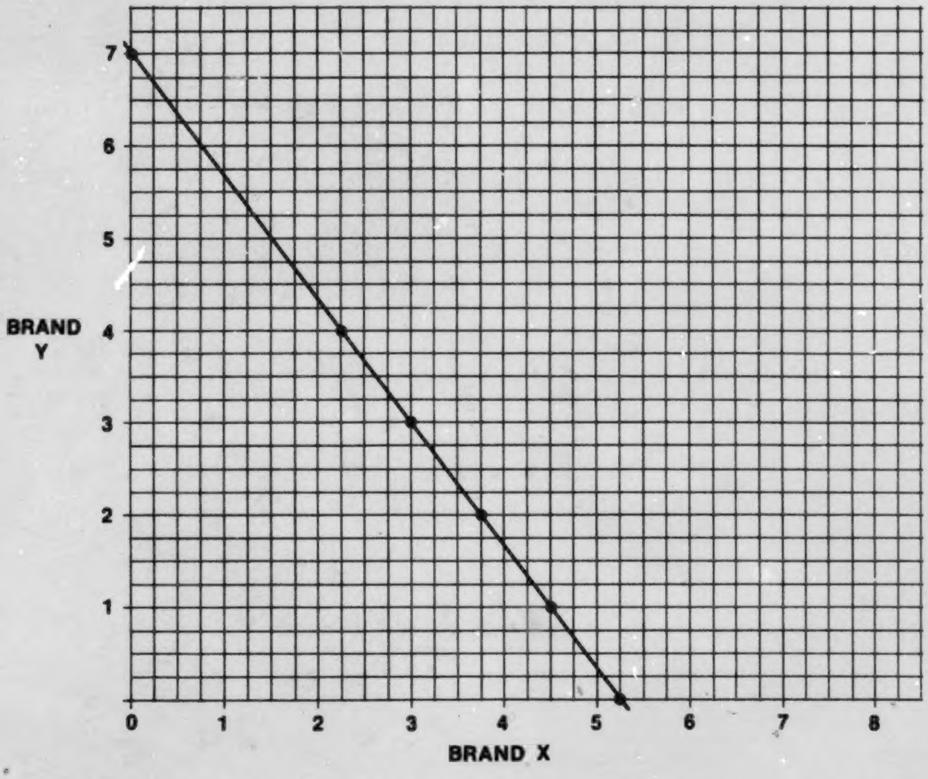


T: What do you notice about these points?

S: They seem to be in a line.

N 35

Draw the line using a straightedge.



T: Let's pick a point on the line and see if that point corresponds to a mixture that costs \$2.10.

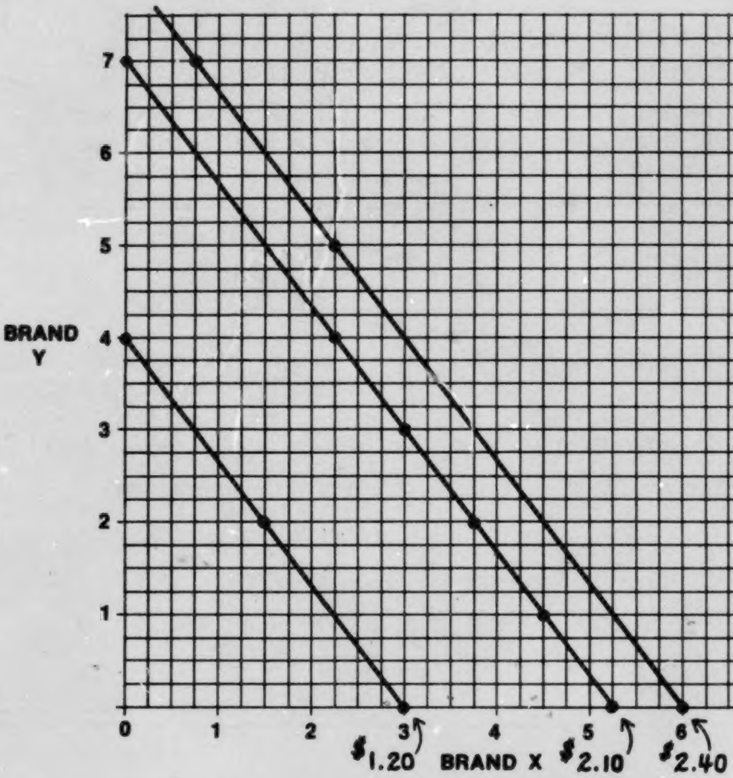
Pick a point and ask the students to calculate the cost for the mixture represented by that point. In one of our development classes the point (1.5, 5) was selected.

T: How much does one and a half scoops of Brand X and five scoops of Brand Y cost?

S: The cost is \$2.10: $1.5 \times 0.4 = 0.6$ and $5 \times 0.3 = 1.5$; $0.6 + 1.5 = 2.1$.

T: It seems that each point on this line represents a mixture that costs $2.10. For this reason, we call it the "$2.10-line."

Label the line "$2.10" and ask the students to find the $2.40-line and the $1.20-line. As students complete this task on their worksheet, call on neat, faster workers to locate points and draw the lines on your poster.



T: What do you notice about these cost lines?

S: They are parallel.

S: The price increases as we look at lines further to the right.

Save the poster with the three cost lines on it for use in Lesson N9.5. Also, save the students' copies of Worksheet N 48.

9.5 Linear Programming #2

CAPSULE LESSON SUMMARY

Review the story and problem of Lesson N9.4. Examine each of the minimum daily requirements (MDR) for the various nutrients and find corresponding lines on the grid. Locate a region for mixtures all of which satisfy the MDR. Use the fact that the cost lines are parallel to locate the least expensive mixture satisfying the MDR.

MATERIALS

Teacher: World of Numbers Posters #3 and #4 for IG-VI; translator
Student: (Optional) Worksheet N48; colored pencils

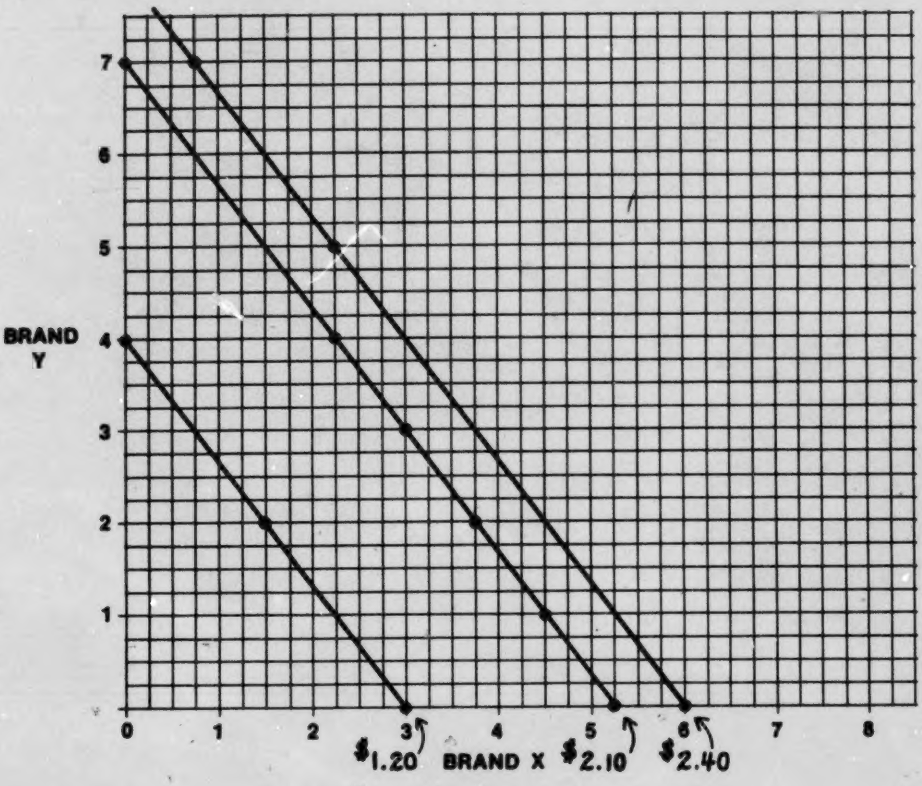
NOTE: World of Numbers Poster #3 was used in Lesson N9.4 and should now have three cost lines drawn on it. The same is true for the students' copies of Worksheet N48.

DESCRIPTION OF LESSON

T: Do you remember the story of Dr. Dantzig and the Wolfe Kennel?

Let the students recall the story and the problem of providing a nutritionally adequate, low cost diet for Great Danes. Summarize the pertinent information in a table on the board and display World of Numbers Poster #3 with the three cost lines drawn on it from use in Lesson N9.4. If you wish, you could distribute the students' copies of Worksheet N48. Then ask them to follow the collective lesson by doing similar work on the worksheet.

	Units of Carbohydrates	Units of Vitamins	Units of Protein	Cost
Brand X	1	4	5	$0.40
Brand Y	3	4	1	$0.30
MDR	8	20	7	▨



T: Dr. Dantzig must find the least expensive diet for a Great Dane which meets the minimum daily requirement for carbohydrates, for vitamins, and for protein.

We were looking at the costs for various mixtures of dog food. (Point to the poster.) What do these lines represent?

S: They are cost lines. Each mixture has a cost and mixtures that cost the same are represented by points on the same line.

S: All of the points on a cost line represent mixtures costing the same.

T: What did we notice about these three cost lines?

S: They are parallel.

T: Yes; if the lines were not parallel, what would happen?

S: The lines would cross.

T: What would it mean if two cost lines cross?

S: If two of the lines cross, then the mixture represented by the point where they cross would have two different costs.

Students may find it difficult to verbalize this idea, so direct them as necessary.

T: About where would we find the $2.50-line?

S: A little to the right of the $2.40-line.

Invite a student to trace approximately where the line would be on the poster.

T: How could we locate it exactly?

S: Find some mixtures that cost $2.50. The points which represent such mixtures are on the $2.50-line.

S: One scoop of Brand X and seven scoops of Brank Y cost $2.50, so the point (1, 7) is on the $2.50-line. Use a translator to draw a line parallel to the other lines and through the point (1, 7).

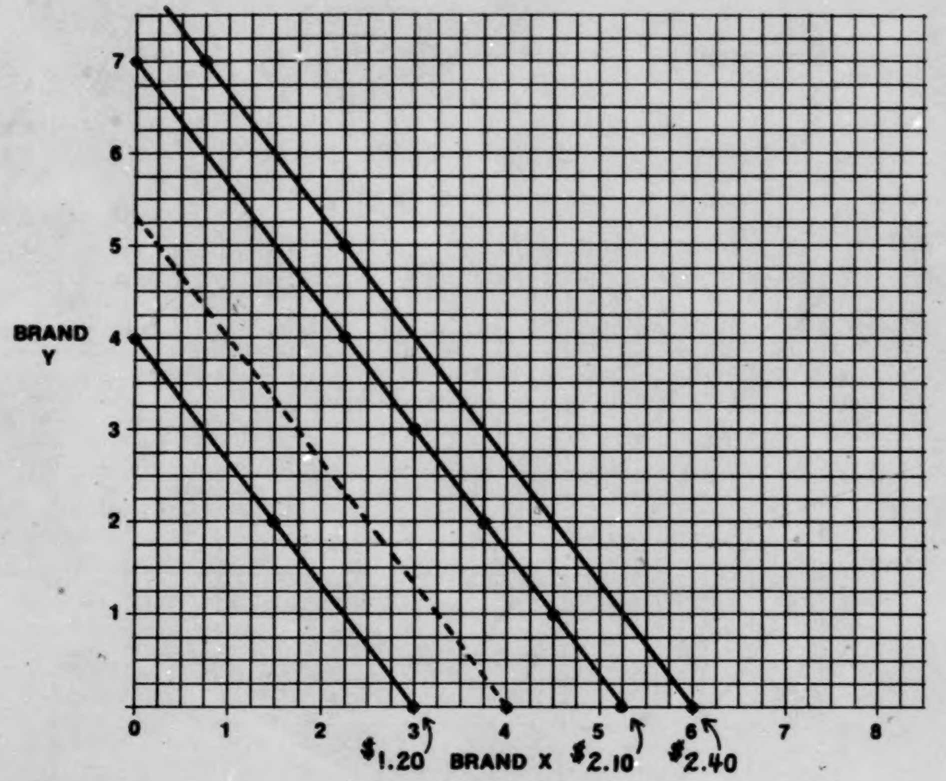
T: About where would we find the $1.80-line?

S: Between the $1.20-line and the $2.10-line, but closer to the $2.10-line.

Invite a student to trace approximately where the line would be on the poster.

T: I will draw a cost line on the grid and you tell me which one it is.

Draw a line as shown (dotted) in the next illustration.



Give the students a couple minutes to think before asking for a response.

S: $1.60; because it goes through the point (4; 0) and four scoops of Brand X cost $1.60 (4 × 0.40 = 1.60).

N 37

T: Dr. Dantzig is still looking for a mixture of dog food that will satisfy the minimum daily requirements and will be the lowest in cost. Which mixture do you think she should recommend?

Record some of the students' responses on the board for future reference.

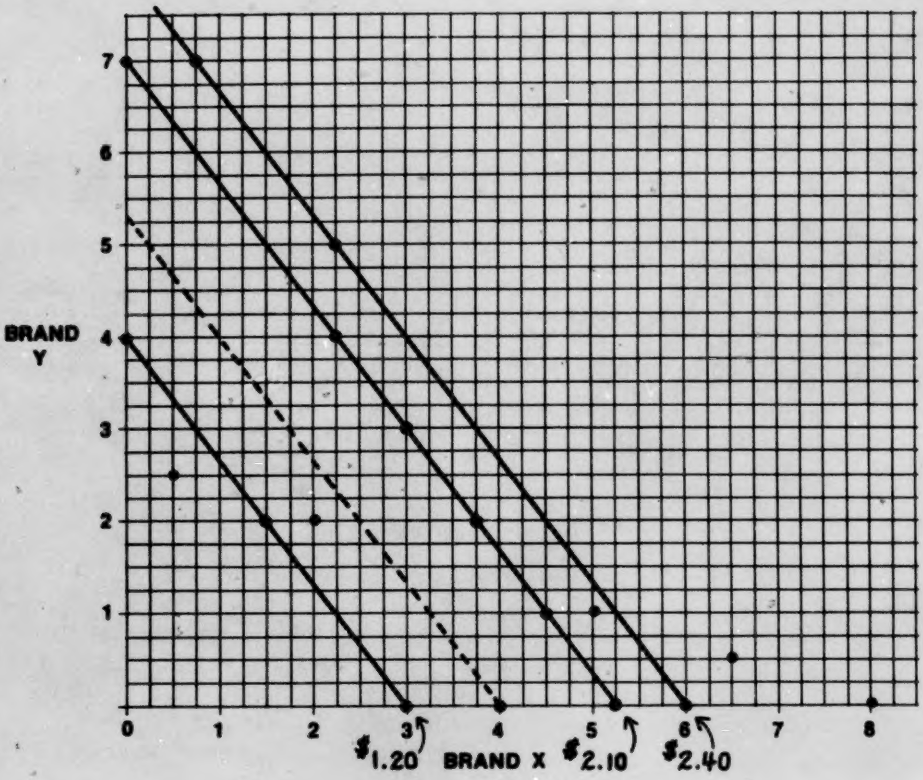
T: Now that we know something about the cost, let's look at the mixtures that satisfy the various minimum daily requirements (refer to the table). We can begin with carbohydrates. Give me some points representing mixtures that exactly satisfy the MDR for carbohydrates.

S: (5, 1), because (5 × 1) + 3 = 8.

S: (2, 2), because (2 × 1) + (2 × 3) = 8.

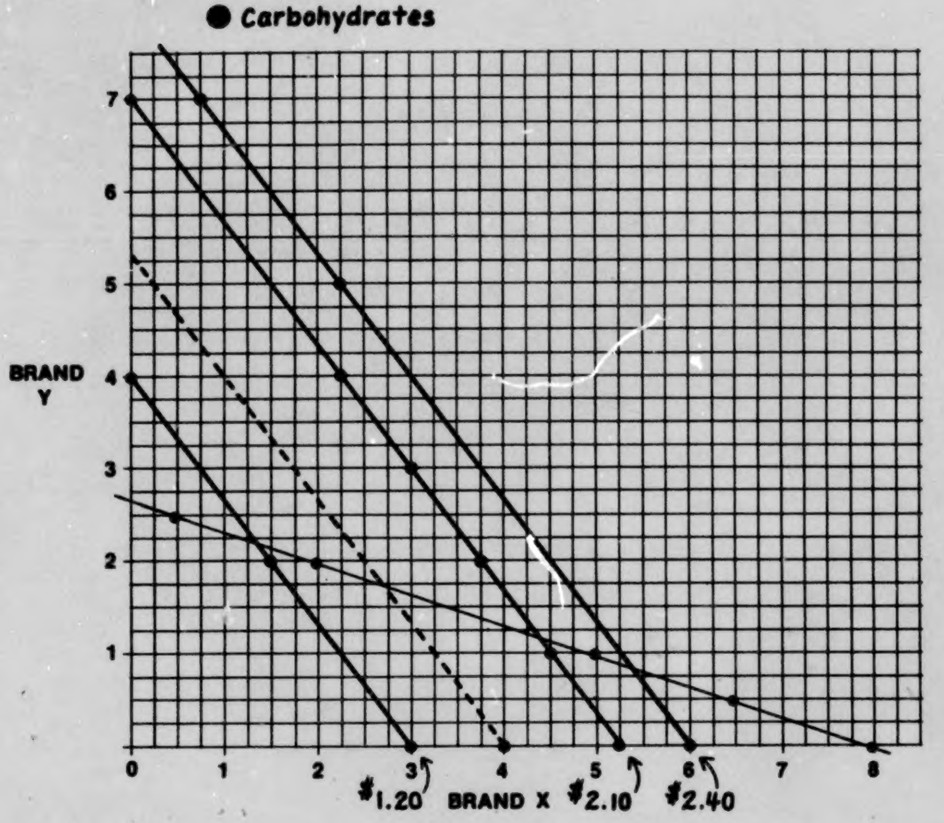
T: Good. We'll mark those points on the grid in red.

Continue this activity until there are four or five red dots.



S: The red dots are in a line.

Draw the carbohydrate line in red.



T: What can we say about points on this red line?

S: They are for mixtures that provide exactly 8 units of carbohydrates.

T (sweeping the appropriate region with your hand): What can we say about points above this red line?

S: Those points are for mixtures that provide more than 8 units of carbohydrates.

T: What about the points below the red line?

S: They are for mixtures that provide less than 8 units of carbohydrates.

N 37

T: What information do we get from the red line crossing the cost lines?

S: The point where the red line crosses a cost line is a mixture with cost given by the cost line and with 8 units of carbohydrates.

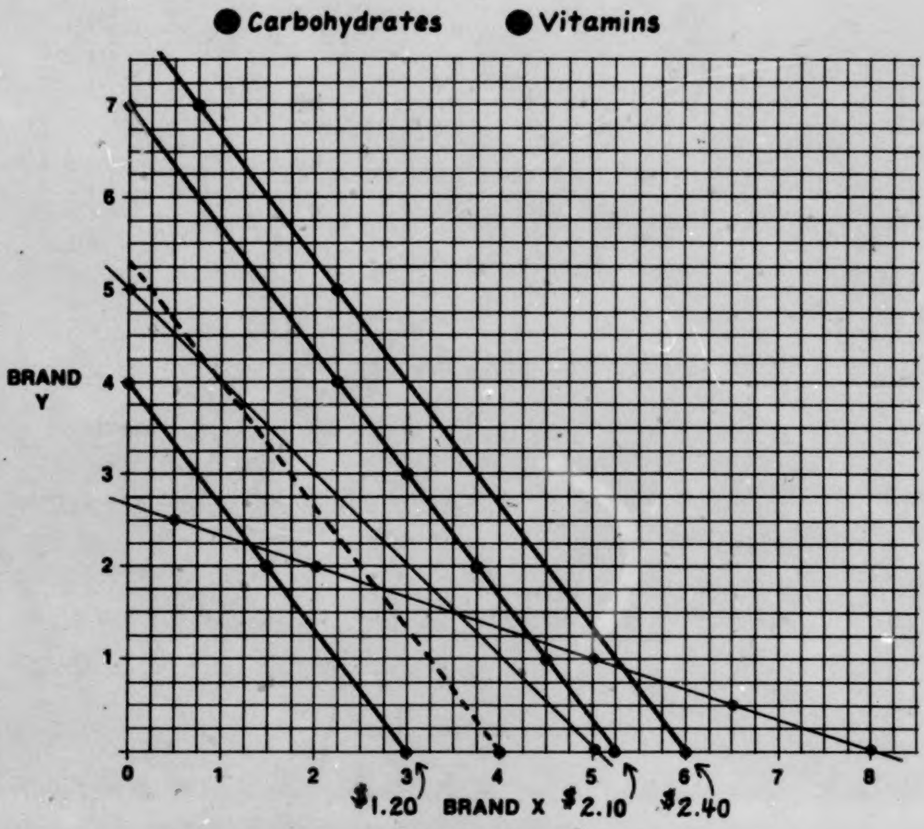
T: Let's now consider the MDR for vitamins. Give me some points for mixtures which provide exactly 20 units of vitamins?

If necessary, remind the class that now you are considering vitamins only with no restrictions on carbohydrates or protein.

S: (5, 0) or (0, 5), because either five scoops of Brand X or five scoops of Brand Y alone provide 20 units of vitamins.

S: You can draw a line through those two points.

Draw the vitamin line in blue.



S: The red line and the blue line cross.

T: Yes, what does it mean?

S: The crossing point, (3.5, 1.5), is for a mixture that provides exactly 8 units of carbohydrates and 20 units of vitamins.

T: Very good; what about the points that are above the red line and above the blue line?

S: They are for mixtures which provide more than the MDR for both carbohydrates and vitamins.

T: What do we have to consider next?

S: A protein line.

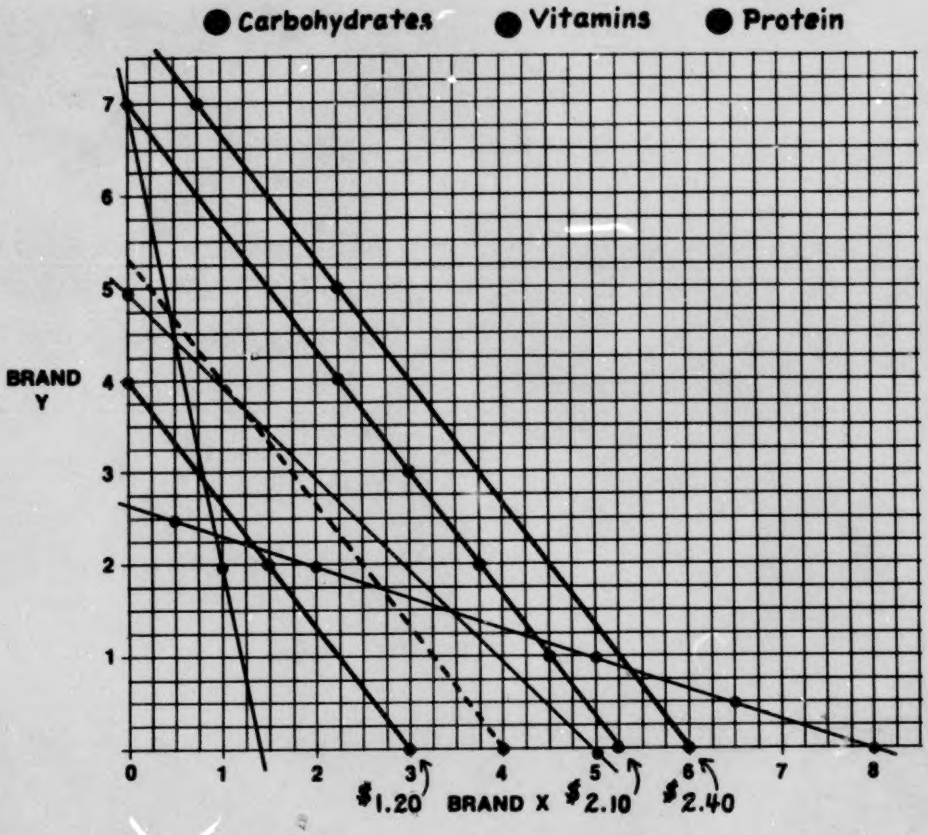
T: Give me some points for mixtures which provide exactly 7 units of protein.

S: (0, 7).

S: (1, 2).

N 37

Locate the points and draw the protein line in green.



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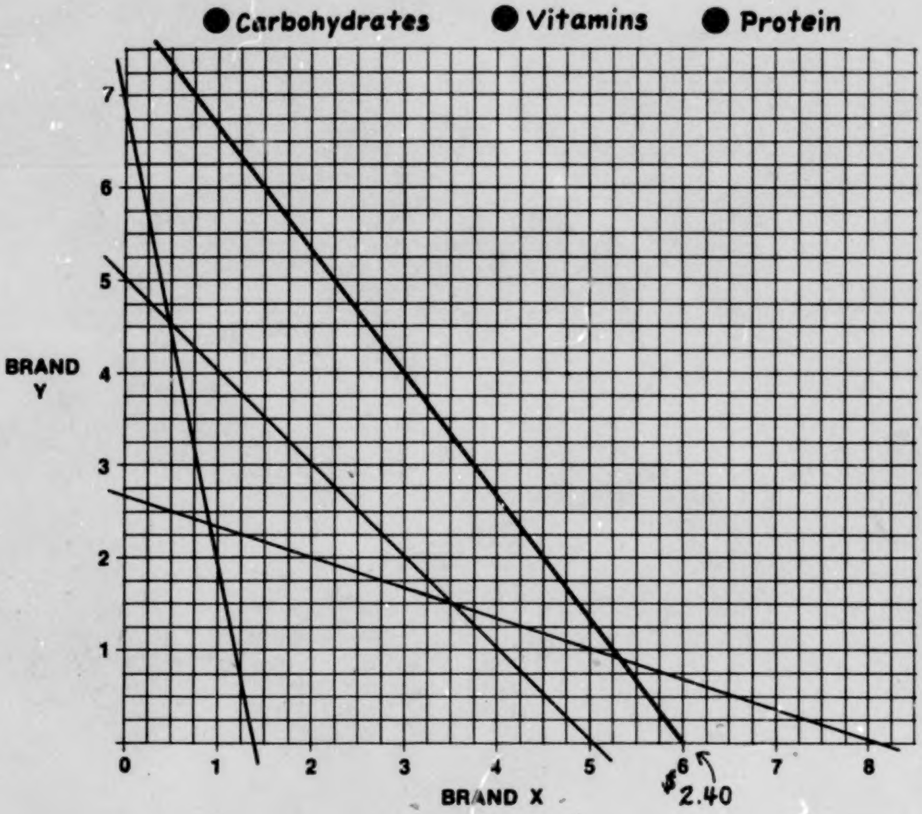
If in the process of getting the three nutrient lines on your poster the picture has become messy or hard to read, display World of Numbers Poster #4. This poster has the carbohydrate line, the vitamin line, and the protein line on it. In addition, the $2.40-line is on it for reference.

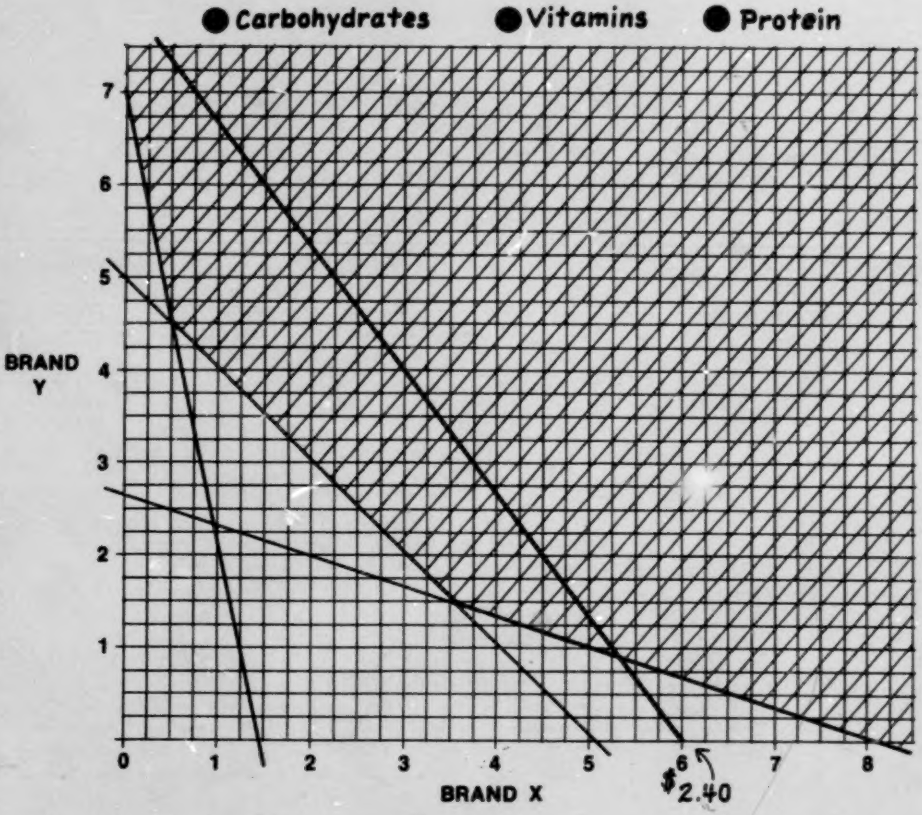


T: Where are the points for mixtures which satisfy (possibly in excess) all of the minimum daily requirements?

Invite a student to indicate where such points are on the poster.

N 37

S: All of the points that are above the red line and above the blue line and above the green line including the border.



T: Where is the point for a mixture that satisfies the MDR and is least expensive?

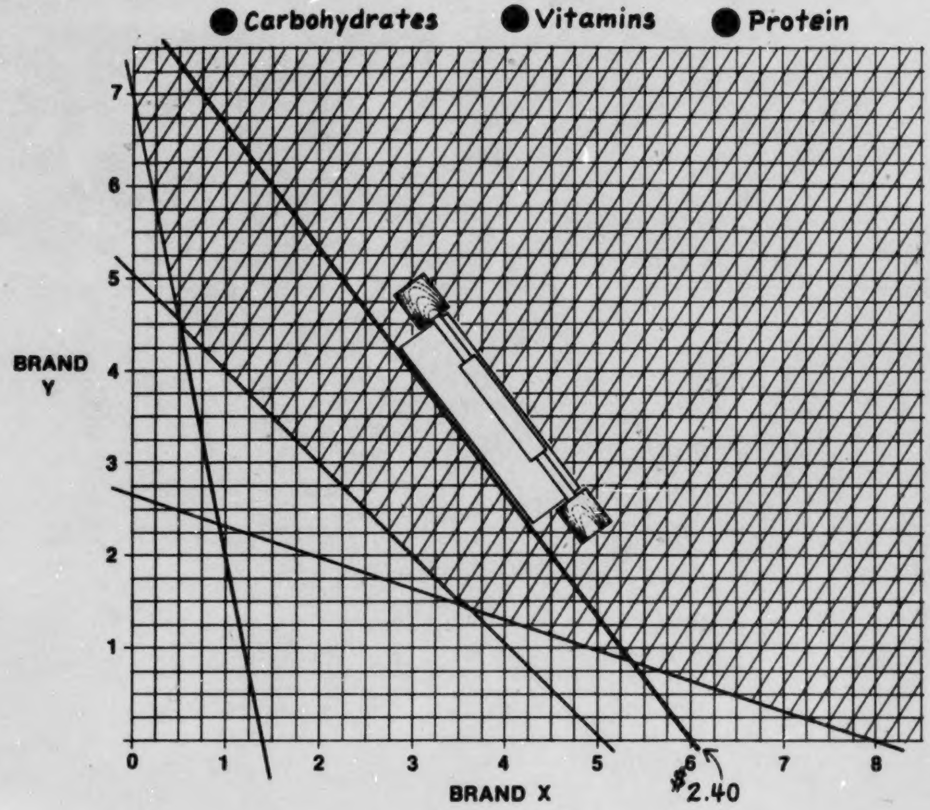
Check any suggestion that is made.

S: (3.5, 1.5); where the blue line and the red line cross.

S: That mixture costs $1.85 and it meets all the requirements.

T: Do you think that is is the least expensive? Let's check.

Place your translator on the $2.40-line as shown below.



T: What happens when I roll the translator down toward 0?

S: It indicates lines for smaller costs.

S: We must find the lowest point which still is in the region for mixtures satisfying the MDR.

Roll the translator to go through any points that have been suggested.

S: The lowest point is where the green line crosses the blue line.

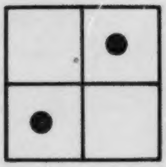
T: Yes; what is the cost of that mixture?

S: $\frac{1}{2}$ scoop Brand X and $4\frac{1}{2}$ scoops Brand Y; that's $1.55.

APPENDIX: STANDARD CONFIGURATION AND STANDARD MINICOMPUTER MOVES

A number can be represented by putting checkers[†] on the squares of the Minicomputer. Each square has a numerical (positional) value, so that the number represented by a checker depends on which square the checker is placed. If several checkers are put on the Minicomputer, the number shown is the sum of the numbers represented by the checkers. A number can be put on the Minicomputer in a variety of ways, but the representation which uses at most one checker on each square and uses checkers to represent a digit 9 or less on each board is usually the easiest to read. In this case, we say that the number is in standard configuration. Standard configurations for the numbers 1-9 soon become as familiar to the students as the usual numerals so that they no longer need to do any mental calculation for such configurations.

For example,



. . . is seen as a name for 6 without doing the calculation "4 + 2 = 6".

Doing calculations on the Minicomputer involves putting one or more checkers on the various squares and making "plays" to get a specific configuration. In this context, a play is an exchange of checkers in a configuration which does not change the number represented. Some plays are natural to the design of the Minicomputer; we call these standard or usual plays (these are described in detail in this section). There are advantages to establishing such a set of plays:

[†] In this appendix we refer only to whole numbers and, therefore, use only regular (positive) checkers.

- They become routine, and a large group of students with varying abilities can follow a calculation without needing to verify every play which is made.
- They lead to standard configurations of numbers.

These are the standard Minicomputer moves.

1 + 1 = 2
10 + 10 = 20
100 + 100 = 200
1,000 + 1,000 = 2,000

two checkers on the white square
of a board
=
one checker on the red square
of the same board

2 = 1 + 1
20 = 10 + 10
200 = 100 + 100
2,000 = 1,000 + 1,000

one checker on the red square
of a board
=
two checkers on the white square
of the same board

2 + 2 = 4
20 + 20 = 40
200 + 200 = 400
2,000 + 2,000 = 4,000

two checkers on the red square
of a board
=
one checker on the purple square
of the same board

4 = 2 + 2
40 = 20 + 20
400 = 200 + 200
4,000 = 2,000 + 2,000

one checker on the purple square
of a board
=
two checkers on the red square
of the same board

4 + 4 = 8
40 + 40 = 80
400 + 400 = 800
4,000 + 4,000 = 8,000

two checkers on the purple square
of a board
=
one checker on the brown square
of the same board

8 = 4 + 4
80 = 40 + 40
800 = 400 + 400
8,000 = 4,000 + 4,000

one checker on the brown square
of a board
=
two checkers on the purple square
of the same board

8 + 2 = 10
80 + 20 = 100
800 + 200 = 1,000
8,000 + 2,000 = 10,000

one checker on each of the brown
and red squares of a board
=
one checker on the white square
of the next board to the left

10 = 8 + 2
100 = 80 + 20
1,000 = 800 + 200
10,000 = 8,000 + 2,000

one checker on the white square
of a board
=
one checker on each of the brown
and red squares of the next board
to the right[†]

[†] This provides the link that motivates the introduction of the decimal numbers. In particular, we have that one checker on the white square of the ones' board equals one checker on each of the brown and red squares of a special board (the tenths' board - or as it is often referred to in the lessons, the dimes' board) located to the right of the ones' board and separated from it by a special barrier.

WORKBOOKS

Comprehensive School Mathematics Program

CSMP Mathematics for the Intermediate Grades, Part VI

Workbooks

CEMREL, Inc.

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WORKBOOK

INTRODUCTION

There are many opportunities for the students to work individually during the course of the lessons described in the four content strands. In the workbook strand, however, it is this individualized work which becomes the chief end of most of the lessons. It has been our goal in this strand to provide the students with opportunities

- to review many of the ideas they have met in the content strands;
- to apply their acquired knowledge to new situations requiring various kinds of strategic thinking;
- to learn how to read and use mathematics workbooks and story-workbooks.

The W-strand includes the following workbooks.

Collage of Problems #1
Collage of Problems #2
Collage of Problems #3
Collage of Problems #4
Collage of Problems #5
Collage of Problems #6

Each of these six workbooks has problems of varying levels of difficulty all in one booklet. Approximately the first ten pages of each booklet are easy problems, the next ten to twelve pages are average problems, and the last ten pages are challenging problems.

The W-strand also includes the following storybook, A VERY STRANGE NEIGHBORHOOD.

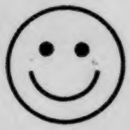
We suggest that you devote one day a week to this strand as indicated on the suggested schedule of lessons given in the General Introduction. We also recommend that you follow this schedule more or less as given, even if many of your students have not finished a particular workbook after the two periods that are allotted to it. As a general rule, students at this age do not enjoy remaining on the same topic for too long. It is a basic principle of the CSMP spiral approach that a variety of experiences is preferable to an attempt to master a given topic by continued "plugging away" at it. This principle applies as much (if not more so) to the Workbook strand as it does elsewhere in the curriculum. If you do feel, however, that more time is needed for a particular workbook, you could allow students to continue their work on that workbook during one (or more) of the adjustment days or during their free time.

The placement of the various workbooks in the suggested schedule of lessons was made keeping in mind where lessons from the other content strands occur. The types of problems in each workbook reflect topics covered in earlier lessons and assume information has previously been introduced. Hence, you are warned to be particularly careful when considering possible modification of this suggested schedule. Whatever changes you might like to make, we strongly urge that you use one day each week for workbooks.

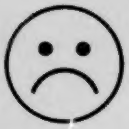
A collective lesson accompanies the storybook, A VERY STRANGE NEIGHBORHOOD. There are also short collective (warm-up) lessons suggested for some of the "Collage of Problems" workbooks. Following any such suggestions for a particular workbook, there is an answer key for that workbook. All answer keys are presented in reduced format, each page usually containing reduced copies of four of the pages of the original booklets, with solutions written in.

For each workbook, we suggest that all students start work at the easiest level of difficulty and then work through as many pages as they can handle during the two scheduled lessons. You can expect that, under normal circumstances, about two-thirds of the students will work through between 15 and 25 pages, and a few will reach the most challenging problems at the end of the booklet.

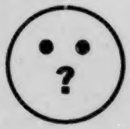
The workbooks provide an excellent instrument with which you can assess the progress of your students on a week-by-week basis. To use this instrument effectively certainly does not require that you check every page and every problem for each student. You should develop the sort of procedure for checking students' work with which you are most comfortable. For example, you might check one or more specific pages, discuss some particular mistakes with individual students, and let them correct their work before continuing. Or you might simply look carefully at the first few pages of the workbooks to see that the students understand the general idea of the problems. In our experience, an effective procedure is to indicate on the front cover of each checked workbook an assessment of the students' work on each attempted page. For this purpose, you might use the following symbols.



meaning "well done" or "correct"

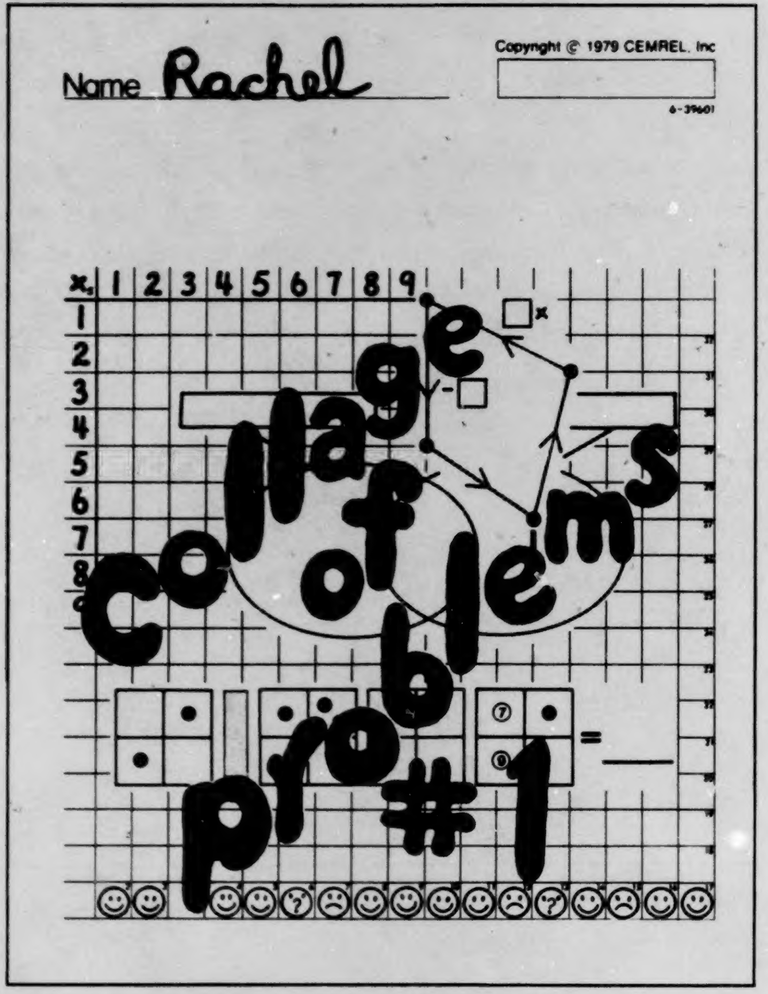


meaning "something wrong, try again"



meaning "did you forget this problem?" or "do you need a hint?"

On the next page we show the front cover of one student's Collage of Problems #1 workbook after it has been checked.


Name Rachel
Copyright © 1979 CEMREL, Inc
6-39601
Collage of problems #1

NOTE: This is the way Rachel's workbook was returned to her at the beginning of the second lesson devoted to this workbook. The blanks in spaces 18-32 indicate that she had not yet worked on these pages. Before Rachel begins on these pages during the second lesson, she should correct her mistakes on pages 7, 12, and 15 and finish her work on pages 6 and 13.

We suggest that you maintain a class record to keep track of

- where each of your students is on any particular workbook day;
- how many pages each student has completed under each title;
- your personal degree of satisfaction with a student's work on each workbook.

In particular you should be on the watch to see if, for any of your students, even the easiest problems in the workbooks are too difficult. Such students probably would benefit a great deal if you were able to devote some time with them on the activities suggested in the books "Math Play Therapy, Volumes I and II".

Here is a list of some of the important points to bear in mind when working in this strand.

- All students start at the beginning of each workbook and progress as far as they can.
- All students begin a workbook on the same day, even if some students have not finished work on the preceding workbook.
- Not all students are expected to complete a given workbook. Only the faster students will reach the most challenging problems. The slower students may succeed to do only the easiest problems although you should not assume that this will automatically be the case — surprises are not at all uncommon in this area.
- Read the introductory material for each workbook and storybook and give a full group lesson when appropriate.

Workbook Progress Chart

Name	Collage of Problems #1	Collage of Problems #2	Collage of Problems #3

Workbook Progress Chart

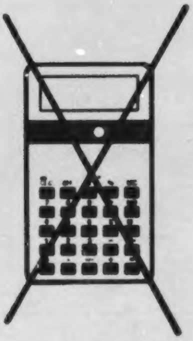
Name	Collage of Problems #4	Collage of Problems #5	Collage of Problems #6

COLLAGE
OF
PROBLEMS
#1

COLLAGE OF PROBLEMS #1

The problems in this workbook are in order of increasing difficulty. Approximately the first ten pages of the workbook have easy problems, the next ten to twelve pages have average problems, and the last ten pages have more challenging problems. It is suggested that all students work through as many pages as they can handle during the two lessons scheduled for the workbook. Not all students are expected to complete the entire workbook; most should complete at least ten pages, many should finish about twenty pages, and only the faster students will reach the most challenging problems at the end of the booklet.

Many problems in the workbook were designed with the availability of hand-calculators in mind. If possible, provide all the students with hand-calculators or at least give students access to hand-calculators.[†] You will find that many students do not need or want to use calculators; however, some students still need the support of the calculator in thinking about these problems. The problems on some pages are to be solved without using a hand-calculator. These pages are indicated by the following symbol.



Begin the first workbook lesson with the brief collective lesson described on the following pages. This should be a brief warm-up activity to review the division algorithm.

[†] See Section 2(f) of the General Introduction for additional comments on the use of hand-calculators and on desirable characteristics of hand-calculators.

W1

Lesson 1

Tell the following story to your class.

T: Mr. Connors works for a newspaper office. Every Sunday morning he distributes papers to 17 boys and girls who sell papers on the street corners. He gives each seller the same number of papers. One Sunday he has 2,950 papers. How many papers does he give to each seller? What could we do to answer this question?

S: Divide 2,950 by 17.

Record the division problem on the board.

$$17\overline{)2{,}950}$$

Invite students to tell you how to solve this division problem. (A division algorithm is described in Lessons N6.3 and N6.5 of IG-V.) The problem could be solved in the following way.

173 R = 9

$17\overline{)2{,}950}$	
$-1{,}700$	100
$1{,}250$	
-680	40
570	
-510	30
60	
-51	3
9	173

These points could be made concerning this algorithm:

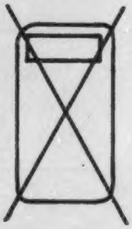
- Estimates ending in zero(s) (e.g., 100, 40, 30) are usually used because it simplifies the required multiplications.
- The most "efficient" estimates need not always be used; for example, in the above problem it is not wrong to use 40 and 30 even though 70 would be more efficient.
- Use the story line as the problem is being solved; for example, "Mr. Connors first decides to give 100 newspapers to each seller. Can he do that? (Yes) How many newspapers does that require? (17 x 100 = 1,700) How many papers would he have left? (2,950 - 1,700 = 1,250)."

Write the following problems on the board and ask the students to solve them. Answers are in the boxes.

$$\begin{array}{r} \boxed{38 \quad R=2} \\ 21\overline{)\,800} \end{array} \qquad \begin{array}{r} \boxed{207 \quad R=27} \\ 39\overline{)\,8,100} \end{array}$$

Solve one or both of the problems collectively only if many students are having difficulty. If only a few students find these problems difficult, briefly help them individually or in a small group after the workbooks are distributed.

Distribute individual copies of "Collage of Problems #1". Draw the following symbol on the board.



T: If you see this symbol on a page, it means that you are not allowed to use a hand-calculator to solve the problems on that page. The symbol first occurs on page 5.

Instruct students to work on their own. If you observe many students struggling with the same problem (or part of a problem), you could have a short collective discussion about that problem.

Collect the workbooks at the end of the lesson. Check the students' work and have the workbooks ready to return for Lesson 2.

Lesson 2

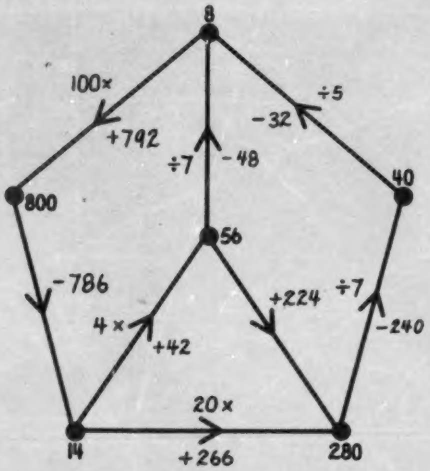
Return the students' copies of "Collage of Problems #1". You may wish to discuss any page that was difficult for most of the students. Then ask the students to correct their mistakes, if any, and to continue working in their workbooks.

If you observe many students struggling with the same problem (or part of a problem), you could have a short discussion of that problem.

At the end of this lesson, collect the workbooks for your progress check. After checking the workbooks, you may wish to ask some students to work further during any free study time or as a homework assignment.

Collage of Problems #1

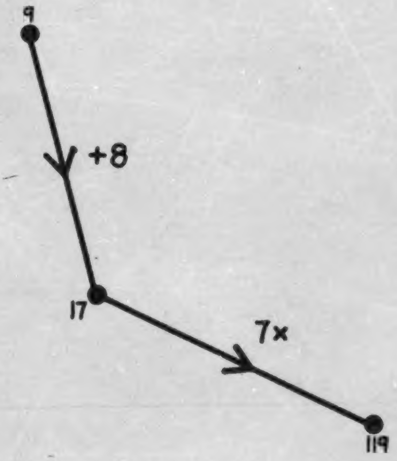
Label the red arrows: try to label some arrows in two ways.


8
100×
+792
÷5
−32
÷7
−48
800
40
56
−786
4×
+42
+224
÷7
−240
20×
14
+266
280

2

(Other solutions are possible.)

Build a road from 9 to 119. Each arrow must be for +☐, −☐, ×☐, or ÷☐, where exactly one of the digits: 1, 2, 3, 4, 5, 6, 7, 8 or 9 is in a box. Build as short a road as you can.



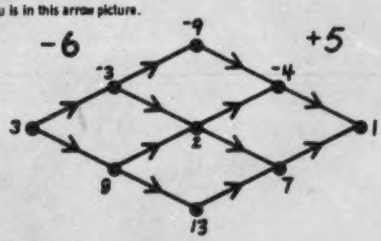
3

(Many solutions are possible; however, a shortest road has two arrows.)

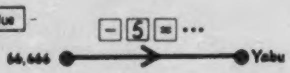
Yabu is a secret number.

First clue

Yabu is in this arrow picture.

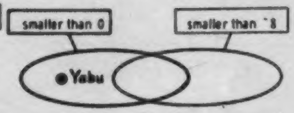

−6
+5
⁻9
⁻3
⁻4
3
2
1
9
7
13

Second clue –


− 5 = …
66,666
Yabu

Yabu could be 1, ⁻4, or ⁻9.

Third clue

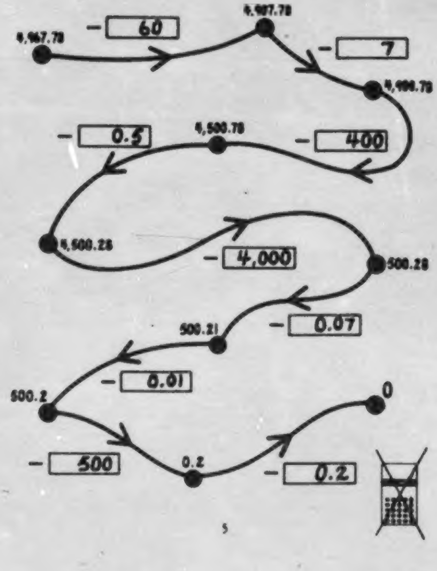

smaller than 0
smaller than ⁻8
Yabu

Yabu is ⁻4.

4

WIPE-OUT

Fill in the boxes for the arrows.


9,967.78
− 60
9,907.78
− 7
9,900.78
− 0.5
9,500.78
− 400
9,500.28
− 4,000
500.28
500.21
− 0.07
− 0.01
500.2
0
− 500
0.2
− 0.2

5

Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculations correct. You may use a digit more than once.

Add

$$\begin{array}{r} 1\,\boxed{3}\,9\,5 \\ 3\,3\,\boxed{4} \\ +\,\boxed{4}\,4\,8\,8 \\ \hline 6\,2\,\boxed{1}\,7 \end{array}$$

Add

340 + 97 + 16,823

17,260

Subtract

$$\begin{array}{r} 7\,\boxed{9}\,8\,6 \\ -\,\boxed{3}\,2\,\boxed{2}\,7 \\ \hline 4\,7\,5\,\boxed{9} \end{array}$$

Subtract

76,092 − 1,459

74,633

Complete each of these tables.

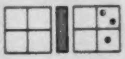
$+_5$	7	13
9	1	2
23	0	1

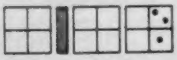
$\times_5$	4	8
7	3	1
9	1	2

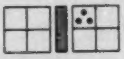
$T_<$	20	$^-4$
8	1	0
$^-8$	1	1

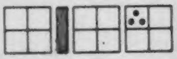
T_D	42	63
7	1	1
9	0	1

Complete.

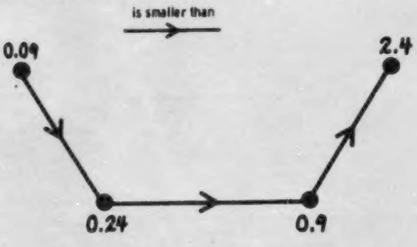

= 0.9


= 0.09


= 2.4


= 0.24

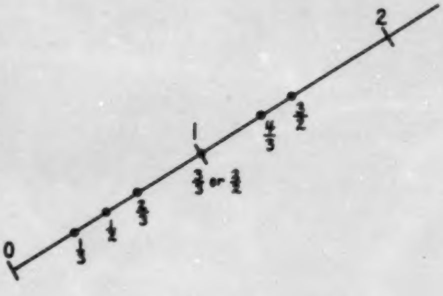
Label these dots with the four numbers above.



Using a ruler, draw a dot for each of these numbers on the number line.

$\frac{1}{3}$ $\frac{2}{3}$ $\frac{3}{3}$ $\frac{4}{3}$

$\frac{1}{2}$ $\frac{2}{2}$ $\frac{3}{2}$



Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculation correct. You may use a digit more than once.

Multiply

$$\begin{array}{r} \boxed{7}\,6\,\boxed{8} \\ \times \quad 3 \\ \hline 2\,3\,\boxed{0}\,4 \end{array}$$

Divide

$\boxed{2}\boxed{6}$ R-$\boxed{1}\boxed{4}$

$43\overline{)1137}$

$\boxed{8}\boxed{6}0 \mid \boxed{2}0$

$\boxed{2}7\boxed{7}$

$\boxed{2}5\boxed{8} \mid \boxed{6}$

$\boxed{1}\boxed{9} \mid \boxed{2}\boxed{6}$

Complete.

Multiply

$$\begin{array}{r} 8\,4 \\ \times 4\,7 \\ \hline 3{,}9\,4\,8 \end{array}$$

Divide

350 R-27

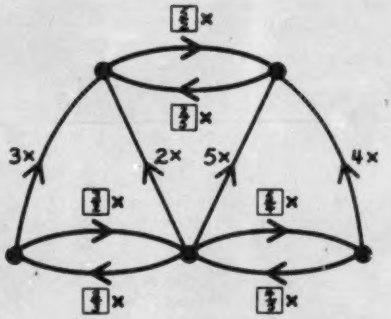
$76\overline{)26{,}627}$

Find all the places in this table for +, where the entry is 6. One is located for you.

+	1	2	3	4	5	6	7	8	9
1					6				
2				6					
3			6						
4		6							
5	6								
6									
7									6
8								6	
9							6		

Did you find the other seven places where the entry is 6? Yes

Label the blue arrows.



Complete these calculations.

$\frac{3}{2} \times 18 = \underline{27}$ $\frac{2}{5} \times 30 = \underline{12}$

$\frac{2}{3} \times 18 = \underline{12}$ $\frac{5}{2} \times 30 = \underline{75}$

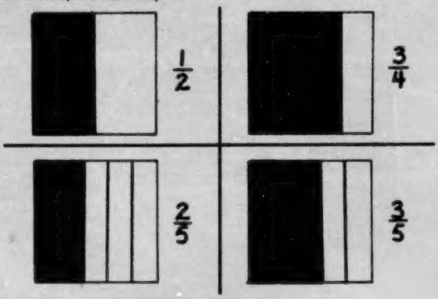
TIMELY QUESTIONS

Write your calculations for each problem in the space provided.

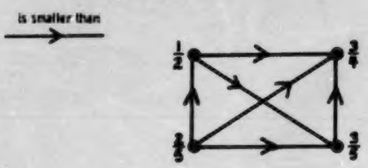
1. Ms. Shorsham works $7\frac{1}{2}$ hours each day and also has a 40 minute lunch break. If she begins work at 7:45 AM, what time does she finish? 3:55 PM (Remember to indicate AM or PM.)

2. If your age were 11 years and 11 months and your mother's age were 34 years and 2 months, how old was your mother when you were born? 22 years, 3 months

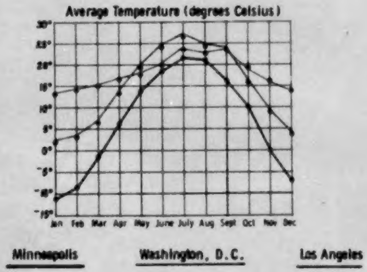
Color in red the indicated fractional part of each square. Use a ruler to divide the squares accurately.



Draw all the possible red arrows between these numbers.



(Other colorings are possible.)



What is the average February temperature in Minneapolis? −8° C

How much colder is Minneapolis than Los Angeles in September? 7°C
in January? 24°C

For how many months is Washington, D.C.'s average temperature below 10°C? Five

In which month is Minneapolis's average temperature closest to Los Angeles's?
June or August How close? 1°C

In which months is Los Angeles at least 10°C warmer than Washington, D.C.?
Jan, Feb, Dec

Fill in the boxes.

$$\frac{1}{2} = \frac{7}{14} = \frac{\boxed{4}}{8} = \frac{3}{\boxed{6}} = \frac{\boxed{5}}{10} = \frac{13}{\boxed{26}}$$

$$\frac{3}{5} = \frac{9}{15} = \frac{\boxed{6}}{10} = \frac{15}{\boxed{25}} = \frac{\boxed{30}}{50} = \frac{21}{\boxed{35}}$$

Complete. You may use the different names for $\frac{1}{2}$ and $\frac{3}{5}$ at the top of this page.

$$\frac{3}{5} + \frac{1}{2} = \underline{\frac{11}{10}} \text{ or } 1\frac{1}{10}$$

$$\frac{3}{5} - \frac{1}{2} = \underline{\frac{1}{10}}$$

HOW OLD?

Lillian is 8 years older than Gina. The sum of their ages is 30 years.
How old is Lillian? 19 Gina? 11

19 + 11 = 30
19 − 11 = 8

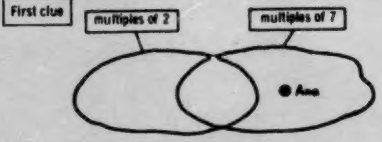
HOW RICH?

Andre has exactly ten coins in his pocket. He has only dimes and quarters.
If he has $2.05, how many quarters does he have? 7 dimes? 3

7 × \$0.25 = \$1.75   \$1.75 + \$0.30 = \$2.05
3 × \$0.10 = \$0.30

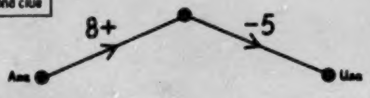
Ana is a secret number smaller than 70.

First clue


multiples of 2
multiples of 7
Ana

Ana could be 7, 21, 35, 49, or 63.

Second clue

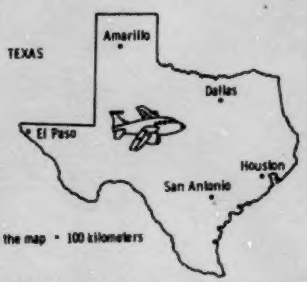


(Ana, Una) could be (7, 10), (21, 24), (35, 38), (49, 52), or (63, 66).

Third clue

↑	40	Una
Ana	40	Una
30	40	Una

(Ana, Una) is (35, 38)


TEXAS
Amarillo
Dallas
El Paso
Houston
San Antonio

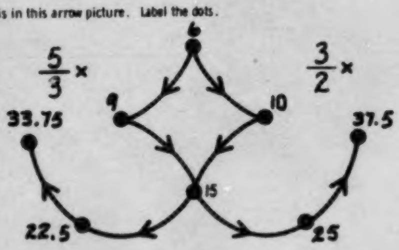
SCALE: 1 cm on the map = 100 kilometers

1. On the map, what is the length of a line segment between El Paso and San Antonio? 8 cm
   What is the real distance between El Paso and San Antonio? 800 km
2. On the map, what is the length of a line segment between Dallas and Houston? 3.5 cm
   What is the real distance between Dallas and Houston? 350 km
3. If an airplane flies 600 kilometers in 1 hour, about how long should the flying time be from El Paso to San Antonio? (Circle the closest answer.)
   50 minutes [✓ 1 hour 20 minutes] 1 hour 50 minutes
4. If an airplane flies 400 kilometers in 1 hour, about how long should the flying time be from Dallas to Houston? (Circle the closest answer.)
   [✓ 50 minutes] 1 hour 10 minutes 1 hour 30 minutes

Vic is a secret number.

First clue

Vic is in this arrow picture. Label the dots.

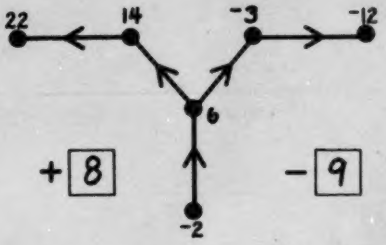

$\frac{5}{3}\times$
$\frac{3}{2}\times$
6
9
10
15
33.75
37.5
22.5
25

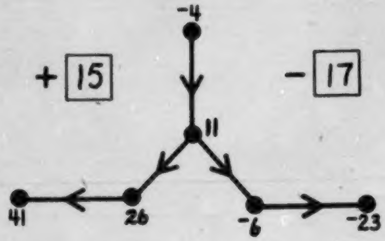
Second clue

$\text{Vic} \downarrow 30 = \text{Vic}$ $\text{Vic}\ T_c\ 23 = 0$

Who is Vic? 25

Fill in the boxes for the arrows. Label the dots.

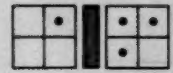

22
14
−3
−12
6
−2
+ 8
− 9


−4
+ 15
− 17
11
41
26
−6
−23

Dano is a secret number.

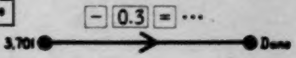
First clue

Dano can be put on this Minicomputer by adding just a ②-checker.



Dano could be 5.6, 5.8, 6.2, 7.0, 7.4, 9.4, 13.4, or 21.4

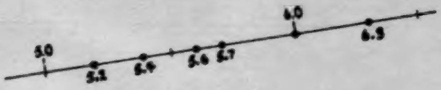
Second clue


−
0.3
= ...
3,701
Dano

Dano could be 5.6, 6.2, 7.4, or 13.4.

Third clue

Dano is one of the dots on this number line.


5.0
5.2
5.4
5.6
5.7
6.0
6.3

Dano is 5.6.

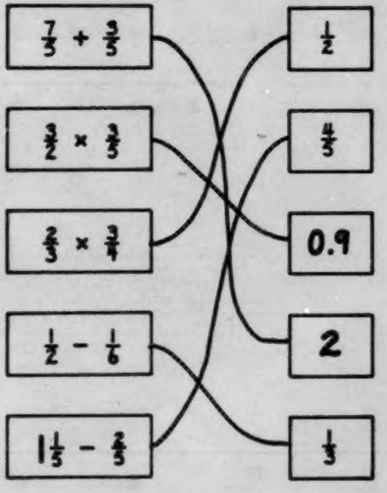
A city council must choose a committee of three people from the five eligible members: Arlene, Bret, Carla, Dinah, and Ed. The selection is not easy due to the many friendships and jealousies among these people.

Arlene will serve on the committee with anyone.
Bret won't serve on the committee with Arlene.
Carla will serve on the committee only if Bret and Dinah also serves.
Dinah won't serve if Arlene or Carla also serves.
Ed won't serve unless Dinah also serves.

Can you select a committee of three people so that everyone is satisfied?

Who is on your committee? Bret, Dinah, Ed

Pair the tags.


$\frac{7}{5}+\frac{3}{5}$
$\frac{3}{2}\times\frac{3}{5}$
$\frac{2}{3}\times\frac{3}{4}$
$\frac{1}{2}-\frac{1}{6}$
$1\frac{1}{5}-\frac{2}{5}$
$\frac{1}{2}$
$\frac{4}{5}$
0.9
2
$\frac{1}{3}$

Each table is for one of these operations.

THE TABLE GAME

$+$	$+_3$	$+_5$	$+_8$
$-$	$-_3$	$-_5$	$-_8$
$\times$	$\times_3$	$\times_5$	$\times_8$
↑	↓	⊓	⊔
$T_=$	$T_{\neq}$	$T_<$	$T_>$
	$T_{\equiv 3}$	$T_{\equiv 4}$	

Label the tables.

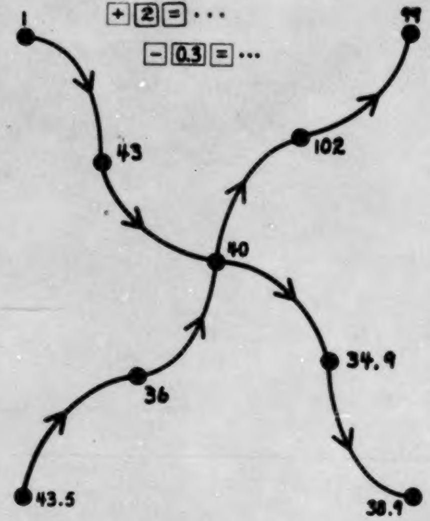
$T_{\equiv_4}$	48	35
7	0	1
6	0	0

⊓	10	12
21	1	3
16	2	4

$\times_8$	3	8
7	1	1
10	0	0

$-_5$	5	28
9	4	1
17	2	4

Label the dots. Many answers are possible.

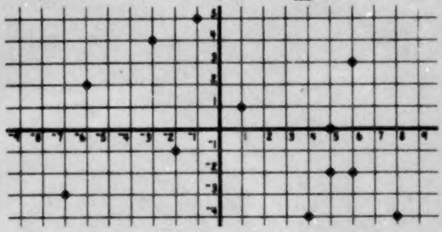

+ 2 = ...
− 0.3 = ...
1
99
43
102
40
34.9
36
43.5
38.9

(Many solutions are possible.)

Tim and Tam are secret numbers.

First clue

(Tim, Tam) is one of the dots on this grid. You do not have to label the dots.

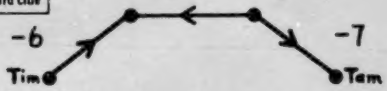


Second clue

Tim + Tam > 3

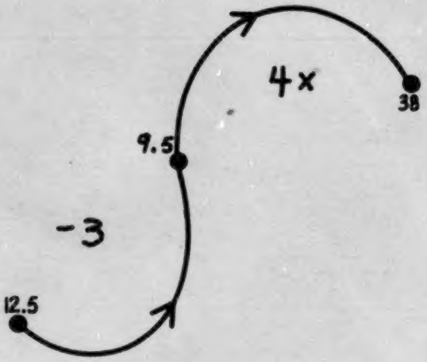
(Tim, Tam) could be ($^{-}1$, 5), (5 , 0), (6 , 3), (6 , $^{-}2$)
or (8 , $^{-}4$).

Third clue



(Tim, Tam) is (6 , $^{-}2$).

Build a road from 12.5 to 38 with two arrows. Each arrow must be for +☐, −☐, x☐, or ÷☐, where exactly one of the digits: 1, 2, 3, 4, 5, 6, 7, 8, or 9, is in a box.

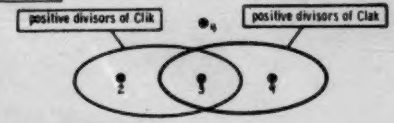

4x
38
9.5
−3
12.5

Clik and Clak are secret whole numbers.

First clue

Clik and Clak are smaller than 50.

Second clue


positive divisors of Clik
positive divisors of Clak
4
2
3
9

Clik could be 6 , 30 , or 42 .

Clak could be 9 , 27 , or 45 .

Third clue

Clik $+_{i}$ Clak = 4

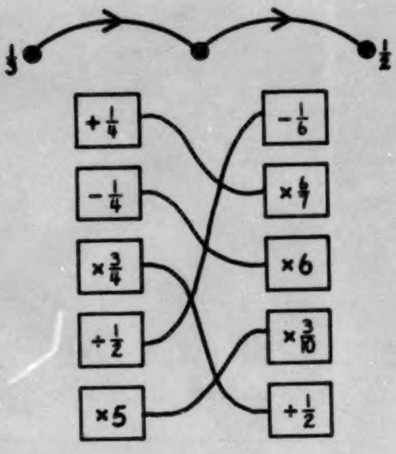
(Clik, Clak) could be (30 , 9) or (42 , 27).

Fourth clue

Clik $-_{i}$ Clak = 0

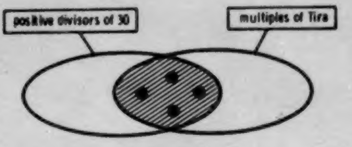
(Clik, Clak) is (42 , 27).

Pair the tags.


$\frac{1}{3}$
$\frac{1}{2}$
$+\frac{1}{4}$
$-\frac{1}{4}$
$\times\frac{3}{4}$
$\div\frac{1}{2}$
$\times 5$
$-\frac{1}{6}$
$\times\frac{6}{7}$
$\times 6$
$\times\frac{3}{10}$
$\div\frac{1}{2}$

Tira is a secret whole number.

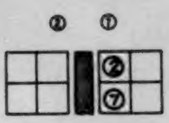
First clue


positive divisors of 30
multiples of Tira

Tira could be 2, 3, or 5.

Second clue

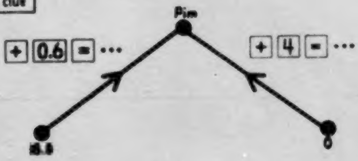
Tira can be put on this Minicomputer with exactly these two checkers.



Tira is 3.

Pim is a secret number.

First clue


Pim
+ 0.6 = ···
+ 4 = ···
18.8
0

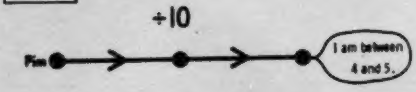
Pim could be 20, 32, 44, 56, 68, 80, 92, 104, and so on.

Second clue

$\text{Pim} =_{5} 7$

Pim could be 32, 92, 152, 212, and so on.

Third clue


÷10
Pim
I am between 4 and 5.

Pim is 452.

A VERY STRANGE NEIGHBORHOOD

A VERY STRANGE NEIGHBORHOOD

Your students are familiar with many different types of numbers and ways to name them; for example, 73, $^-24$, $\frac{3}{4}$, 8.517, and $0.\overset{*}{3}$. The Storybook A VERY STRANGE NEIGHBORHOOD introduces a new kind of number, namely the irrational numbers[†]. During the story, students use a number line to explore the ordering and density of decimal numbers.

DESCRIPTION OF LESSON

T: You already know several different kinds of numbers. Give me examples of some of the different kinds.

Students are likely to suggest examples of whole numbers, negative integers, and rational numbers. Write the students' examples on the board; for instance,

$$17 \qquad {}^-82 \qquad \frac{5}{7} \qquad 8.2$$

If no one gives a name such as "$0.\overset{*}{3}$" or "$5.1\overset{*}{\overline{462}}$", lead the class in this discussion.

Write "$\frac{1}{3}$" on the board.

T: Do you remember a name for $\frac{1}{3}$ which we said was a kind of decimal "nickname"?

S: 0.333... or $0.\overset{*}{3}$.

[†] For your information only, rational numbers are numbers which have fractional names; for example, $\frac{1}{3}$, $^-2\frac{3}{8}$, and 5.02 ($5\frac{2}{100}$) are rational numbers. Irrational numbers do not have fractional names although they lie on the (real) number line.

W 3

Write "$5.1\overset{*}{\overline{462}}$" on the board.

$$5.1\overset{*}{\overline{462}}$$

T: What does this notation mean?

S: The 462 repeats. You could write it as "5.1462462462 ...".

T: You already know many types of numbers. Today we will read a story about a new kind of number.

Distribute copies of the Storybook A VERY STRANGE NEIGHBORHOOD. Read the story with your class. Questions based on the mathematical ideas are suggested below. Use the questions carefully to enhance, not diminish, the spirit of the story. By keeping the discussions brief, you should finish the story in one lesson. At a later time your students may wish to discuss further some of the ideas.

Pages 1-5

Read pages 1-5 together.

T: What are some numbers between 7.7 and 7.8?

S: 7.74 and 7.79.

S: 7.715.

S: $7\frac{3}{4}$.

Page 6

Read page 6 together.

T: In your storybook, point to the mark on the number line for 7.73.

Check a few students to see that they are pointing to the correct mark.

Page 7

Read page 7 together.

T: Are there exactly nine numbers between 7.7 and 7.8?

S: No; there are many more, for example, 7.727 and 7.70622.

S: There are infinitely many numbers between 7.7 and 7.8.

Pages 8-9

Read pages 8-9 together.

T: Marks for many of the numbers between 7.7 and 7.8 are shown on the number line on page 9, but they are not labeled. Do you know some of the numbers these marks are for?

S: 7.732.

T: Where is 7.732 on the number line?

S: Between 7.73 and 7.74; the second mark above 7.73.

Let several students respond.

W 3

Pages 10-11

Read pages 10-11 together.

T: Could the story ever end?

S: No, you can always find more numbers between 7 and 8.

Pages 12-14

Read pages 12-14 together.

T: Look at the spiral and try to figure out its rule. What would be the next number if we continued the spiral?

S: 7.888888.

T: . . . and after that?

S: Then 7.8888882, 7.8888884, and so on.

T: Who can explain the rule?

Several correct explanations are possible, but "add a 2 to the last digit" is not correct.

S: Start at 7. Add 0.2 four times; add 0.02 four times; and so on.

S: Add 2 to the last digit each time, except when the last digit is 8. Then you write another "2" at the end so there is one more decimal place.

Page 15

Read page 15.

T: What is the rule in the upper part of this snake dance?

S: Write a "4" on the end of a number to get the next.

T: What is the rule for the lower part?

S: Take off a "6" from the end of a number to get the next.

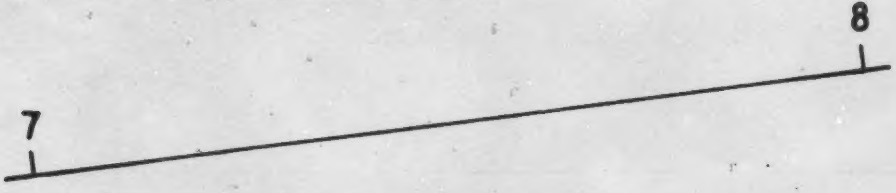
T: What is the smallest number in this snake dance?

S: 7.4.

T: What is the largest number?

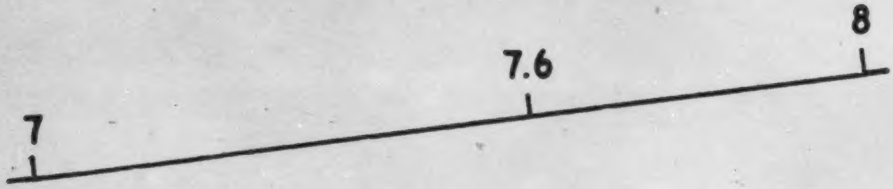
S: There is no largest number, although 7.666666666666 is the largest number shown on the page.

Draw the following picture on the board.

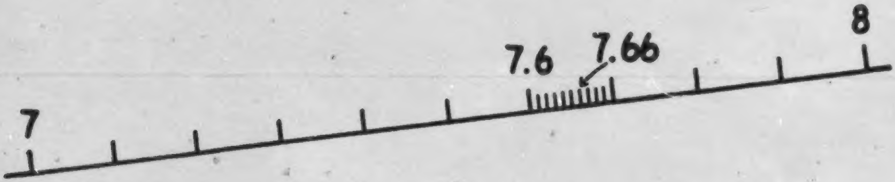


T: Who can point to where 7.6 is on this number line?

It will be helpful for the student to divide the segment from 7 to 8 into ten pieces of equal length.



T: Where is 7.66 on this number line?



T: Where is 7.666?

S: Just to the right of 7.66.

S: Between 7.66 and 7.67.

T: . . . and 7.6666?

S: Between 7.666 and 7.667.

T: These numbers, 7.6, 7.66, 7.666, and so on, are getting closer and closer to some number. That number has a fractional name. What number is it?

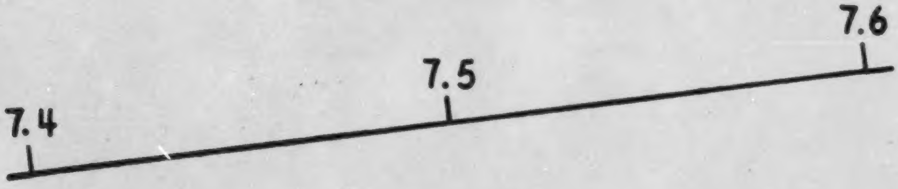
S: $7\frac{2}{3}$, because $0.666\ldots = 0.\overset{*}{6} = \frac{2}{3}$.

If no one suggests $7\frac{2}{3}$, remind the class that $0.3 = 0.333\ldots = \frac{1}{3}$ and ask about a fraction for $0.666\ldots = 0.\overset{*}{6}$. In this way, lead to the answer "$7.666\ldots = 7\frac{2}{3}$.

T: If we start at 7.6 on the spiral and follow the arrows backwards, we get closer and closer to $7\frac{2}{3}$.

Page 16

Read page 16 together and draw this part of the number line on the board.



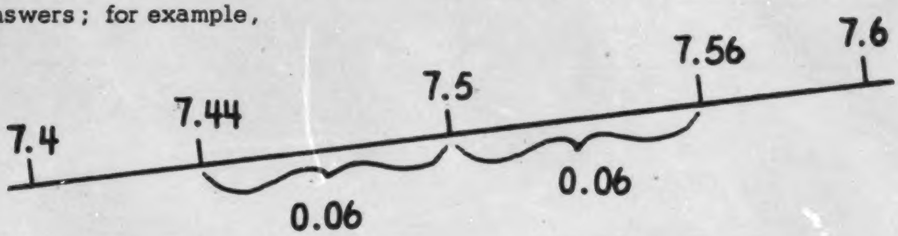
T: 7.5 is exactly halfway between 7.4 and 7.6. What are two other numbers that 7.5 is exactly halfway between?

S: 7.44 and 7.56.

S: 7.3 and 7.7.

S: 7 and 8.

Use the idea of distance from 7.5 to generate or confirm some of the answers; for example,



Depending on student interest, you may wish to challenge them to find pairs of numbers closer to 7.5 which have 7.5 as their midpoint; for example, 7.49 and 7.51, 7.495 and 7.505, 7.499 and 7.501.

Pages 17-24

Read pages 17-24 together.

W 3

T: This sad number is singing a strange song: "Bing, bang; bing, bing, bang; bing, bing, bing, bang; and so on." Which number could be sending this message?

Accept a few suggestions as to what the sounds could mean.

Pages 25-26

Read pages 25-26 together. Refer to the last number, 7.5155155515555, on the red arrow road.

T: What would be the next number on the red arrow road?

S: 7.51551555155551.

T: How can we get the next number?

S: Put five "5"'s on the end of the new number.

T: . . . and the next numbers?

S: Then put another "1" at the end.

S: Then put six "5"'s.

T: Who can explain the pattern?

S: You alternate between putting on one "1" and putting on a group of "5"'s. Each time you put on "5"'s you put on one more than the last time.

T: And we could keep on going. Could we use the "*" notation to write a short name for the number we get by going on and on?

After a few attempts, students are likely to conclude that it is impossible since no one part of the number is repeated; that is, one more "5" is used each time. Students might suggest some more elaborate abbreviations, for example, "$7.5\overset{+5}{\overline{155}}$". Accept these suggestions as possible notations, but mention that there is no standard notation for this type of number and that it would be impractical to invent a new notation for each non-repeating pattern that we could imagine.

T: This is an example of a new kind of number. It does not have a decimal "nickname" because when we try to use decimal writing there is no pattern that repeats.
Now can anyone explain the "bing-bang" song?

S: The number was trying to call its name. Each bing is for a "1" and each bang is for a "0".

Pages 27-30

Complete the story by reading pages 27-30 together.

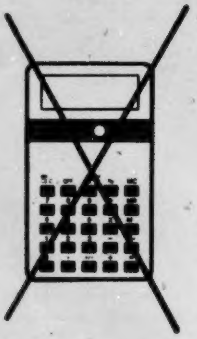
If time allows, invite students to suggest a few more examples of this new type of number.

COLLAGE
OF
PROBLEMS
#2

COLLAGE OF PROBLEMS #2

The problems in this workbook are in order of increasing difficulty. Approximately the first ten pages of the workbook have easy problems, the next ten to twelve pages have average problems, and the last ten pages have more challenging problems. It is suggested that all students work through as many pages as they can handle during the two lessons scheduled for the workbook. Not all students are expected to complete the entire workbook; most should complete at least ten pages, many should finish about twenty pages, and only the faster students will reach the most challenging problems at the end of the booklet.

Many problems in the workbook were designed with the availability of hand-calculators in mind. If possible, provide all the students with hand-calculators or at least give students access to hand-calculators.[†] You will find that many students do not need or want to use calculators; however, some students still need the support of the calculator in thinking about these problems. The problems on some pages are to be solved without using a hand-calculator. These pages are indicated by the following symbol.



Begin the first workbook lesson with the brief collective lesson described on the following pages. This brief warm-up activity introduces a new type of problem.

[†] See Section 2(f) of the General Introduction for additional comments on the use of hand-calculators and on desirable characteristics of hand-calculators.

W 4

Lesson 1

Write the following information and problems on the board. Answers are in the boxes.

$$< \qquad = \qquad >$$

$$1{,}658 + 749 \ \boxed{<}\ 926 + 1{,}658$$

$$311 - 76 \ \boxed{<}\ 311 - 47$$

$$\frac{1}{5} \ \boxed{>}\ \frac{1}{20}$$

$$568 + 87 \ \boxed{=}\ 68 + 587$$

T (pointing to "< ", "=", and ">", in turn): One of these symbols belongs in each box. Can you determine the symbol needed to make each number story correct? Do not use pencil and paper. Try to solve each problem without doing any calculation.

Invite students to solve the problems and explain their answers.

S: 1,658 + 749 is less than 926 + 1,658 because 749 is less than 926 and each number (749 and 926) is being added to 1,658.

S: 311 - 76 is less than 311 - 47. Each number, 76 and 47, is being subtracted from 311. Since 76 is larger than 47, you take away more if you subtract 76. So 311 - 76 is less than 311 - 47.

S: $\frac{1}{5}$ of something is larger than $\frac{1}{20}$ of the same thing, so $\frac{1}{5}$ is larger $\frac{1}{20}$.

S: 568 + 87 equals 68 + 587 since in each case you could add "500 + 68 + 87".

Distribute individual copies of "Collage of Problems #2" and instruct students to work on their own. If you observe many students struggling with the same problem (or part of a problem), you could have a short collective discussion about that problem.

Collect the workbooks at the end of the lesson. Check the students' work and have the workbooks ready to return for Lesson 2.

Lesson 2

Return the students' copies of "Collage of Problems #2". You may wish to discuss any page that was difficult for most of the students. Then ask the students to correct their mistakes, if any, and to continue working in their workbooks.

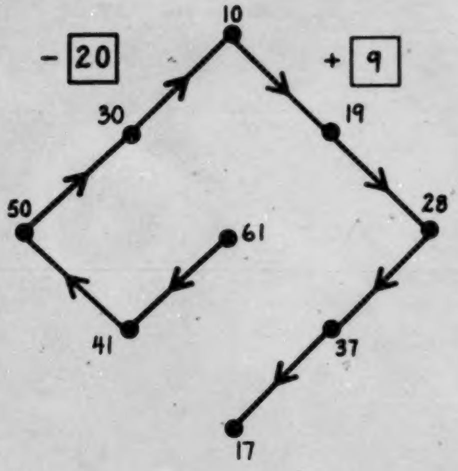
If you observe many students struggling with the same problem (or part of a problem), you could have a short discussion of that problem.

At the end of this lesson, collect the workbooks for your progress check. After checking the workbooks, you may wish to ask some students to work further during any free study time or as a homework assignment.

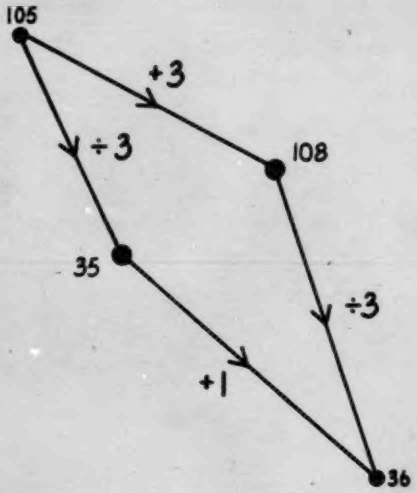
Collage of Problems #2

Fill in the boxes for the arrows.

Label the dots.


- 20
+ 9
10
30
19
50
28
61
41
37
17

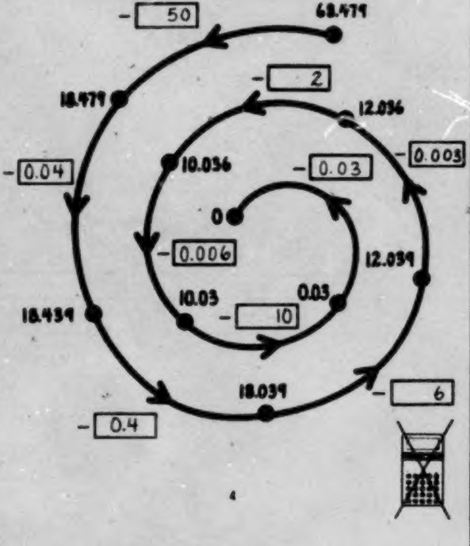
Build a road from 105 to 36. Each arrow must be for + ☐, - ☐, x ☐, or ÷ ☐, where exactly one of the digits: 1, 2, 3, 4, 5, 6, 7, 8, or 9 is in a box. Build as short a road as you can.


105
+3
÷3
108
35
÷3
+1
36

(Many solutions are possible; however, a shortest road has two arrows.)

WIPE - OUT

Fill in the boxes for the arrows.


- 50
68.479
- 2
18.479
12.036
- 0.003
- 0.04
10.036
- 0.03
0
- 0.006
12.039
18.439
10.03
0.03
- 10
18.039
- 6
- 0.4

Find all the places in this table for — where the entry is ⁻2. One is located for you.

—	1	2	3	4	5	6	7	8	9
1			⁻2						
2				⁻2					
3					⁻2				
4						⁻2			
5							⁻2		
6								⁻2	
7									⁻2
8									
9									

Did you find the other six places where the entry is ⁻2? Yes

Fill in each box with " < ", " > ", or " = ". Try to solve these problems without doing any calculation.

$782 + 149 \boxed{<} 782 + 188$

$621 - 17 \boxed{>} 621 - 84$

$567 + 138 + 495 \boxed{>} 316 + 567 + 138$

$71 - 25 \boxed{=} 73 - 27$

$2{,}447 + 828 \boxed{=} 447 + 2{,}828$

$1{,}143 - 660 \boxed{<} 1{,}532 - 660$

$\frac{1}{3} \boxed{>} \frac{1}{8}$

$\frac{5}{7} \boxed{>} \frac{2}{7}$

TIMELY QUESTIONS

Show your work in the space provided. You may use a hand-calculator to do the calculations.

1. Lloyd takes a 5 hour and 40 minute bus ride from Chicago to St. Louis to see a football game. If he arrives in St. Louis at 1:30 PM, what time did he leave Chicago? 7:50 AM (Don't forget to indicate "AM" or "PM".)

2. Your heart beats about 80 times per minute. About how many times does it beat in one day? 115,200
one year? 42,048,000

80 × 60 × 24 = 115,200 beats per day

115,200 × 365 = 42,048,000 beats per year

Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculations correct. You may use a digit more than once.

Subtract

$$\begin{array}{r} \boxed{5}390 \\ -17\boxed{6}8 \\ \hline 3\boxed{6}2\boxed{2} \end{array}$$

Multiply

$$\begin{array}{r} \boxed{3}83 \\ \times \boxed{7} \\ \hline 26\boxed{8}1 \end{array}$$

Complete the calculations. Show your work.

Subtract

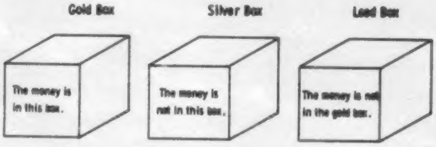
12,802 – 4,769

= 8,033

Multiply

$$\begin{array}{r} 476 \\ \times 26 \\ \hline 12{,}376 \end{array}$$

Exactly one of these boxes contains a $100 bill.

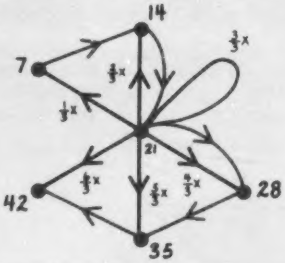

Gold Box
Silver Box
Lead Box
The money is in this box.
The money is not in this box.
The money is not in the gold box.

Exactly one of the statements on the boxes is true.

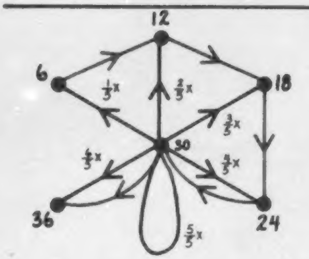
Which box contains the $100 bill? Silver

Which box has the true statement written on it? Lead

Label the dots and fill in the box for the blue arrows.


14
7
21
42
35
28
$\frac{2}{3}x$
$\frac{3}{3}x$
$\frac{1}{3}x$
$\frac{4}{3}x$
$\frac{5}{3}x$
$\frac{4}{3}x$

$+\boxed{7}$


12
6
18
30
36
24
$\frac{1}{3}x$
$\frac{2}{3}x$
$\frac{3}{5}x$
$\frac{4}{5}x$
$\frac{4}{5}x$
$\frac{5}{5}x$

$+\boxed{6}$

Complete these two division calculations.

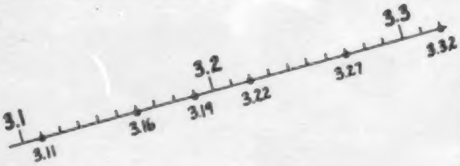
133 R=19
48)6,403

706 R=5
21)14,831

Lim is a secret number.

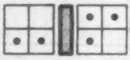
First clue

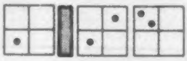
Lim is one of the dots on this number line. Label the dots.

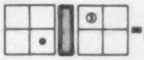

3.1
3.11
3.16
3.19
3.2
3.22
3.27
3.3
3.32

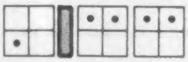
Second clue

Lim is one of these numbers.


= 4.4

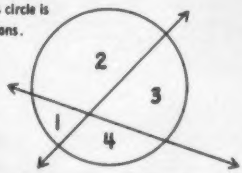

= 2.76


= 3.4

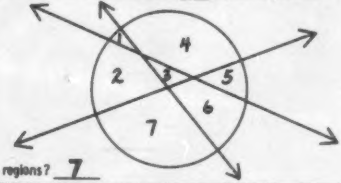

= 3.32

Who is Lim? 3.32

Using two lines, this circle is divided into four regions.

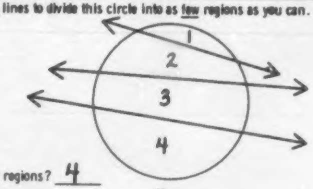


Use three lines to divide this circle into as many regions as you can.



How many regions? 7

Use three lines to divide this circle into as few regions as you can.



How many regions? 4

Complete.

$$\frac{1}{4} = \frac{\boxed{2}}{8} = \frac{\boxed{5}}{20} = \frac{3}{\boxed{12}} = \frac{10}{\boxed{40}}$$

$$\frac{3}{5} = \frac{\boxed{9}}{15} = \frac{\boxed{15}}{25} = \frac{30}{\boxed{50}} = \frac{12}{\boxed{20}}$$

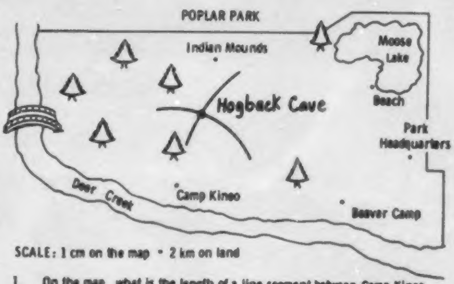
$$\frac{2}{3} = \frac{\boxed{4}}{6} = \frac{\boxed{10}}{15} = \frac{8}{\boxed{12}} = \frac{40}{\boxed{60}}$$

Complete.

$\frac{3}{5} + \frac{1}{4} = \underline{\frac{17}{20}}$ $\frac{3}{5} - \frac{1}{4} = \underline{\frac{7}{20}}$

$\frac{2}{3} + \frac{3}{5} = \underline{\frac{19}{15} \text{ or } 1\frac{4}{15}}$ $\frac{2}{3} - \frac{3}{5} = \underline{\frac{1}{15}}$

$\frac{2}{3} + \frac{1}{4} = \underline{\frac{11}{12}}$ $\frac{2}{3} - \frac{1}{4} = \underline{\frac{5}{12}}$



SCALE: 1 cm on the map = 2 km on land

1. On the map, what is the length of a line segment between Camp Kineo and Moose Lake Beach? 9 cm
   What is the real distance between Camp Kineo and Moose Lake Beach? 18 km

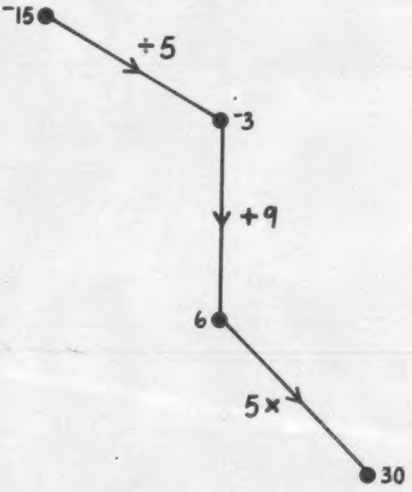
2. On the map, what is the length of a line segment between Beaver Camp and Park Headquarters? 3.3 cm
   What is the real distance from Park Headquarters to Beaver Camp? 6.6 km

3. If a hiker walks 4 km in one hour, how long should she take to walk from Camp Kineo to Moose Lake Beach? 4 hours, 30 minutes
   From Moose Lake Beach to the Indian Mounds? 3 hours, 15 minutes

4. Hogback Cave in Poplar Park is 6 km from Camp Kineo and 14 km from Beaver Camp. Find the location of Hogback Cave, draw a dot, and label it.

(This is most easily done using a compass.)

Build a road from ⁻15 to 30. Each arrow must be for +☐, -☐, x☐, or ÷☐, where exactly one of the digits: 1, 2, 3, 4, 5, 6, 7, 8, or 9 is in a box. Build as short a road as you can.



(Many solutions are possible; however, a shortest road has three arrows.)

A PAIR OF PROBLEMS

1. There are 120 sixth graders in Jackson Middle School. If there are 12 more girls than boys in the sixth grade, how many girls are there? 66 boys? 54

$66 - 54 = 12$

$66 + 54 = 120$

2. On a summer day in Tuscaloosa, there are 9 hours more of daylight than of darkness. On that day, how many hours of daylight are there? $16\frac{1}{2}$ of darkness? $7\frac{1}{2}$

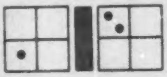
$16\frac{1}{2} - 7\frac{1}{2} = 9$

$16\frac{1}{2} + 7\frac{1}{2} = 24$

Zim is a secret number.

First clue

Zim can be put on this Minicomputer by adding one regular positive checker.

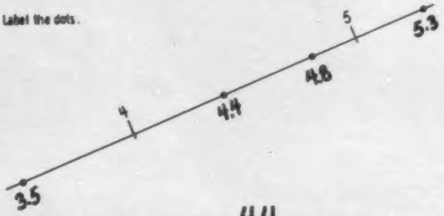


Zim could be 3.7, 3.8, 4.0, 4.4, 4.6, 5.6, 7.6, or 11.6.

Second clue

Zim is one of the dots on this number line. (You may use a ruler if you wish.)

Label the dots.



5
5.3
4.8
4
4.4
3.5

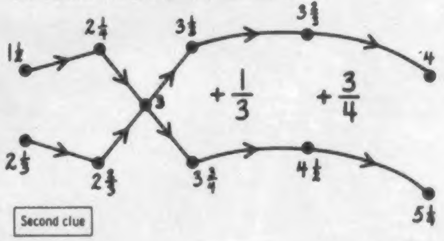
Who is Zim? 4.4

18

Za and Zu are secret numbers.

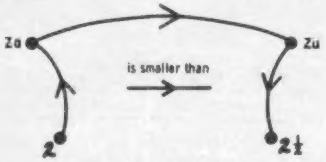
First clue

Za and Zu are in this arrow picture. Label the dots.



1½
2¼
3⅓
3⅔
4
3
$+\frac{1}{3}$
$+\frac{3}{4}$
2⅓
2⅔
3¾
4½
5¼

Second clue

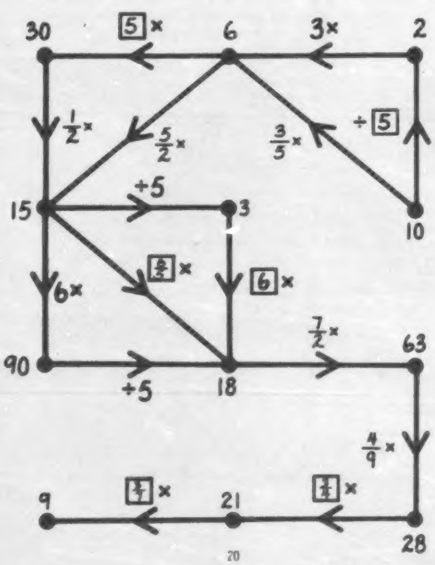


Za
Zu
is smaller than
2
2½

Za is 2¼. Zu is 2⅓.

19

Label the dots and fill in the boxes for the arrows.



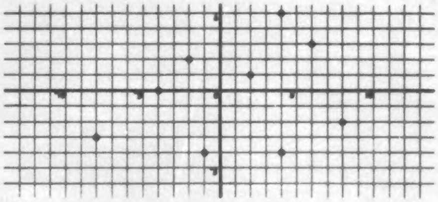
30
5 ×
6
3×
2
½×
5/2×
3/5×
÷ 5
15
÷5
3
10
6/5 ×
6 ×
6×
7/2×
90
÷5
18
63
4/9×
9
3/7 ×
21
3/4 ×
28

20

Eks and Zee are secret numbers.

First clue

(Eks, Zee) is one of the dots on this grid picture. You do not have to label the dots.



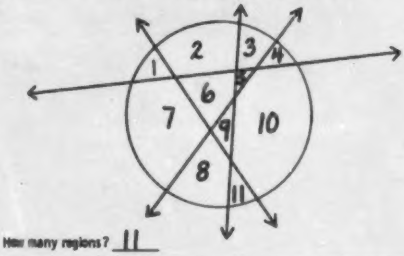
Second clue

T<	Zee	3
Eks	1	1
⁻1	0	1

(Eks, Zee) is (⁻8, ⁻3).

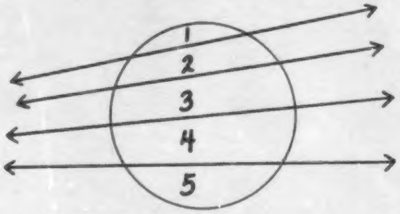
21

Use four lines to divide this circle into as many regions as you can.


1
2
3
4
5
6
7
8
9
10
11

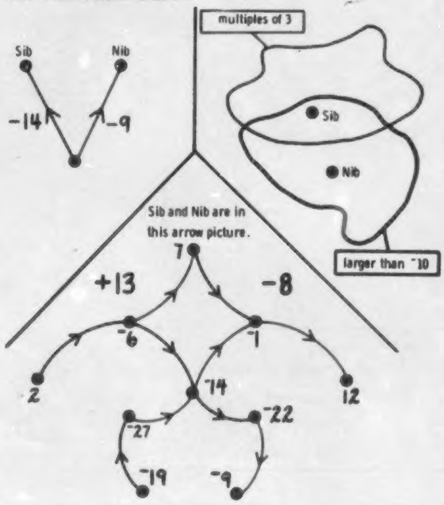
How many regions? 11

Use four lines to divide this region into as few regions as you can.



How many regions? 5

Sib and Nib are secret numbers.


Sib
Nib
-14
-9
multiples of 3
Sib
Nib
Sib and Nib are in this arrow picture.
larger than ⁻10
7
+13
-8
⁻6
⁻1
2
12
⁻14
⁻22
⁻27
⁻19
⁻9

Sib is ⁻6. Nib is ⁻1.

# is for one of these operations.

[illegible]

+	+[illegible]	+[illegible]	+[illegible]
−	−[illegible]	−[illegible]	−[illegible]
×	×[illegible]	×[illegible]	×[illegible]
↑	↓	⊓	⊔
T[illegible]	T[illegible]	T[illegible]	T[illegible]
	T[illegible]	T[illegible]	

First clue

There are no 0's is this table for #.

*	5	11
7		
5		

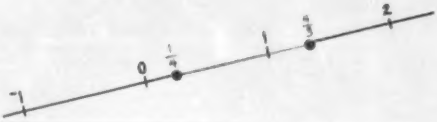
# could be $+$, $\times$, $\times_5$, $T_{≡2}$, $\times_{10}$, ↑, ↓, ⊓, or ⊔.

Second clue

There is exactly one 2 in this table for #.

*	2	6
4	8	4
7	4	2

# is $\times_{10}$.


⁻1
0
$\frac{1}{4}$
1
$\frac{4}{3}$
2

Complete the tables and circle the entries that belong to the blue segment.

+	$\frac{1}{2}$	$\frac{3}{4}$
$\frac{3}{4}$	$\frac{5}{4}$	$\frac{6}{4}$
$\frac{4}{5}$	$\frac{13}{10}$	$\frac{31}{20}$
$\frac{2}{5}$	$\frac{9}{10}$	$\frac{23}{20}$

−	$\frac{1}{2}$	$\frac{3}{4}$
$\frac{3}{4}$	$\frac{1}{4}$	0
$\frac{4}{5}$	$\frac{3}{10}$	$\frac{1}{20}$
$\frac{2}{5}$	$\frac{⁻1}{10}$	$\frac{⁻7}{20}$

Each table is for one of these operations.

THE TABLE GAME

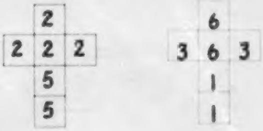
+	$+_{[illegible]}$	$+_{[illegible]}$	$+_{[illegible]}$
−	$-_{[illegible]}$	$-_{[illegible]}$	$-_{[illegible]}$
x	$x_{[illegible]}$	$x_{[illegible]}$	$x_{[illegible]}$
↑	↓	⊓	⊔
$T_{[illegible]}$	$T_{[illegible]}$	$T_<$	$T_>$
	$T_{[illegible]}$	$T_{[illegible]}$	

Label the tables.

⊓	9	11
5	~~2~~	1
12	X	~~8~~

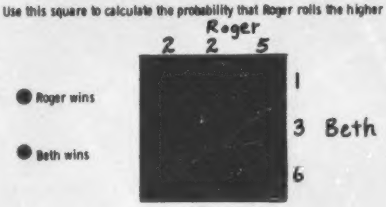
⁻5	8	14
25	~~7~~	1
6	~~2~~	2

These are the maps of two number cubes.



Roger and Beth play a game with these two cubes; higher number wins. Roger rolls the red cube and Beth rolls the blue cube.

Use this square to calculate the probability that Roger rolls the higher number.



What is Roger's probability of winning? $\frac{4}{9}$

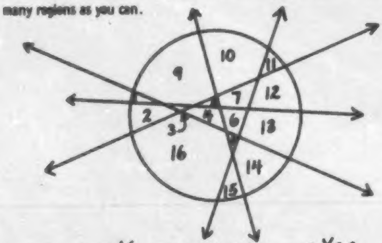
Who is favored in this game, Roger or Beth? Beth

Ask your teacher to check your answers on pages 13 and 22. Use correct answers to fill in the boxes in this table:

Number of line segments	2	3	4	5	6
Maximum number of regions	4	$\boxed{7}$	$\boxed{11}$		

Do you see a pattern? Yes Predict the largest number of regions into which you can divide a circle using five line segments. 16

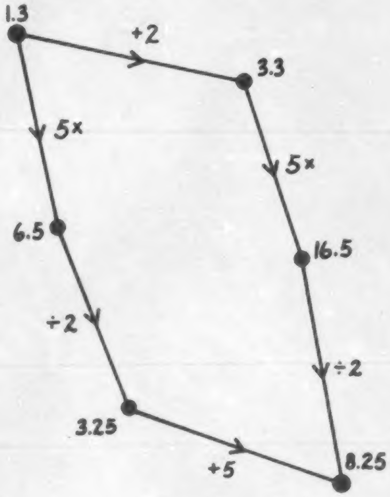
Check your prediction by using five line segments to divide this circle into as many regions as you can.



How many regions? 16 Was your prediction correct? Yes

Using the same pattern, predict the largest number of regions into which you could divide a circle using six line segments? 22

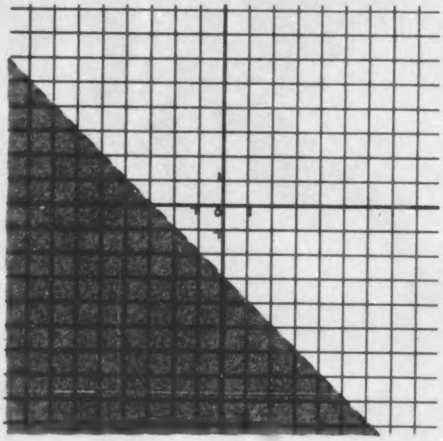
Build a road from 1.3 to 8.25. Each arrow must be for +☐, -☐, x☐, or ÷☐, where exactly one of the digits: 1, 2, 3, 4, 5, 6, 7, 8, or 9 is in a box. Use at most four arrows.



(Many solutions are possible; however, a shortest road has three arrows.)

Lin + Lon < ⁻3

Shade a region on the grid to indicate all couples that (Lin, Lon) could be.



It is impossible to make two of these shapes (without dents).

A: A triangle with at least two right angles.

B: A five-sided shape with exactly one right angle and at least two acute angles.

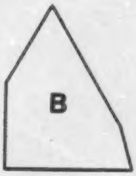
C: A five-sided shape with at least four acute angles.

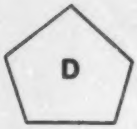
D: A five-sided shape with five obtuse angles.

Circle the letters of the two shapes which are impossible to make.

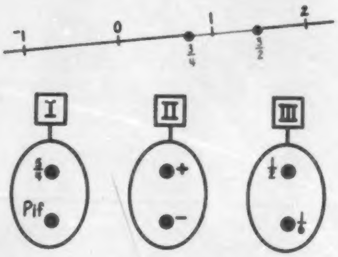
Ⓐ B Ⓒ D

Draw the other two shapes.





(Many solutions are possible.)



One symbol is chosen at random from each string to form the name of a number, for example, "$\frac{5}{4} - \frac{1}{6}$". Choose Pif to make the probability that the number named is in the blue segment exactly $\frac{1}{2}$.

You may use the tables below to help you find who Pif could be. Many solutions are possible. (Letting Pif = $\frac{1}{4}$ results in the following tables. The numbers on the blue segment are circled in blue.)

+	$\frac{1}{2}$	$\frac{1}{6}$
$\frac{5}{4}$	$\frac{7}{4}$	$\frac{17}{12}$
Pif	$\frac{3}{4}$	$\frac{5}{12}$

−	$\frac{1}{2}$	$\frac{1}{6}$
$\frac{5}{4}$	$\frac{3}{4}$	$\frac{13}{12}$
Pif	$^{-}\frac{1}{4}$	$\frac{1}{12}$

Pif could be $\frac{1}{4}$.

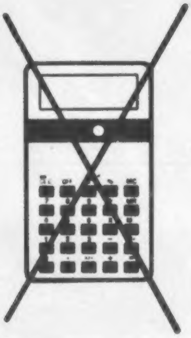
(Pif could be $\frac{1}{4}$ or 2 or any number between $\frac{1}{4}$ and $\frac{7}{12}$ or any number between $\frac{5}{3}$ and 2.)

COLLAGE
OF
PROBLEMS
#3

COLLAGE OF PROBLEMS #3

The problems in this workbook are in order of increasing difficulty. Approximately the first ten pages of the workbook have easy problems, the next ten to twelve pages have average problems, and the last ten pages have more challenging problems. It is suggested that all students work through as many pages as they can handle during the two lessons scheduled for the workbook. Not all students are expected to complete the entire workbook; most should complete at least ten pages, many should finish about twenty pages, and only the faster students will reach the most challenging problems at the end of the booklet.

Many problems in the workbook were designed with the availability of hand-calculators in mind. If possible, provide all the students with hand-calculators or at least give students access to hand-calculators.[†] You will find that many students do not need or want to use calculators; however, some students still need the support of the calculator in thinking about these problems. The problems on some pages are to be solved without using a hand-calculator. These pages are indicated by the following symbol.



Begin the first workbook lesson with the brief collective lesson described on the following pages. This should be a brief warm-up activity to introduce a new type of problem.

[†] See Section 2(f) of the General Introduction for additional comments on the use of hand-calculators and on desirable characteristics of hand-calculators.

W 6

Lesson 1

Write these problems on the board.

86.935 + 67.38 = 1 5 4 3 1 5

3,710.417 − 3,190.837 = 5 1 9 5 8

3.2 × 7.9 = 2 5 2 8

680.15 × 7.4 = 5 0 3 3 1 1

T: Each of these calculations is correct, except that the decimal point is missing from each answer. Can you determine where to place the decimal point in the first problem? Try to solve the problem without using paper and pencil; you might try using estimation.

S: The answer for the first problem is 154.315. Since 86.935 is about 90 and 67.38 is about 70 and 90 + 70 = 160, the answer should be slightly less than 160. It must be 154.315.

Invite students to solve the other problems similarly. The correct answers are: 519.58; 25.28; 5,033.11.

Distribute individual copies of "Collage of Problems #3" and instruct students to work on their own. If you observe many students struggling with the same problem (or part of a problem), you could have a short collective discussion about that problem.

Collect the workbooks at the end of the lesson. Check the students' work and have the workbooks ready to return for Lesson 2.

Lesson 2

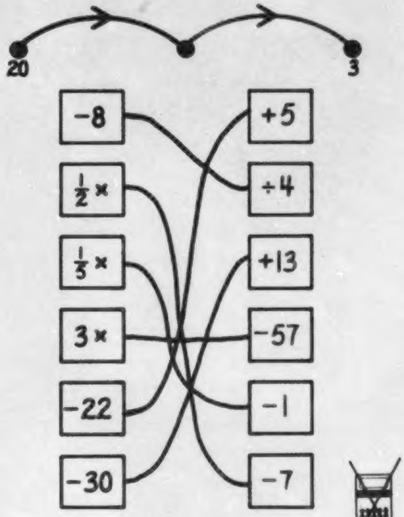
Return the students' copies of "Collage of Problems #3". You may wish to discuss any page that was difficult for most of the students. Then ask the students to correct their mistakes, if any, and to continue working in their workbooks.

If you observe many students struggling with the same problem (or part of a problem), you could have a short discussion of that problem.

At the end of this lesson, collect the workbooks for your progress check. After checking the workbooks, you may wish to ask some students to work further during any free study time or as a homework assignment.

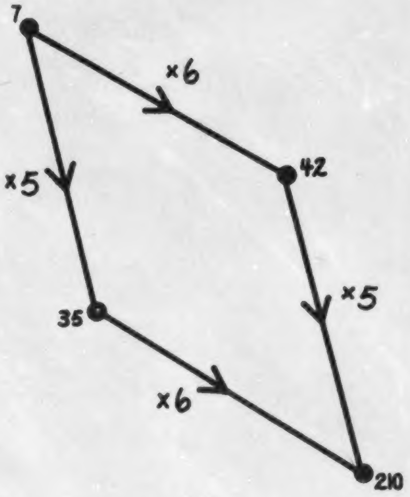
Collage of Problems #3

Pair the tags.


20
3
-8
$\frac{1}{2}$ ×
$\frac{1}{3}$ ×
3 ×
-22
-30
+5
÷4
+13
-57
-1
-7

2

Build a road from 7 to 210. Each arrow must be for +☐, -☐, x☐, or ÷☐, where exactly one of the digits: 1, 2, 3, 4, 5, 6, 7, 8, 9 is in a box. Build as short a road as you can.


7
×6
42
×5
35
×5
×6
210

3

(Many solutions are possible; however, a shortest road has two arrows.)

All these calculations are correct, except that a decimal point is missing in each answer. Place a decimal point in each answer to make a true number story.

787.6 + 35.87 = 823.47

613.82 + 492.48 = 1106.3

86.15 − 12.483 = 73.667

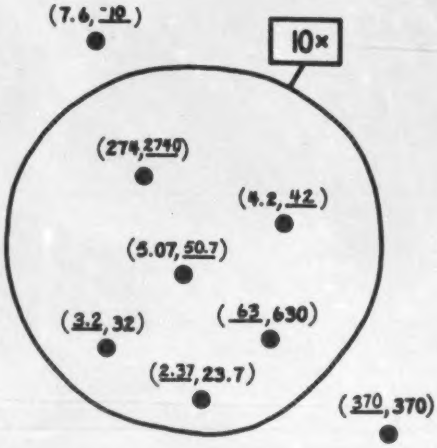
420.36 − 381.76 = 38.6

6.2 × 4.5 = 27.9

31.53 × 4.97 = 156.7041

4

Fill in the blanks.

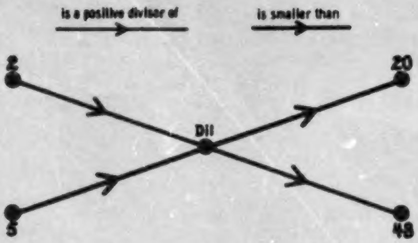

(7.6, ⁻10)
10×
(274, 2740)
(4.2, 42)
(5.07, 50.7)
(3.2, 32)
(.63, 630)
(2.37, 23.7)
(370, 370)

5

(Many solutions are possible for the dots outside the red string.)

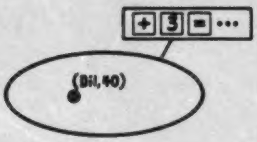
Dil is a secret whole number.

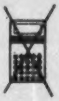
First clue



Dil could be 6, 8, 12, or 16.

Second clue





Dil is 16.

Fill in each box with " < ", " > ", or " = ". Try to solve these problems without doing any calculation.

$736 + 587 \ \boxed{>} \ 587 + 426$

$1{,}018 - 650 \ \boxed{<} \ 1{,}018 - 200$

$500 + 700 \ \boxed{=} \ 520 + 680$

$82 \div 7 \ \boxed{<} \ 82 \div 3$

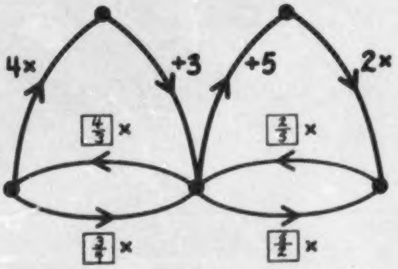
$70 \times 36 \ \boxed{=} \ 7 \times 360$

$58 \times 67 \ \boxed{<} \ 71 \times 58$

$\frac{2}{3} \ \boxed{>} \ \frac{2}{5}$

$\frac{3}{2} \ \boxed{<} \ \frac{5}{2}$

Fill in the boxes for the arrows.



Complete these calculations.

$\frac{4}{3} \times 12 = \underline{16}$

$\frac{2}{5} \times 20 = \underline{8}$

$\frac{3}{4} \times 12 = \underline{9}$

$\frac{5}{2} \times 20 = \underline{50}$

Find all of the places in this table for ↑ where the entry is 4 and find all the places where the entry is 7. One of each is located for you.

↑	1	2	3	4	5	6	7	8	9
1				4			7		
2				4			7		
3				4			7		
4	4	4	4	4			7		
5							7		
6							7		
7	7	7	7	7	7	7	7		
8									
9									

1. When a printer prints the page numbers of a 100 page book, how many "9" 's are printed? 20

When a printer prints the page numbers of a 1,000 page book, how many "9" 's are printed? 300

2. Shawn and Vicki Rogers ran in The Tri-City Marathon. Shawn took 2 hours and 40 minutes to complete the race and Vicki took 2 hours and 57 minutes. If Shawn finished the race at 1:10 PM, what time did he start? 10:30 AM

If Vicki started at the same time as Shawn, what time did Vicki finish?
1:27 PM

Fill in the boxes with these numbers.

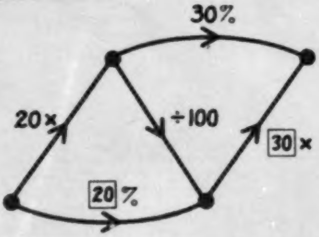
$$\frac{5}{2} \qquad \frac{5}{3} \qquad \frac{17}{4}$$

$$\frac{28}{5} \qquad \frac{7}{8} \qquad \frac{37}{10}$$

$$0 < \boxed{\tfrac{7}{8}} < 1 < \boxed{\tfrac{5}{3}} < 2 < \boxed{\tfrac{5}{2}} < 3$$

$$3 < \boxed{\tfrac{37}{10}} < 4 < \boxed{\tfrac{17}{4}} < 5 < \boxed{\tfrac{28}{5}} < 6$$

Fill in the boxes for the arrows.



Complete.

30% of $30 = \boxed{9}$ $\qquad$ 30% of $\boxed{100} = 30$

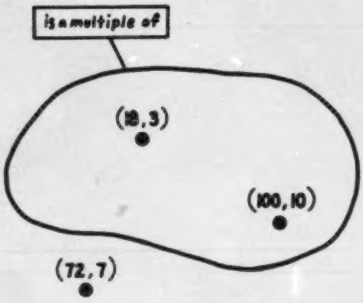
30% of $40 = \boxed{12}$ $\qquad$ 20% of $\boxed{150} = 30$

20% of $40 = \boxed{8}$ $\qquad$ 20% of $\boxed{75} = 15$

The red string is for one of these relations.

÷6	+10
is larger than	is a multiple of
$\boxed{-}\,\boxed{2}\,\boxed{=}$ •••	$\boxed{-}\,\boxed{5}\,\boxed{=}$ •••

Label the string.



Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculations correct. You may use a digit more than once.

Subtract

$$\begin{array}{r} 2\,\boxed{2}\,7\,4\,3 \\ -\;8\,\boxed{6}\,\boxed{6}\,7 \\ \hline 1\,4\,0\,7\,\boxed{6} \end{array}$$

Divide

$$\begin{array}{r|l} \multicolumn{2}{r}{\boxed{2}\boxed{3}\ R=\boxed{1}\boxed{2}} \\ 67\,)\,1\,5\,5\,3 & \\ -\,\boxed{1}\boxed{3}\boxed{4}\,0 & \boxed{2}\,0 \\ \hline \boxed{2}\,1\,\boxed{3} & \\ -\,2\,\boxed{0}\boxed{1} & \boxed{3} \\ \hline \boxed{1}\boxed{2} & \boxed{2}\boxed{3} \end{array}$$

Complete the calculations. Show your work.

Subtract

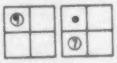
4,703 − 4,278

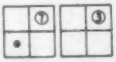
425

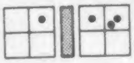
Divide

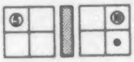
$$\begin{array}{r} 307\ R=14 \\ 42\,)\,12,908 \end{array}$$

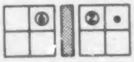
Put each number on the Minicomputer by adding exactly one regular positive checker.

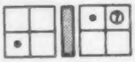

= 742

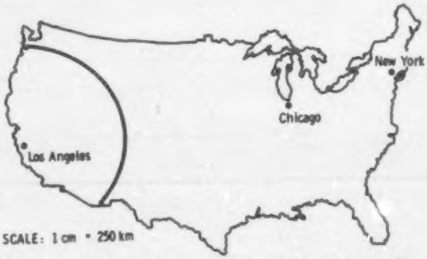

= 312


= 5.6


= 44.1


= 26


= 5.6



SCALE: 1 cm = 250 km

1. Find the length of a line segment and the real distance between each pair of cities.

	Line Segment	Real Distance
Los Angeles - Chicago	11 cm	2,750 km
Chicago - New York	4.5 cm	1,125 km

2. If an airplane flies 400 km in one hour, what is the flying time from Los Angeles to Chicago? (Circle the closest answer.)

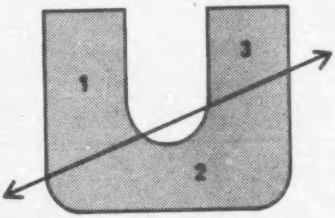
3 hours 5 hours [✓ 7 hours] 9 hours

. . . from Chicago to New York? (Circle the closest answer.)

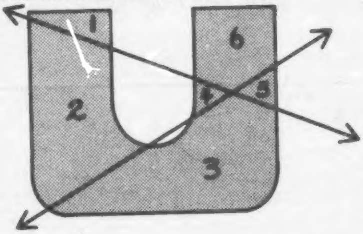
$1\frac{1}{4}$ hours [✓ $2\frac{3}{4}$ hours] $3\frac{1}{2}$ hours $4\frac{1}{2}$ hours

3. The city, Truth or Consequences, is 1,000 km from Los Angeles and is in the United States. Using a compass, indicate in red the places where Truth or Consequences could be.

Using one line, this shape is divided into three regions.



Use two lines to divide this shape into as many regions as you can.

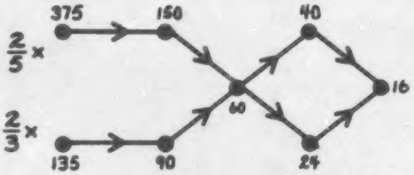


How many regions? 6

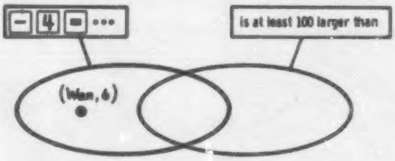
Wan is a secret number.

First clue

Wan is in this arrow picture.


$\frac{2}{5}\times$
$\frac{2}{3}\times$
375
150
40
60
16
135
90
24

Second clue


− 4 = ...
is at least 100 larger than
(Wan, 6)

Wan is 90.

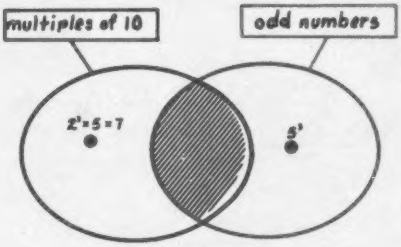
The red tag is one of these:

- multiples of 5
- multiples of 7
- odd numbers
- prime numbers
- multiples of 10
- multiples of 25

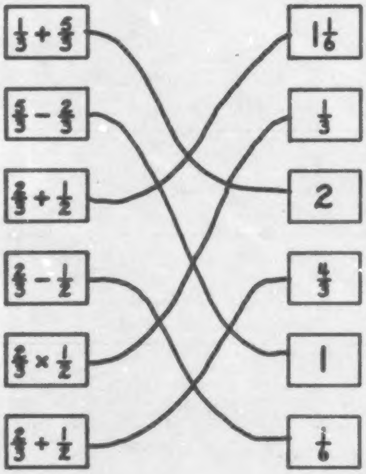
The blue tag is one of these:

- multiples of 5
- multiples of 7
- odd numbers
- prime numbers
- multiples of 10
- multiples of 25

Label the strings.


multiples of 10
odd numbers
$2^2 \times 5 \times 7$
5^3

Pair the tags.


$\frac{1}{3}+\frac{5}{3}$
$\frac{5}{3}-\frac{2}{3}$
$\frac{2}{3}+\frac{1}{2}$
$\frac{2}{3}-\frac{1}{2}$
$\frac{2}{3}\times\frac{1}{2}$
$\frac{2}{3}+\frac{1}{2}$
$1\frac{1}{6}$
$\frac{1}{3}$
2
$\frac{4}{3}$
1
$\frac{1}{6}$

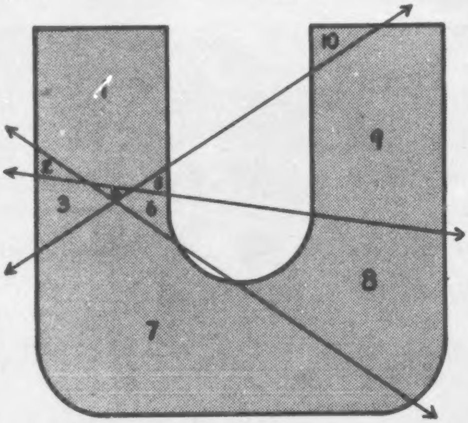
After a burglary, the police questioned three men: Archie, Gus, and Paul. The police knew that exactly one of these men was guilty. The suspects made the following statements:

Archie: Gus did it.
Gus: Paul did it.
Paul: Archie lied when he said Gus did it.

If exactly one of these statements is false, who was the guilty man? Paul

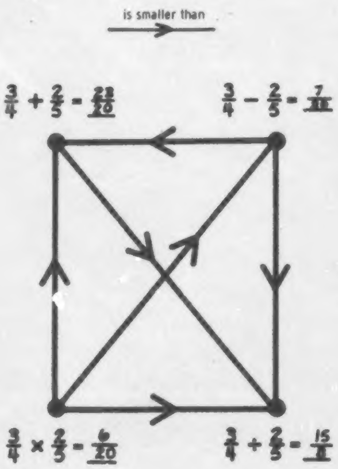
Whose statement was false? Archie's

Use three lines to divide this shape into as many regions as you can.


1
2
3
4
5
6
7
8
9
10

How many regions? 10

Complete the calculations. Draw all the missing red arrows between the dots.


is smaller than

$\frac{3}{4} + \frac{2}{5} = \frac{23}{20}$

$\frac{3}{4} - \frac{2}{5} = \frac{7}{20}$

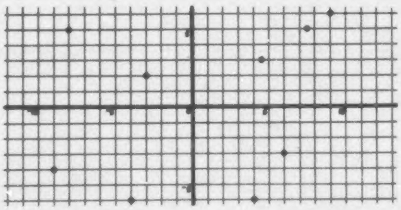
$\frac{3}{4} \times \frac{2}{5} = \frac{6}{20}$

$\frac{3}{4} \div \frac{2}{5} = \frac{15}{8}$

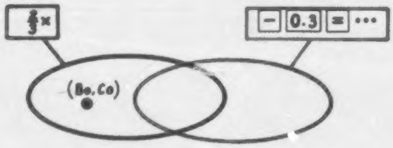
Bo and Co are secret numbers.

First clue

(Bo, Co) is one of the dots on this grid. You do not have to label the dots.



Second clue


$\frac{2}{3}\times$
− 0.3 = •••
(Bo, Co)

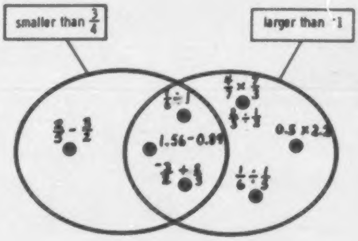
(Bo, Co) is (7.5, 5).

Label the dots. One of the numbers has two names listed. Write both names for that number beside its dot.

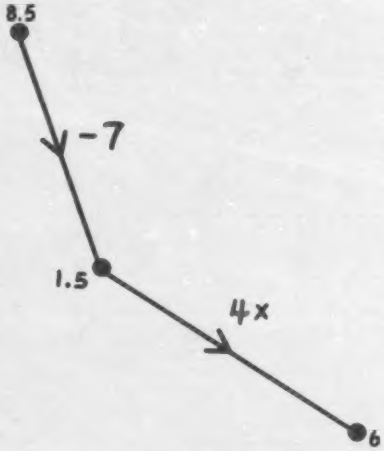
$\frac{2}{3} \div \frac{1}{2}$ $\quad$ $^{-}\frac{3}{2} + \frac{2}{3}$ $\quad$ 0.5×2.2

$1.56 - 0.89$ $\quad$ $\frac{1}{6} \div \frac{1}{3}$

$\frac{1}{6} \div 1$ $\quad$ $\frac{2}{5} - \frac{3}{2}$ $\quad$ $\frac{4}{7} \times \frac{7}{3}$

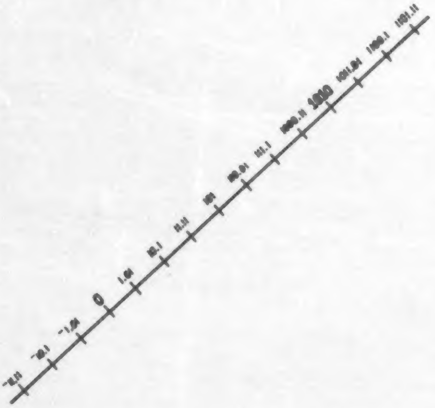

smaller than $\frac{3}{4}$
larger than $^{-}1$
$\frac{2}{5} - \frac{3}{2}$
$\frac{1}{6} \div 1$
$1.56 - 0.89$
$^{-}\frac{3}{2} + \frac{2}{3}$
$\frac{4}{7} \times \frac{7}{3}$
$\frac{2}{3} \div \frac{1}{2}$
0.5×2.2
$\frac{1}{6} \div \frac{1}{3}$

Build a road from 8.5 to 6 with exactly two arrows. Each arrow must be for +☐, −☐, x☐, ÷☐, where exactly one of the digits: 1, 2, 3, 4, 5, 6, 7, 8, or 9 is in a box.



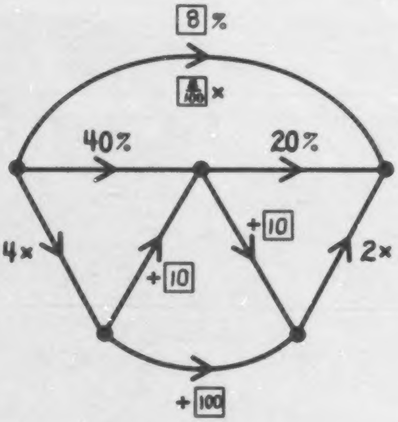
26

Label this number line in the binary code.


⁻11.11
⁻10.1
⁻1.01
0
1.01
10.1
11.11
101
110.01
111.1
1000.11
1010
1011.01
1100.1
1101.11

27

Fill in the boxes for the arrows.


8 %
8/100 ×
40%
20%
4×
+ 10
+ 10
2×
+ 100

28

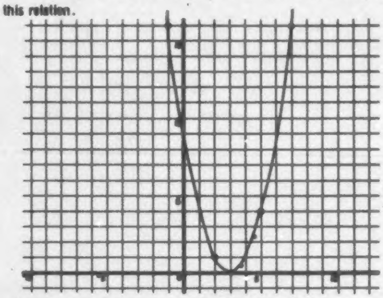
$$\boxed{a \rightarrow (a-3)^2}$$

Fill in the blanks. Each ordered pair belongs to the red relation. One is done for you.

(5,4)   (7, 16)   (2, 1)

(⁻1, 16)   ($3\frac{1}{3}$, $\frac{1}{9}$)   ($\frac{9}{2}$, $\frac{9}{4}$)

For each ordered pair, draw a dot on the grid. Sketch the Cartesian graph for this relation.



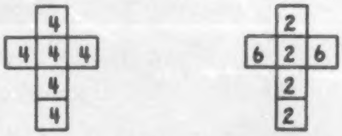
Name two more ordered pairs which belong to the red relation.

(0 , 9) and (4 , 1)

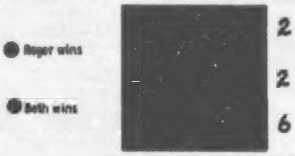
29

(Many other ordered pairs belong to the red relation.)

These are the maps of two number cubes.

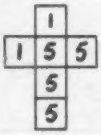

4
4 4 4
4
4
2
6 2 6
2
2

Roger rolls the red cube and Beth rolls the blue cube. Use this square to calculate the probability that Roger wins by rolling the higher number.


● Roger wins
● Beth wins
2
2
6

What is Roger's probability of winning over Beth? $\frac{2}{3}$

Using only the numbers, 1, 3, and 5, label this block cube for Brenda so that Brenda is favored over Roger, but Beth is favored over Brenda.



What is Brenda's probability of beating Roger? $\frac{2}{3}$

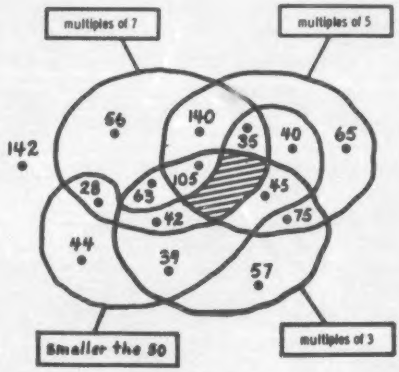
What is Beth's probability of beating Brenda? $\frac{5}{9}$

Draw a fourth string for smaller than 50.

Put these numbers in the string picture.

28 35 39 40 42 44 45 56

57 63 65 75 105 140 142


multiples of 7
multiples of 5
142
56
140
35
40
65
28
63
105
45
42
75
44
39
57
Smaller the 50
multiples of 3

One region in your picture should be empty. Hatch it.

$\frac{1}{4}$ 123 $^-25$

1.8 $^-16$ 1.28 $\frac{1}{5}$

Names for the above numbers can be written using just these symbols. Each symbol may be used as often as you wish.

3 5 − + ( ) 2

Example: $2^3 + 2^5 = 8 + 32 = \frac{1}{4}$

Write names for the other numbers above.

$5^3 - 2 = 123$

$(3-3) - 5^2 = {^-25}$

$3^2 \div 5 = \frac{9}{5} = 1.8$

$3^2 - 5^2 = {^-16}$

$5 \div 25 = \frac{5}{25} = \frac{1}{5}$

$(2^5 \div 5) \div 5 = 1.28$

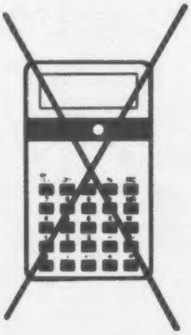
(Other solutions are possible.)

COLLAGE
OF
PROBLEMS
#4

COLLAGE OF PROBLEMS #4

The problems in this workbook are in order of increasing difficulty. Approximately the first ten pages of the workbook have easy problems, the next ten to twelve pages have average problems, and the last ten pages have more challenging problems. It is suggested that all students work through as many pages as they can handle during the two lessons scheduled for the workbook. Not all students are expected to complete the entire workbook; most should complete at least ten pages, many should finish about twenty pages, and only the faster students will reach the most challenging problems at the end of the booklet.

Many problems in the workbook were designed with the availability of hand-calculators in mind. If possible, provide all the students with hand-calculators or at least give students access to hand-calculators.[†] You will find that many students do not need or want to use calculators; however, some students still need the support of the calculator in thinking about these problems. The problems on some pages are to be solved without using a hand-calculator. These pages are indicated by the following symbol.



[†] See Section 2(f) of the General Introduction for additional comments on the use of hand-calculators and on desirable characteristics of hand-calculators.

Lesson 1

Distribute individual copies of "Collage of Problems #4" and instruct students to work on their own. If you observe many students struggling with the same problem (or part of a problem), you could have a short collective discussion about that problem.

Collect the workbooks at the end of the lesson. Check the students' work and have the workbooks ready to return for Lesson 2.

Lesson 2

Return the students' copies of "Collage of Problems #4". You may wish to discuss any page that was difficult for most of the students. Then ask the students to correct their mistakes, if any, and to continue working in their workbooks.

If you observe many students struggling with the same problem (or part of a problem), you could have a short discussion of that problem.

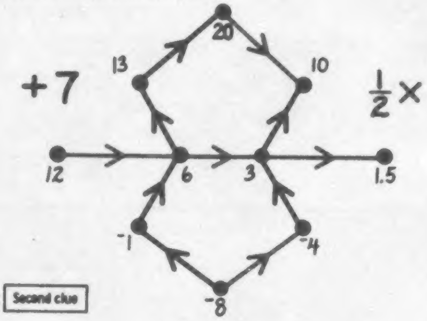
At the end of this lesson, collect the workbooks for your progress check. After checking the workbooks, you may wish to ask some students to work further during any free study time or as a homework assignment.

Collage of Problems #4

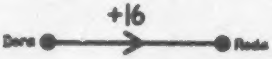
Dora and Roda are secret numbers.

First clue

Dora and Roda are in this arrow picture.


+7
$\frac{1}{2}\times$
20
13
10
12
6
3
1.5
⁻1
⁻4
⁻8

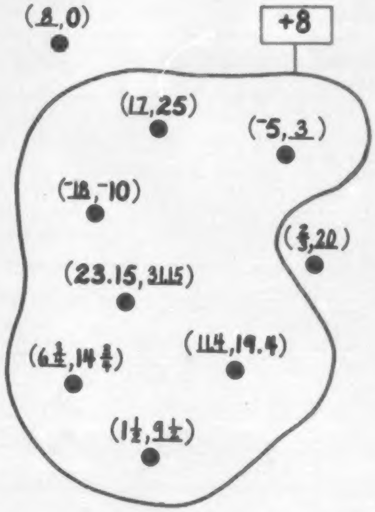
Second clue


+16
Dora
Roda

(Dora, Roda) is (⁻4, 12).

2

Fill in the blanks.

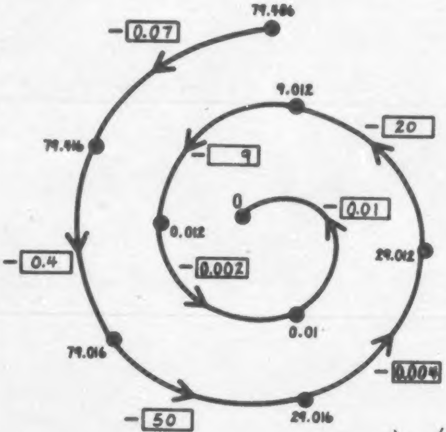

+8
(8,0)
(17,25)
(⁻5,3)
(⁻18,⁻10)
($\frac{2}{3}$,20)
(23.15,31.15)
($6\frac{3}{4}$,$14\frac{3}{4}$)
(11.4,19.4)
($1\frac{1}{2}$,$9\frac{1}{2}$)

3

(Many solutions are possible for the two dots outside the blue string.)

WIPE-OUT

Fill in the boxes for the arrows.


79.486
−0.07
79.416
−0.4
79.016
−50
29.016
−0.004
29.012
−20
9.012
−9
0.012
−0.002
0.01
−0.01
0

4

Fill in each box with " < ", " > ", or " = ". Try to solve these problems without doing any calculation.

55 + 9 $\boxed{<}$ 55 + 7

73 + 6 $\boxed{<}$ 87 + 6

23 + 3 $\boxed{=}$ 280 + 30

80% of 40 $\boxed{>}$ 20

23% of 18 $\boxed{<}$ 18

100% of 37 $\boxed{=}$ 37

$\frac{3}{2}\times 175$ $\boxed{>}$ 175

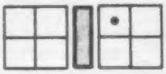
$\frac{2}{3}\times 175$ $\boxed{<}$ 175

5

Liba is a secret number.

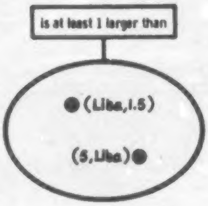
First clue

Liba can be put on this Minicomputer by adding exactly one regular positive checker.



Liba could be 0.9, 1.0, 1.2, 1.6, 1.8, 2.8, 4.8, or 8.8.

Second clue



Liba is 2.8.

Complete the calculations. Show your work.

Add

$$\begin{array}{r} \boxed{5}.7 \\ 30.\boxed{3}8 \\ 7.09 \\ +26.5\boxed{6} \\ \hline \boxed{6}9.73 \end{array}$$

Add

163.7 + 38 + 84.53

286.23

Subtract

$$\begin{array}{r} \boxed{8}9.60 \\ -2\boxed{5}.\boxed{8}3 \\ \hline 63.7\boxed{7} \end{array}$$

Subtract

803.42 − 539.6

263.82

Complete.

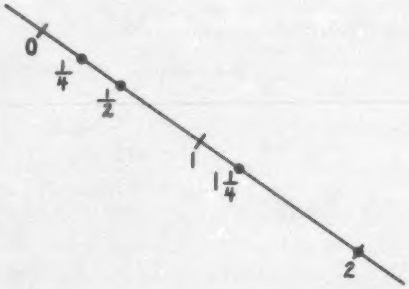
$\frac{5}{4} + \frac{3}{4} = \frac{8}{4} = 2$

$\frac{3}{4} + \frac{1}{2} = \frac{5}{4} = 1\frac{1}{4}$

$\frac{5}{4} - \frac{3}{4} = \frac{2}{4} = \frac{1}{2}$

$\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

Use the numbers above to label the dots on this number line.



Fill in the blanks. Recall that Ⓡ is the "rounding" operation.

Examples: 11 Ⓡ 3 = 12 because the nearest multiple of 3 to 11 is 12.
4.3 Ⓡ 0.5 = 4.5 because using $\boxed{0.5}\ \boxed{+}\ \boxed{=}$ . . .
the closest we get to 4.3 is 4.5.

17 Ⓡ 4 = 16

60 Ⓡ 7 = 63

38 Ⓡ 5 = 40

172 Ⓡ 20 = 180

283 Ⓡ 10 = 280

44,527 Ⓡ 10 = 45,530

283 Ⓡ 100 = 300

44,527 Ⓡ 100 = 44,500

13.58 Ⓡ 1 = 14

7.66 Ⓡ 0.1 = 7.7

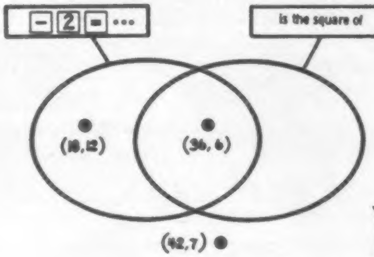
The red tag is one of these:

- is at least 20 larger than
- is a multiple of
- is the square of
- $\frac{2}{3}\times$
- $\boxed{-}\boxed{2}\boxed{=}\cdots$
- $\boxed{-}\boxed{4}\boxed{=}\cdots$

The blue tag is one of these:

- is at least 20 larger than
- is a multiple of
- is the square of
- $\frac{2}{3}\times$
- $\boxed{-}\boxed{2}\boxed{=}\cdots$
- $\boxed{-}\boxed{4}\boxed{=}\cdots$

Label the strings.



Each calculation is correct, except that a decimal point is missing in the answer. Place a decimal point in each answer to make a true number story.

376 + 14.29 + 265.8 = 6 5 6.0 9

408.27 + 255.68 + 3,862.75 = 4 5 2 6.7

3,461.3 − 512.76 = 2 9 4 8.5 4

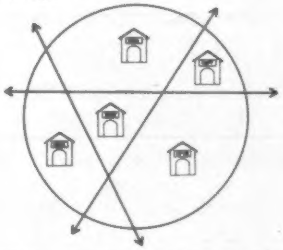
7.4 × 53.27 = 3 9 4.1 9 8

31.35 × 8.06 = 2 5 2.6 8 1

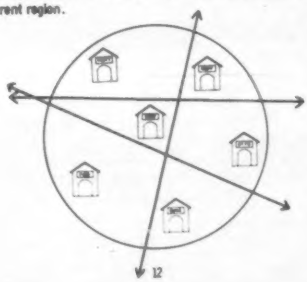
648.69 ÷ 2.1 = 3 0 8.9

83.3 ÷ 19.6 = 4.2 5

Using three straight fences (lines), this circle is divided so that each dog house is a different region.



Use three straight fences (lines) to divide this circle so that each dog house is in a different region.



(Other solutions are possible.)

Complete the calculations. Show your work.

Multiply

$$\begin{array}{r} 7.36 \\ \times\ 1.3 \\ \hline 9.568 \end{array}$$

Multiply

97.2 × 6.7

= 651.24

Divide

$$\begin{array}{r} 85.2 \\ 7\overline{)596.4} \end{array}$$

Divide

2356.2 ÷ 63

= 37.4

Alice, Brenda, and Carl are all different ages. They make the following statements.

Alice: I am older than Carl.
Brenda: Carl is not the oldest.
Carl: I am older than Brenda.

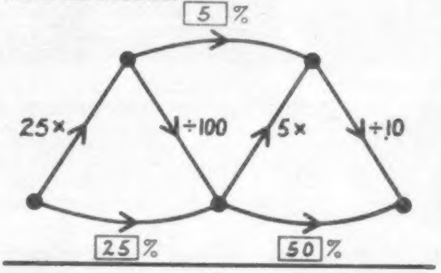
Problem 1: If all three statements are true,

who is the oldest? Alice
who is the youngest? Brenda

Problem 2: If exactly one of the statements is false,

who is the youngest? Carl
whose statement is false? Carl's

Fill in the boxes for the arrows.



Complete.

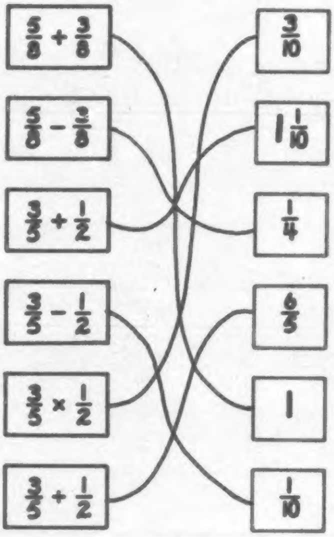
25% of $20 = \boxed{5}$ $\quad$ 5% of $40 = \boxed{2}$

$\frac{1}{4} \times 20 = \boxed{5}$ $\quad$ 5% of $\boxed{800} = 40$

25% of $\boxed{80} = 20$ $\quad$ 50% of $40 = \boxed{20}$

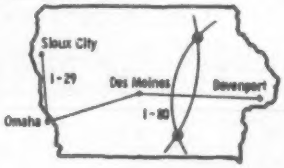
$\frac{1}{4} \times \boxed{80} = 20$ $\quad$ 50% of $\boxed{80} = 40$

Pair the tags.



IOWA

SCALE: 1 cm = 50 km



Complete.

	Length of Line Segment on Map	Real Distance by Road
Davenport to Des Moines	5 cm	250 km
Des Moines to Omaha	4 cm	200 km
Omaha to Sioux City	2.8 cm	140 km

Ms. Peltin drives from Omaha to Davenport on I-80. If she averages 90 km in one hour, about how long is her drive? 5 hr.

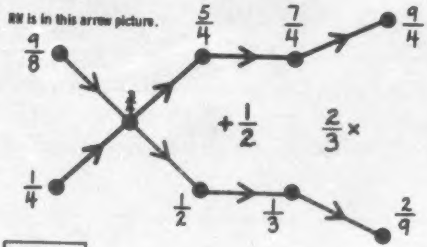
Ms. Peltin's car uses one liter of gasoline every 10 km. The gasoline costs $0.30 for one liter. How much gasoline does she use on her trip from Omaha to Davenport? 45 liters How much does that much gasoline cost? $13.50

By air, Cedar Falls, Iowa is 325 km from Sioux City and 185 km from Davenport. Draw dots at the two places where Cedar Falls could be. You may use a compass.

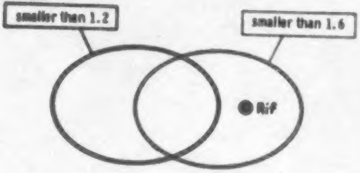
Rif is a secret number.

First clue

Rif is in this arrow picture.


$\frac{9}{8}$
$\frac{5}{4}$
$\frac{7}{4}$
$\frac{9}{4}$
$+\frac{1}{2}$
$\frac{2}{3}\times$
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{2}{9}$

Second clue


smaller than 1.2
smaller than 1.6
Rif

Who is Rif? $\frac{5}{4}$ (=1.25)

TWO AT A TIME

1. Pamela is 14 years older than her brother Andrew. She also is 3 times as old as Andrew. How old is Pamela? 21 Andrew? 7

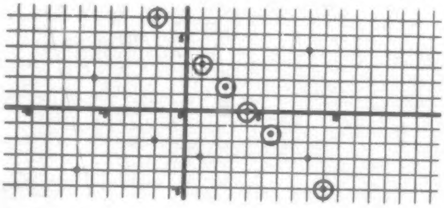
2. Only $\frac{1}{3}$ of the students in Mr. Cunetto's class are boys. If there are 9 boys in his class, how many students are in his class? 27

How many girls are in his class? 18

Fee and Gee are secret numbers.

First clue

(Fee, Gee) is one of the dots on this grid. You do not have to label the dots.

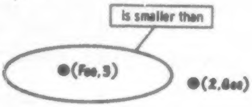


Second clue

Fee + Gee = 4

On the grid, circle the six dots that (Fee, Gee) could be.

Third clue


is smaller than
(Fee,3)
(2,Gee)

(Fee, Gee) is (2.5, 1.5).

THE RELATIONS GAME

Find an ordered pair, (Vim, Vam), which belongs to at least three of these relations. Many solutions are possible.

2×	is the square of
+2	[✓ is at least 20 smaller than]
[✓ $\frac{3}{2}\times$]	is a multiple of
[✓ $\boxed{+}\boxed{3}\boxed{=}\cdots$]	$\boxed{-}\boxed{5}\boxed{=}\cdots$

(Vim, Vam) could be (48, 72).

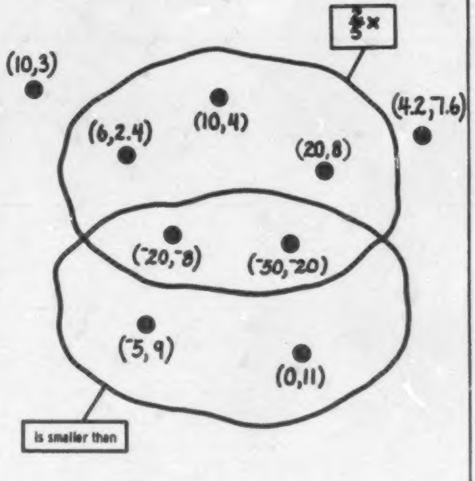
Circle all the relations that your choice of (Vim, Vam) belongs to.

(Many solutions are possible.

Find all the places in this table where the entry is 2.

×	1	2	3	4	5	6	7	8	9
1		2			2			2	
2	2			2			2		
3									
4		2			2			2	
5	2			2			2		
6									
7		2			2			2	
8	2			2			2		
9									

The dots are for ordered pairs of numbers.
Label the dots. One is done for you. Many answers are possible.
Hint: Some answers require negative numbers.



(Many solutions are possible.)

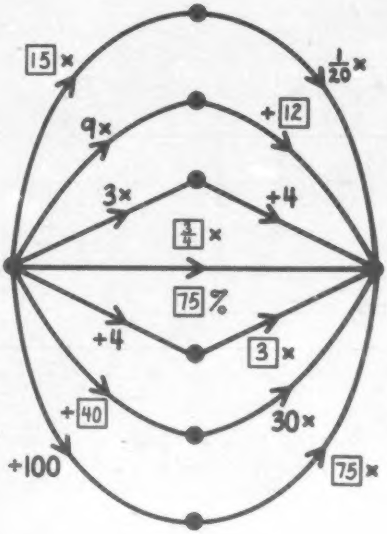
Mr. and Mrs. Harris and their two children, Audrey and Glen, must cross a river. The only boat available is a small canoe which can hold at most one adult or the two children. All the Harris' know how to paddle the canoe. Describe how the family can use the canoe to cross the river. The alligators in the river prevent anyone from swimming. Try to use as few trips as possible.

Above each arrow, indicate who is in the boat. You may not need all the arrows.

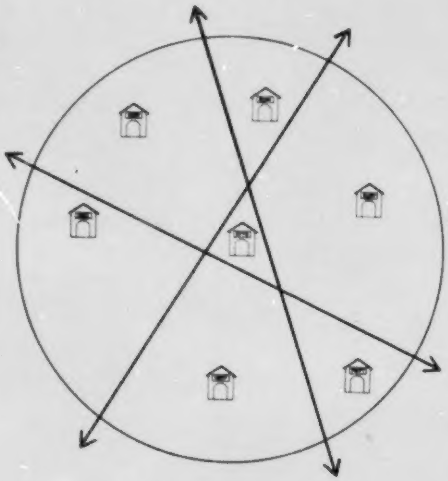
First trip across	Audrey, Glen →	Fourth trip across	Mrs. Harris →
First return trip	Audrey ←	Fourth return trip	Audrey ←
Second trip across	Mr. Harris →	Fifth trip across	Audrey, Glen →
Second return trip	Glen ←	Fifth return trip	←
Third trip across	Audrey, Glen →	Sixth trip across	→
Third return trip	Glen ←	Sixth return trip	←
		Seventh trip across	→

(Other solutions are possible; however, a shortest solution uses five trips across the river.)

Fill in the boxes for the arrows.

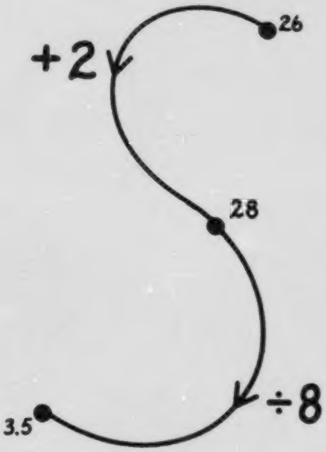


Use three straight fences (lines) to divide this circle so that each doghouse is in a different region.



(Other solutions are possible.)

Build a road from 26 to 3.5 with exactly two arrows. Each arrow may be for +☐, -☐, x☐, or ÷☐, where exactly one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 is in a box.



(Other solutions are possible.)

First clue

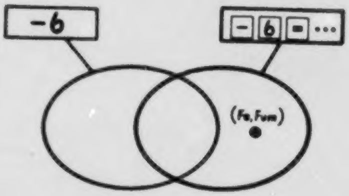
Fo and Fum are secret numbers between ⁻20 and 20.

Second clue

$+_5$	7	Fo
24	1	3
Fum	0	2

Fo could be ⁻16, ⁻11, ⁻6, ⁻1, 4, 9, 14, or 19.
Fum could be ⁻17, ⁻12, ⁻7, ⁻2, 3, 8, 13, or 18.

Third clue



(Fo, Fum) is (19, ⁻17).

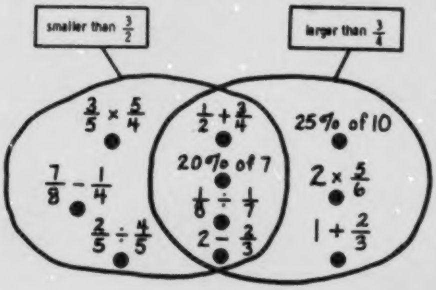
Put these numbers in the string picture.

$\frac{1}{2} + \frac{3}{4}$ $\frac{3}{5} \times \frac{5}{4}$ $2 \times \frac{5}{6}$

25% of 10 20% of 7

$\frac{7}{8} - \frac{1}{4}$ $\frac{1}{8} \div \frac{1}{7}$ $\frac{2}{5} \div \frac{4}{5}$

$1 + \frac{2}{3}$ $2 - \frac{2}{3}$


smaller than 3/2
larger than 3/4
3/5 × 5/4
7/8 − 1/4
2/5 ÷ 4/5
1/2 + 3/4
20% of 7
1/8 ÷ 1/7
2 − 2/3
25% of 10
2 × 5/6
1 + 2/3

$*$ is one of these operations.

$+$	$+_5$	$\sqcap$	$\sqcup$
$-$	$-_5$	$T_<$	$T_>$
$\times$	$\times_5$	T_8	T_{14}
$\uparrow$	$\downarrow$	$T_{=2}$	$T_{=4}$

First clue

$$9 * 3 = 3 * 9$$

$*$ could be $+$, $\times$, $+_5$, $\times_5$, $\sqcap$, $\sqcup$, $\uparrow$, $\downarrow$, $T_{=4}$ or $T_{=2}$.

Second clue

$$4 * 6 = 0$$

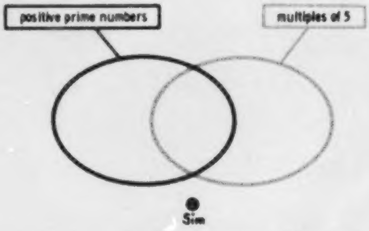
$*$ could be $+_5$ or $T_{=4}$.

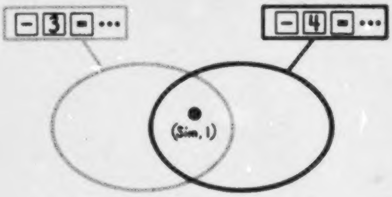
Third clue

$$^{-}7 * 29 = 1$$

$*$ is $T_{=4}$.

Sim is a secret number between 2 and 80.

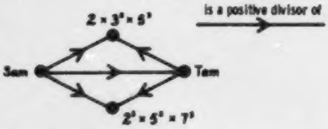

positive prime numbers
multiples of 5
Sim



Who is Sim? 49

Sam and Tam are two different secret numbers.

First clue


is a positive divisor of
$2 \times 3^4 \times 5^2$
Sam
Tam
$2^3 \times 5^2 \times 7^3$

(Sam, Tam) could be (1,2), (1,5), (1,10), (1,25), (1,50), (2,10), (2,50)(5,10), (5,25), (5,50), (10,50), (25,50)

(Hint: There are between ten and fifteen possibilities for (Sam, Tam).

Second clue

$$\frac{1}{Sam} + \frac{1}{Tam} < \frac{1}{4}$$

(Sam, Tam) could be (5,25), (5,50), (10,50), (25,50).

Third clue

$$Sam^3 > Tam^2$$

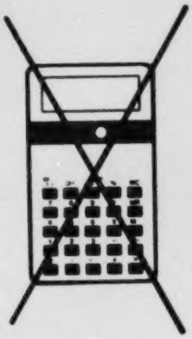
(Sam, Tam) is (25, 50).

COLLAGE
OF
PROBLEMS
#5

COLLAGE OF PROBLEMS #5

The problems in this workbook are in order of increasing difficulty. Approximately the first ten pages of the workbook have easy problems, the next ten to twelve pages have average problems, and the last ten pages have more challenging problems. It is suggested that all students work through as many pages as they can handle during the two lessons scheduled for the workbook. Not all students are expected to complete the entire workbook; most should complete at least ten pages, many should finish about twenty pages, and only the faster students will reach the most challenging problems at the end of the booklet.

Many problems in the workbook were designed with the availability of hand-calculators in mind. If possible, provide all the students with hand-calculators or at least give students access to hand-calculators.[†] You will find that many students do not need or want to use calculators; however, some students still need the support of the calculator in thinking about these problems. The problems on some pages are to be solved without using a hand-calculator. These pages are indicated by the following symbol.



[†] See Section 2(f) of the General Introduction for additional comments on the use of hand-calculators and on desirable characteristics of hand-calculators.

Lesson 1

Distribute individual copies of "Collage of Problems #5" and instruct students to work on their own. If you observe many students struggling with the same problem (or part of a problem), you could have a short collective discussion about that problem.

Collect the workbooks at the end of the lesson. Check the students' work and have the workbooks ready to return for Lesson 2.

Lesson 2

Return the students' copies of "Collage of Problems #5". You may wish to discuss any page that was difficult for most of the students. Then ask the students to correct their mistakes, if any, and to continue working in their workbooks.

If you observe many students struggling with the same problem (or part of a problem), you could have a short discussion of that problem.

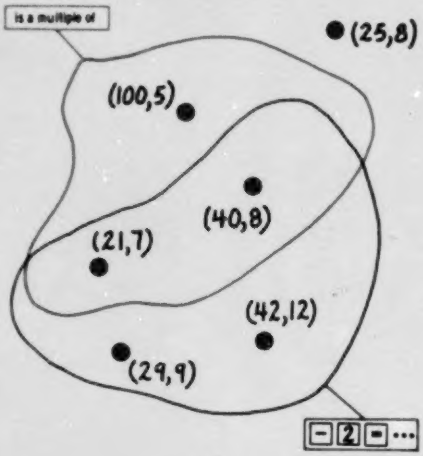
At the end of this lesson, collect the workbooks for your progress check. After checking the workbooks, you may wish to ask some students to work further during any free study time or as a homework assignment.

Collage of Problems #5

Put these ordered pairs into the string picture.

(21, 7) (42, 12) (40, 8)

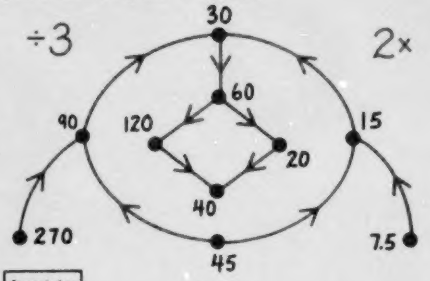
(25, 8) (100, 5) (29, 9)


is a multiple of
(25,8)
(100,5)
(40,8)
(21,7)
(42,12)
(29,9)
−2=…

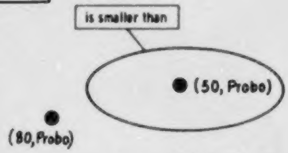
Probo is a secret number.

First clue

Probo is in this arrow picture. Label the dots.

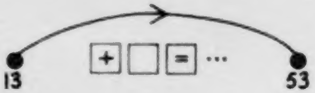

÷3
2×
30
60
90
120
15
20
40
270
45
7.5

Second clue

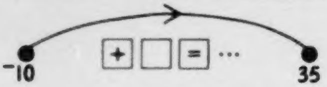

is smaller than
(50, Probo)
(80, Probo)

Probo is 60.

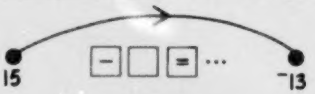
For each problem, list at least four whole numbers which could be in the box.



1, 2, 4, 5
(8, 10, 20, and 40 are also correct.)



1, 3, 5, 9
(15 and 45 are also correct.)

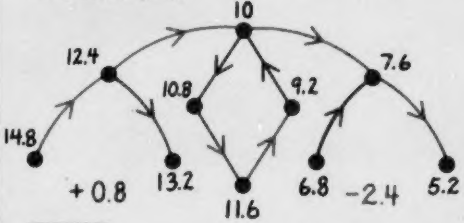


1, 2, 4, 7
(14 and 28 are also correct.)

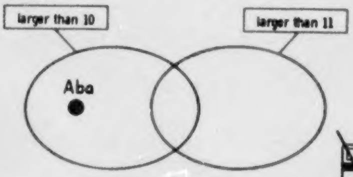
Aba is a secret number.

First clue

Aba is in this arrow picture.


10
12.4
7.6
10.8
9.2
14.8
13.2
11.6
6.8
5.2
+0.8
−2.4

Second clue


larger than 10
larger than 11
Aba

Who is Aba? 10.8

Complete the calculations. Show your work.

Add

$$\begin{array}{r} 6.\boxed{6}4 \\ 7\boxed{0}.7 \\ +\ \boxed{6}1.9\boxed{8} \\ \hline \boxed{1}39.32 \end{array}$$

Add

$0.08 + 7.153 + 64$

71.233

Subtract

$$\begin{array}{r} \boxed{6}6.92 \\ -\ 27.\boxed{6}5 \\ \hline 3\boxed{9}.2\boxed{7} \end{array}$$

Subtract

$65.7 - 8.46$

57.24

Fill in the boxes with these numbers.

$\frac{8}{3}$ $\frac{9}{2}$ $\frac{5}{8}$

$\frac{5}{11}$ $\frac{13}{11}$ $\frac{37}{10}$

$0 < \boxed{\frac{5}{11}} < \frac{1}{2} < \boxed{\frac{5}{8}} < 1 < \boxed{\frac{13}{11}} < 2$

$2 < \boxed{\frac{8}{3}} < 3 < \boxed{\frac{37}{10}} < 4 < \boxed{\frac{9}{2}} < 5$

Write each number as a product of positive prime numbers.
One is done for you.

$280 = 2 \times 2 \times 2 \times 5 \times 7 = 2^3 \times 5 \times 7$

(42) $= 2 \times 3 \times 7$

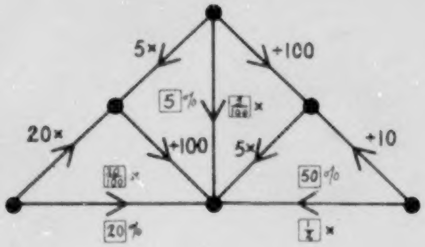
(90) $= 2 \times 3 \times 3 \times 5 = 2 \times 3^2 \times 5$

$275 = 5 \times 5 \times 11 = 5^2 \times 11$

$392 = 2 \times 2 \times 2 \times 7 \times 7 = 2^3 \times 7^2$

Circle the two numbers in the above list which are divisors of $2 \times 3^2 \times 5 \times 7$.

Fill in the boxes for the arrows.



Complete.

20% of 30 = $\boxed{6}$

20% of $\boxed{150}$ = 30

5% of 80 = $\boxed{4}$

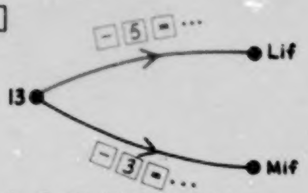
5% of $\boxed{1{,}600}$ = 80

50% of 24 = $\boxed{12}$

50% of $\boxed{48}$ = 24

(Lif, Mif) is a secret ordered pair.

First clue



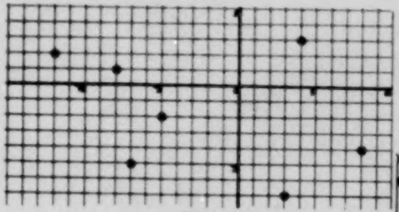
− 5 = …
Lif
13
− 3 = …
Mif

Lif could be 8, 3, $^-2$, $^-7$, $^-12$, $^-17$, $^-22$, and so on.

Mif could be 10, 7, 4, 1, $^-2$, $^-5$, $^-8$, and so on.

Second clue

(Lif, Mif) is one of the dots on this grid.



(Lif, Mif) = ($^-7$, $^-5$)

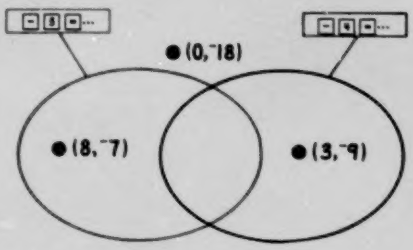
The red lag is one of these:

is larger than
+ =…
− 2 =…
− 3 =…
− 4 =…
− 5 =…

The blue lag is one of these:

is larger than
+ =…
− 2 =…
− 3 =…
− 4 =…
− 5 =…

Label the strings.

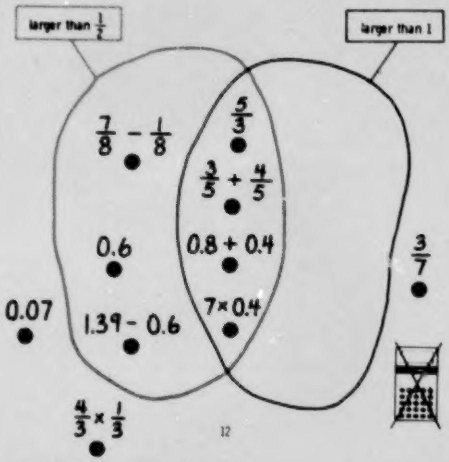


Put each of these numbers in the string picture.

$\frac{5}{3}$ $\frac{3}{7}$ $\frac{3}{5}+\frac{4}{5}$ $\frac{7}{8}-\frac{1}{8}$

$\frac{4}{3}\times\frac{1}{3}$ 0.6 0.07

0.8 + 0.4 1.39 − 0.6 7 × 0.4



larger than $\frac{1}{2}$
larger than 1
$\frac{7}{8}-\frac{1}{8}$
$\frac{5}{3}$
$\frac{3}{5}+\frac{4}{5}$
0.6
0.8 + 0.4
$\frac{3}{7}$
0.07
1.39 − 0.6
7 × 0.4
$\frac{4}{3}\times\frac{1}{3}$

Complete the calculations. Show your work.

Multiply

63.8 × 1.76 = 112.288

Divide	Divide
96.5 ÷ 2.5	12,852.6 ÷ 62
38.6	207.3

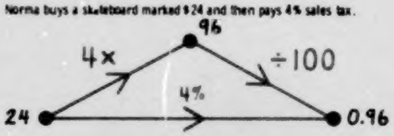
BULLDOG BUS LINES TIMETABLE From: Arolla To: Hill City	
City	Time
Arolla	5:50 AM
Blue Springs	7:30 AM
Cheyenne	8:10 AM
Davis	10:30 AM
El Pardo	1:20 PM
Franklin	3:10 PM
Gaston	5:10 PM
Hill City	7:40 PM

1. Ruby is taking a 4 hour, 40 minute bus ride starting from Davis.
To which city is she going? Franklin
After a 5 hour, 50 minute bus ride, Nick arrives at El Pardo.
From which city did he start? Blue Springs

2. Mr. Holtz drives the bus from Arolla to El Pardo. Ms. Melton drives from El Pardo to Hill City. If each receives $6.00 per hour, how much does Mr. Holtz earn each day? $45.00 Ms. Melton? $38.00

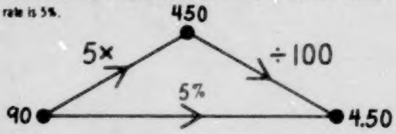
Use the arrow picture to help solve each problem. Label the dots and arrows; then answer the questions.

1. Norma buys a skateboard marked $24 and then pays 4% sales tax.



How much tax does Norma pay? $0.96
What is her total bill? $24.96

2. Ms. Hummel buys a bicycle for her son and her tax is $4.50. The tax rate is 5%.

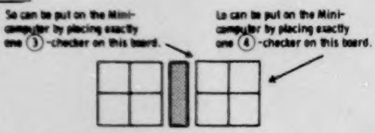


How much did the bicycle cost (without tax)? $90
What is the total bill? $94.50

(The blue arrows may be labeled differently.)

(So, Lo) is a secret ordered pair of numbers.

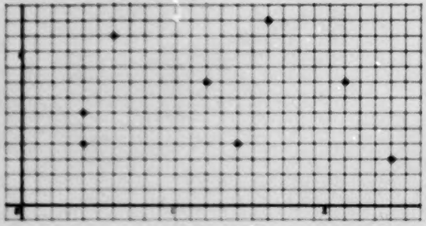
First clue


So can be put on the Mini-computer by placing exactly one ③-checker on this board.
Lo can be put on the Mini-computer by placing exactly one ④-checker on this board.

So could be 0.3, 0.6, 1.2, or 2.4.
Lo could be 0.4, 0.8, 1.6, or 3.2.

Second clue

(So, Lo) is one of the dots on this grid.



(So, Lo) = (1.2, 0.8)

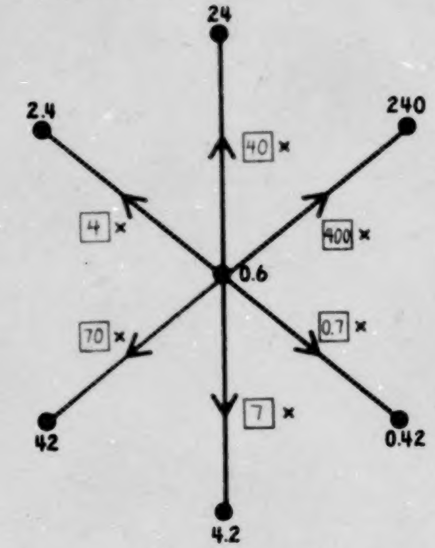
A farmer must carry a fox, a goose, and a basket of cabbage across a river. The farmer's rowboat is large enough for the farmer and only one item. The farmer cannot leave the goose and cabbage together because the goose will eat the cabbage. Also, the fox and goose cannot be left together (for obvious reasons). Describe how the farmer can get all three items across the river safely. Try to use as few trips as possible.

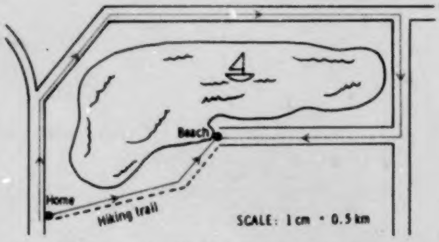
Write above each arrow what the farmer carries. You may not need all the arrows.

First trip across	→	goose
First return trip	←	nothing
Second trip across	→	cabbage
Second return trip	←	goose
Third trip across	→	fox
Third return trip	←	nothing
Fourth trip across	→	goose
Fourth return trip	←	
Fifth trip across	→	

(Other solutions are possible; however, a shortest solution uses four trips across.)

Fill in the boxes for the arrows.





Laticia enjoys running on the hiking trail (blue) from her home to the beach. Her brother, Tony, prefers riding his bicycle around the lake along the route marked in red.

Complete.

	Length of route on map	Real distance traveled
Laticia (blue)	8 cm	4 km
Tony (red)	34 cm	17 km

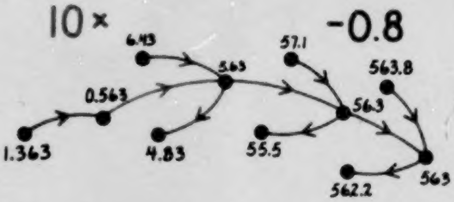
Laticia runs at an average of 10 km in one hour and Tony rides 20 km in one hour. About how long does it take Laticia to run on the hiking trail from her home to the beach? 24 minutes

About how long does Tony take to ride his bicycle on the roads from his home to the beach? 51 minutes

Vun is a secret number.

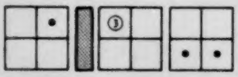
First clue

Vun is in this arrow picture.



Second clue

Vun can be put on this Minicomputer by adding just this checker: ③.



Vun is 6.43.

EARL'S
PARKING
LOT

$0.30 for each $\frac{1}{4}$ hour

How much does it cost to park a car for 1 hour? $1.20

10 hours? $12.00

About how long was a car parked if the charge is $0.90? 45 minutes ($\frac{3}{4}$ hour)

$3.00? $2\frac{1}{2}$ hours (2 hrs. 30 min)

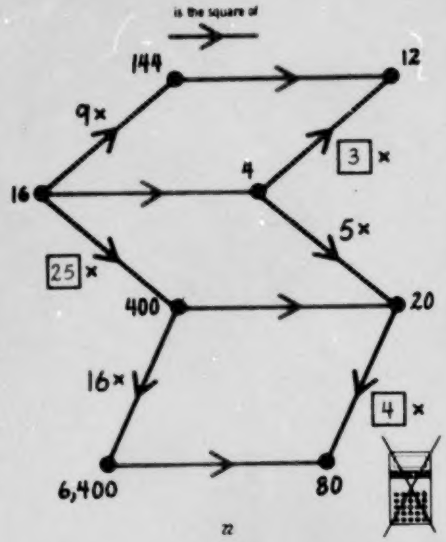
Sharon parked her car from 11:20 AM to 1:20 PM. How much must she pay? $2.40

Douglas parked his car from 5:45 PM to 7:00 PM. How much must he pay? $1.50

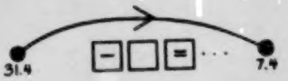
Carlos parked his car from 7:50 PM. When he left the lot, he had to pay $0.60. About what time did he leave the lot? 8:20 PM

The dots are for positive numbers.

Label the dots and fill in the boxes for the blue arrows.

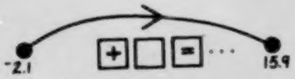

is the square of
144
12
9×
16
4
3 ×
5×
25 ×
400
20
16×
4 ×
6,400
80

For each problem, list at least three whole numbers which could be in the box.



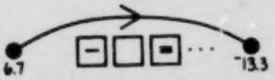
1, 2, 3

(4, 6, 8, 12, and 24 are also correct.)



1, 2, 3

(6, 9, and 18 are also correct.)



1, 2, 4

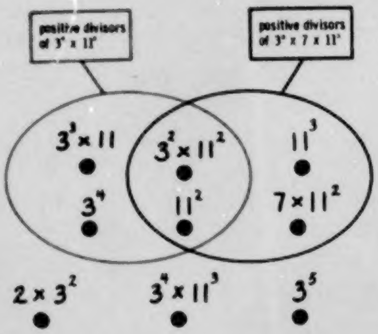
(5, 10, and 20 are also correct.)

Put these numbers in the string picture.

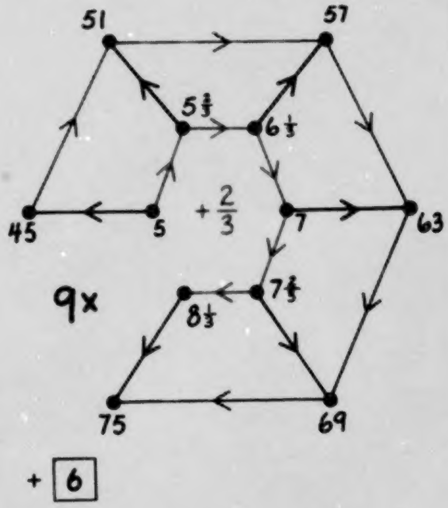
2×3^2 11^3 $3^2 \times 11^2$

11^2 $3^4 \times 11^3$ $3^3 \times 11$

3^4 3^5 7×11^2


positive divisors of 3⁴ x 11²
positive divisors of 3² x 7 x 11³
3³×11
3²×11²
11³
3⁴
11²
7×11²
2×3²
3⁴×11³
3⁵

Label the dots and fill in the box for the gray arrows.


51
57
5⅔
6⅓
45
5
+ 2/3
7
63
9x
7⅔
8⅓
75
69
+ 6

Find an ordered pair, (Zen, Zan), which belongs to at least three of these relations. Many solutions are possible.

$\frac{3}{2}\times$	$\frac{5}{2}\times$
$\frac{2}{3}\times$	is at least 20 larger than
$\boxed{-}\boxed{3}\boxed{=}\cdots$	$\boxed{+}\boxed{2}\boxed{=}\cdots$
$\boxed{-}\boxed{5}\boxed{=}\cdots$	$\boxed{+}\boxed{3}\boxed{=}\cdots$

(Zen, Zan) could be (90, 60).

Circle all the relations that your choice of (Zen, Zan) belongs to.

(Many solutions are possible.)

1. Toby is now three times as old as Leo. In eight years, Toby will be only twice as old as Leo. How old is Toby now? 24 Leo? 8

2.

STATE THEATER

NOW PLAYING: THE WILD AND CRAZY GUYS!

Adults: $3.00
Children: $2.00

Theater owner: How many customers did we have tonight?
Ticket seller: Only 80.
Owner: How many adults?
Seller: I forgot to count how many.
Owner: Do you know how much money we received?
Seller: Yes, $184.
Owner: Good, then I can calculate how many adults came.

How many adults came to the movie? 24 How many children? 56

$$\boxed{a * b = (\tfrac{1}{2}\times a) - b}$$

Example: $6 * 5 = (\frac{1}{2}\times 6) - 5 = 3 - 5 = {}^{-}2$

Complete.

$12 * 3 = \boxed{3}$ $\quad 2.5 * 0.5 = \boxed{0.75}$

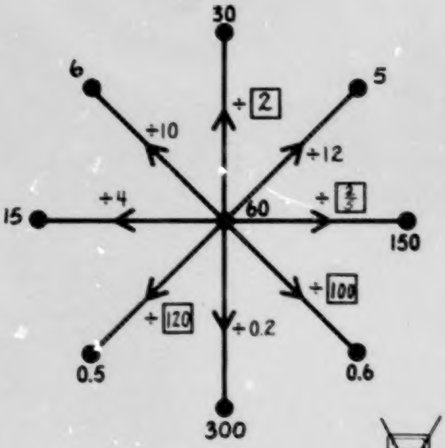
$23 * 8 = \boxed{3.5}$ $\quad 7 * 5 = \boxed{{}^{-}1.5}$

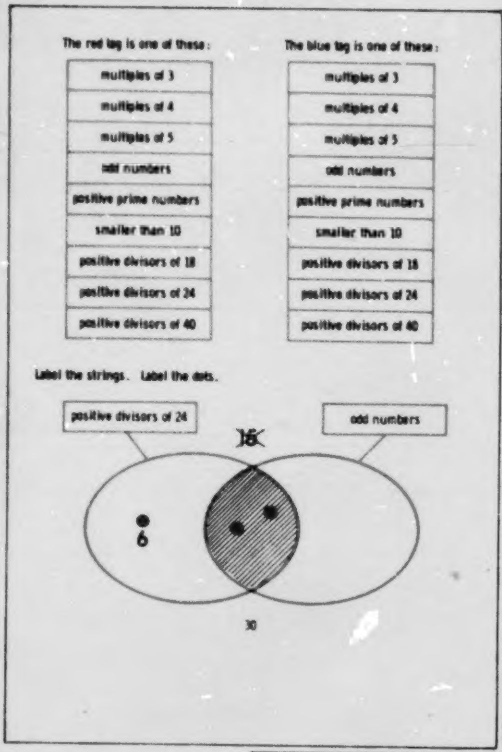
$10 * \boxed{3.5} = 1.5$ $\quad \boxed{30} * 7 = 8$

$26 * \boxed{10} = 3$ $\quad \boxed{34} * 10 = 7$

$\frac{2}{3} * \frac{1}{4} = \boxed{\frac{1}{12}}$ $\quad \frac{4}{5} * \boxed{\frac{3}{5}} = \frac{{}^{-}1}{5}$

Label the dots and fill in the boxes for the arrows.





The red tag is one of these:

- multiples of 3
- multiples of 4
- multiples of 5
- odd numbers
- positive prime numbers
- smaller than 10
- positive divisors of 18
- positive divisors of 24
- positive divisors of 40

The blue tag is one of these:

- multiples of 3
- multiples of 4
- multiples of 5
- odd numbers
- positive prime numbers
- smaller than 10
- positive divisors of 18
- positive divisors of 24
- positive divisors of 40

Label the strings. Label the dots.

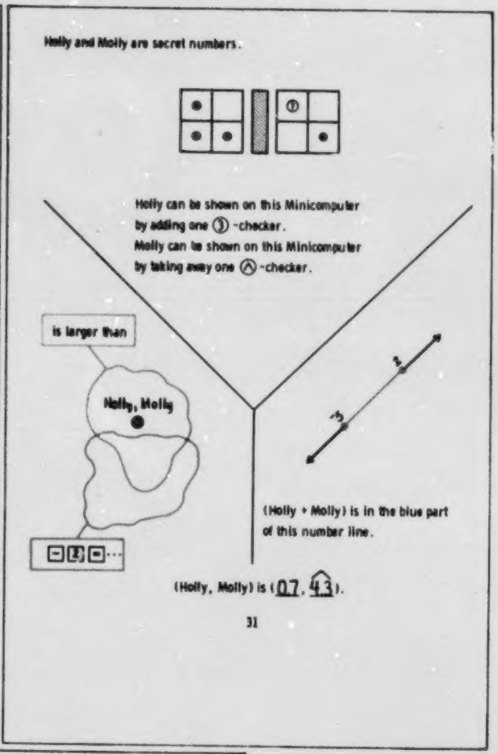
positive divisors of 24

odd numbers

~~15~~

6

30



Holly and Molly are secret numbers.

Holly can be shown on this Minicomputer by adding one ③ -checker.
Molly can be shown on this Minicomputer by taking away one Ⓐ -checker.

is larger than

Holly, Molly

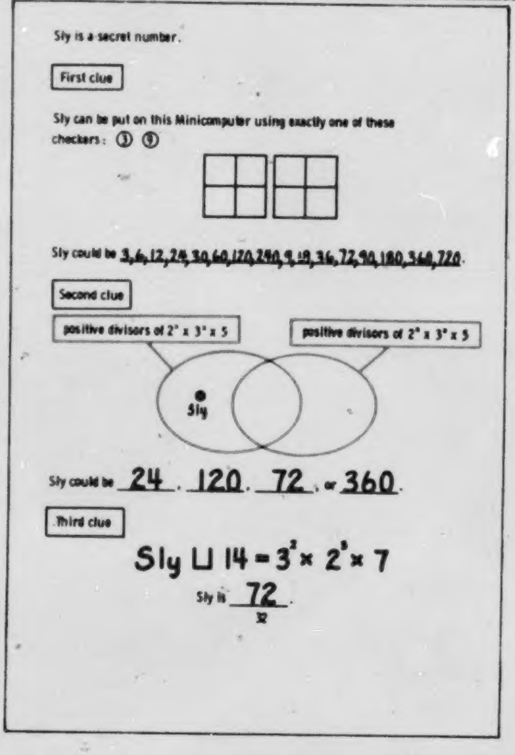
-3

2

(Holly + Molly) is in the blue part of this number line.

(Holly, Molly) is (07, 43).

31



Sly is a secret number.

First clue

Sly can be put on this Minicomputer using exactly one of these checkers: ③ ⑨

Sly could be 3, 6, 12, 24, 30, 60, 120, 240, 9, 18, 36, 72, 90, 180, 360, 720.

Second clue

positive divisors of 2" x 3" x 5

positive divisors of 2" x 3" x 5

Sly

Sly could be 24, 120, 72, or 360.

Third clue

$$Sly \sqcup 14 = 3^2 \times 2^3 \times 7$$

Sly is 72.

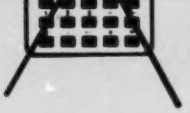
32

COLLAGE
OF
PROBLEMS
#6

COLLAGE OF PROBLEMS #6

The problems in this workbook are in order of increasing difficulty. Approximately the first ten pages of the workbook have easy problems, the next ten to twelve pages have average problems, and the last ten pages have more challenging problems. It is suggested that all students work through as many pages as they can handle during the two lessons scheduled for the workbook. Not all students are expected to complete the entire workbook; most should complete at least ten pages, many should finish about twenty pages, and only the faster students will reach the most challenging problems at the end of the booklet.

Many problems in the workbook were designed with the availability of hand-calculators in mind. If possible, provide all the students with hand-calculators or at least give students access to hand-calculators.[†] You will find that many students do not need or want to use calculators; however, some students still need the support of the calculator in thinking about these problems. The problems on some pages are to be solved without using a hand-calculator. These pages are indicated by the following symbol.



[†] See Section 2(f) of the General Introduction for additional comments on the use of hand-calculators and on desirable characteristics of hand-calculators.

W 12 &13

Lesson 1

Distribute individual copies of "Collage of Problems #6" and instruct students to work on their own. If you observe many students struggling with the same problem (or part of a problem), you could have a short collective discussion about that problem.

Collect the workbooks at the end of the lesson. Check the students' work and have the workbooks ready to return for Lesson 2.

Lesson 2

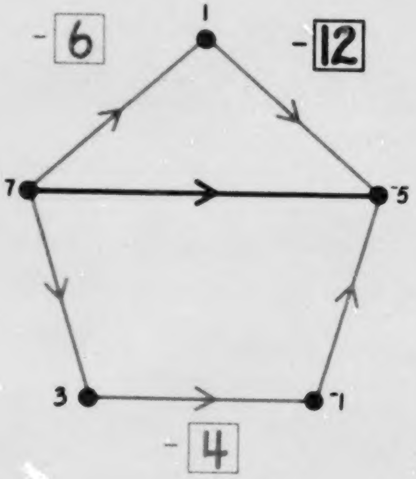
Return the students' copies of "Collage of Problems #6". You may wish to discuss any page that was difficult for most of the students. Then ask the students to correct their mistakes, if any, and to continue working in their workbooks.

If you observe many students struggling with the same problem (or part of a problem), you could have a short discussion of that problem.

At the end of this lesson, collect the workbooks for your progress check. After checking the workbooks, you may wish to ask some students to work further during any free study time or as a homework assignment.

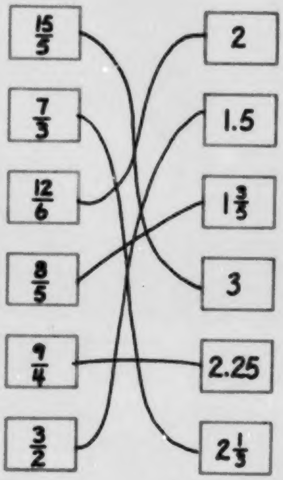
Collage of Problems #6

Fill in the boxes for the arrows. Label the dots.


-6
1
-12
7
ˉ5
3
ˉ1
-4

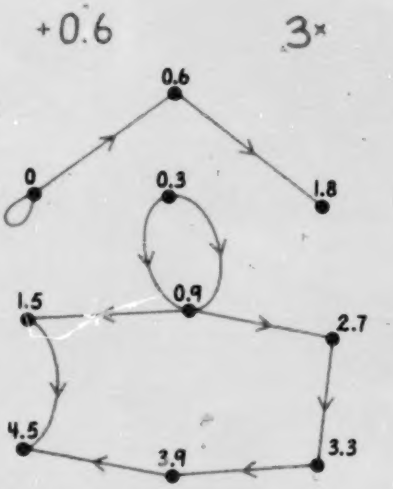
2

Pair the tags.


15/5
7/3
12/6
8/5
9/4
3/2
2
1.5
1 3/5
3
2.25
2 1/3

3

Draw all the red arrows and all the blue arrows you can between the dots in this picture.


+0.6
3×
0.6
0
0.3
1.8
1.5
0.9
2.7
4.5
3.9
3.3

4

Complete the calculations. Show your work.

Add

$$\begin{array}{r} \boxed{3}.68 \\ \boxed{7}0.8\boxed{3} \\ +23.\boxed{5}4 \\ \hline 98.05 \end{array}$$

Add

6.28 + 183 + 9.764

199.044

Subtract

$$\begin{array}{r} 4\boxed{1}.733 \\ -4.\boxed{6}\boxed{8}9 \\ \hline 37.04\boxed{4} \end{array}$$

Subtract

328 - 63.2

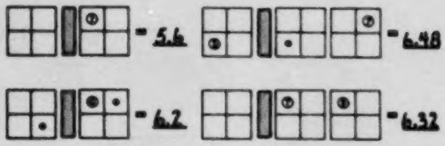
264.8

5

Vim is a secret number.

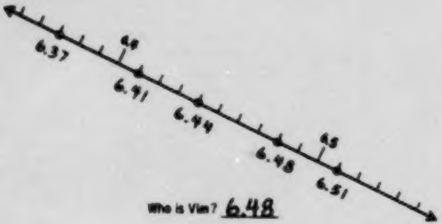
First clue

Vim is one of these numbers.


= 5.6
= 6.48
= 6.2
= 6.32

Second clue

Vim is one of the dots on this number line. Label the dots.


6.37
6.4
6.41
6.44
6.48
6.5
6.51

Who is Vim? 6.48

All of these calculations are correct, except that the decimal points are missing in the answers. Place a decimal point in each answer to make a true number story.

64.8 + 395.37 + 426 + 2.77 = 8 8 8.9 4

650.27 + 5.045 + 3,188.6 + 38.125 = 3 8 8 2.0 4

207.621 − 8.93 = 1 9 8.6 9 1

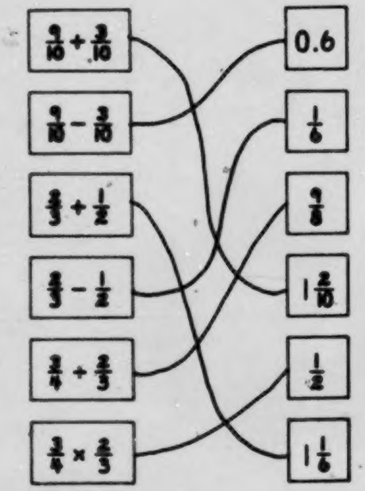
27,825.276 − 27,654.876 = 1 7 0.4

62.3 × 21.8 = 1 3 5 8.1 4

104.15 × 7.38 = 7 6 8.6 2 7

28 × 0.47 = 1 3.1 6

Pair the tags.


$\frac{9}{10} + \frac{3}{10}$
$\frac{9}{10} - \frac{3}{10}$
$\frac{2}{3} + \frac{1}{2}$
$\frac{2}{3} - \frac{1}{2}$
$\frac{3}{4} \div \frac{2}{3}$
$\frac{3}{4} \times \frac{2}{3}$
0.6
$\frac{1}{6}$
$\frac{9}{8}$
$1\frac{2}{10}$
$\frac{1}{2}$
$1\frac{1}{6}$

Use the given information to help solve these problems.

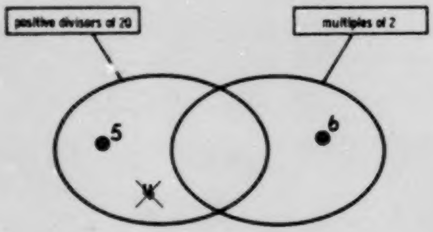
$20\% \text{ of } \$350 = \$70$
$6\% \text{ of } \$350 = \$21$

$26\% \text{ of } \$350 = \boxed{\$91}$
$60\% \text{ of } \$350 = \boxed{\$210}$
$100\% \text{ of } \$350 = \boxed{\$350}$
$\boxed{40}\% \text{ of } \$350 = \$140$
$\boxed{120}\% \text{ of } \$350 = \$420$

$14\% \text{ of } \$350 = \boxed{\$49}$
$18\% \text{ of } \$350 = \boxed{\$63}$
$106\% \text{ of } \$350 = \boxed{\$371}$
$\boxed{10}\% \text{ of } \$350 = \$35$
$\boxed{200}\% \text{ of } \$350 = \$700$

The red tag is one of these:	The blue tag is one of these:
multiples of 2	multiples of 2
multiples of 3	multiples of 3
larger than 10	larger than 10
smaller than 10	smaller than 10
positive divisors of 18	positive divisors of 18
positive divisors of 20	positive divisors of 20

Label the strings.



Fill in each box with "<", ">", or "=".

$$1{,}346 \div 27 \ \boxed{<}\ 1{,}346 \div 11$$

$$\frac{3}{5} \ \boxed{>}\ \frac{3}{11}$$

$$\frac{4}{9} \ \boxed{<}\ 0.5$$

$$\frac{11}{12} \ \boxed{<}\ \frac{12}{11}$$

$$50\% \text{ of } 18 \ \boxed{=}\ \frac{1}{2} \times 18$$

$$20\% \text{ of } 50 \ \boxed{=}\ 50\% \text{ of } 20$$

$$\frac{5}{7} \times 100 \ \boxed{>}\ \frac{3}{7} \times 100$$

$$\frac{1}{2} + \frac{1}{3} \ \boxed{>}\ 0.5 + 0.3$$

Complete the calculations. Show your work.

Multiply

$$4.79 \times 0.603 = 2.88837$$

Divide

$$377.4 \div 1.2$$

Answer: $\boxed{3}\boxed{1}\boxed{4}.\boxed{5}$

R = 0.$\boxed{0}$

Divide

$$204.37 \div 1.5$$

Answer: $\boxed{1}\boxed{3}\boxed{6}.\boxed{2}$

R = 0.$\boxed{0}\boxed{7}$

TUG - OF - WAR

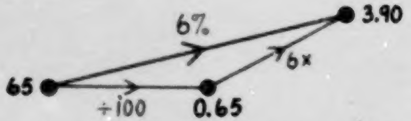
Barb, Alice, Jerry, and Ted enjoy playing tug - of - war. Alice by herself can beat Barb and Jerry together. Alice and Barb together are exactly as strong as Jerry and Ted; neither team can win. But Ted and Barb easily beat Alice and Jerry.

List the children in order of strength.

Ted (strongest), Alice, Barb, Jerry (weakest)

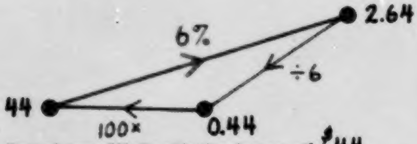
Use the arrow picture to help solve each problem. Label the dots and then solve the problem.

1. Josephine places $65 in a savings account at a bank which gives 6% annual interest.



How much interest does Josephine earn at the end of one year? $3.90

2. Lester placed his money in a savings account at the same bank. At the end of one year, he collected $2.64 interest.



How much money did Lester put in his savings account? $44

(There are other methods of using the arrow pictures to solve these problems.)

VIP Record Store sells albums at 4 for $25. At this rate, how much would a customer pay for:

20 albums? $125

6 albums? $37.50

1 album? $6.25

5 albums? $31.25

How many albums could you buy with $15? 2

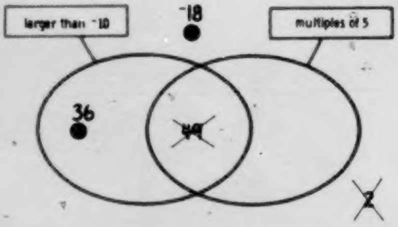
The Discount Store sells albums at 3 for $20. From which store would you buy records? VIP

Explain. 6 albums at VIP cost $37.50.
6 albums at Discount Store cost $40.

(There are many correct explanations.)

The red tag is one of these:	The blue tag is one of these:
multiples of 3	multiples of 3
multiples of 5	multiples of 5
multiples of 7	multiples of 7
odd numbers	odd numbers
square numbers	square numbers
larger than ⁻10	larger than ⁻10
smaller than ⁻10	smaller than ⁻10

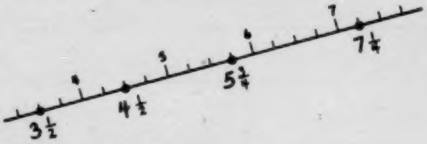
Label the strings.



Tina is a secret number.

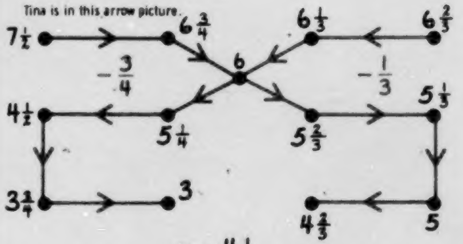
First clue

Tina is one of these dots. Label the dots.



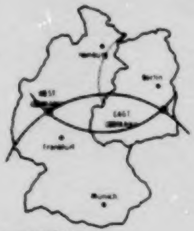
Second clue

Tina is in this arrow picture.



Tina is 4½

(The dots could be labeled with decimal names.)



SCALE: 1 cm on the map = 100 kilometers

On the map, what is the length of a line segment between Munich and Hamburg? 6.5 cm What is the real distance between Munich and Hamburg? 650 km

If an airplane that flies 400 kilometers in 1 hour leaves Munich at 9:30 AM, what time will it arrive in Hamburg? (Circle the closest answer.)

10:00 AM [✓ 11:00 AM] 1:00 PM 4:00 PM

What is the distance between Germany's northernmost point and Germany's southernmost point? 900 km

Köln is 450 km from Munich and 360 km from Hamburg. Draw blue dots at the two places where Köln could be.

Hanover, a city in West Germany, is 240 km from Berlin. Indicate in red places where Hanover could be.

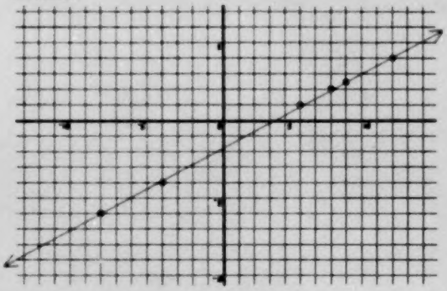
$$\boxed{a \longrightarrow (\tfrac{1}{2} \cdot a) - 2}$$

Fill in the blanks. Each ordered pair belongs to the red relation. One is done for you.

$(8, 2)$ $(12, \underline{4})$ $(6, \underline{1})$

$({}^-8, \underline{{}^-6})$ $({}^-4, \underline{{}^-4})$ $(9, \underline{2\tfrac{1}{2}})$

For each ordered pair, draw a dot on the grid. Sketch the Cartesian graph of this relation.



Young Wilbur visited the zoo one day and later told his sister, Julia, about one area where he saw a mixture of ostriches and zebras. Julia asked, "How many ostriches and zebras did you see?" Wilbur answered, "I won't tell you that, but there were 24 eyes and 30 feet."

Julia quickly calculated the number of ostriches and zebras. Can you?

There were 9 ostriches and 3 zebras.

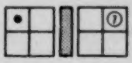
Extra problem: Wilbur saw some other common jungle animals in another part of the zoo. Altogether, these animals had 5 heads and 8 feet. Which animals and how many of each kind could Wilbur have seen? (Hint: the answer is a little tricky.)

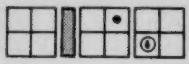
2 tigers and 3 snakes

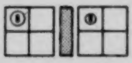
(Many solutions are possible.)

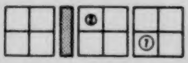
Put each number on the Minicomputer by adding exactly one of these checkers:

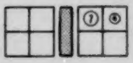
⑤ ⑦ ⑧

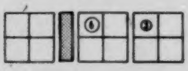

= 10.8


= 0.52

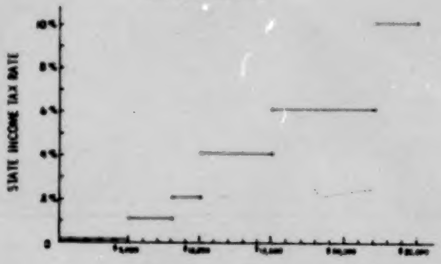

= 71.2


= 1.74


= 7.2


= 5.04

1979 STATE TAX RATE



Annual Income

1. What is the percent tax rate for a person who earns $13,500 in one year? 4% $7,300? 1%

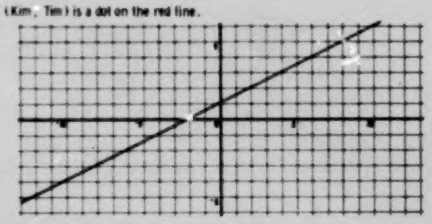
2. Ms. Stein's income tax rate if 6%. What is the largest her annual income could be? $22,000 Smallest? $15,000.01

3. Mr. Dowler earns $12,200 per year. How much income tax does he pay? $488 After paying his income tax, Mr. Dowler puts 5% of his money into a savings account this year (1979). How much does he put in the savings account this year? $585.60

4. Ms. Lapin's income tax rate is 10% and this year (1979) she paid $2,470 in state income tax. What is her annual income? $24,700

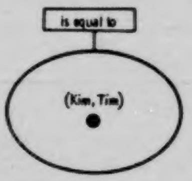
First clue

(Kim, Tim) is a dot on the red line.



(Kim, Tim) could be (6, 4), ($^-6$, $^-2$), (8, 5), ($^-10$, $^-4$), (9, $5\frac{1}{2}$), ($^-9$, $^-3\frac{1}{2}$), and so on.

Second clue



(Kim, Tim) is (2, 2).

Put one of these symbols in each box to make a true sentence.

$=_3$ M $=_4$ D $=_5$

48 + 6 $\boxed{=_4}$ 100 + 18

27 + 98 $\boxed{=_5}$ 5 × 48

$7^3 \times 11^2$ $\boxed{M}$ 7 × 11

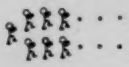
13 $\boxed{D}$ 3 × 5 × 13

$2^2 \times 3 \times 17$ $\boxed{=_3}$ 3 × 5 × 17

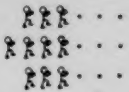
49 $\boxed{D}$ $3 \times 5 \times 7^2$

Paton High School's Marching Band is preparing to play in a city parade. Their director is trying to find which formation they should use.

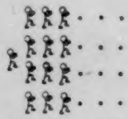
When the band marches in rows of two, poor Waldo is alone in the last row.



When the band marches in rows of three, again poor Waldo is alone in the last row.



When the band marches in rows of four, alas, poor Waldo is alone in the last row.



But when the band marches in rows of five, each row has exactly five people.

What is the smallest number of members the band could have? 25

If the band has between 50 and 140 members, how many students are in the band? 85

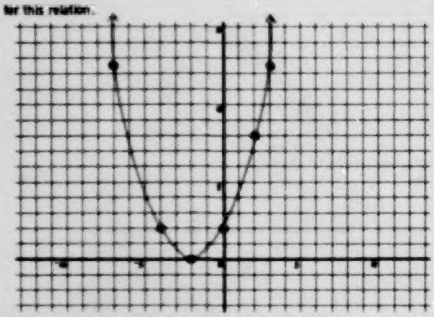
$$\boxed{a \longrightarrow \frac{1}{2} \times (a+2)^2}$$

Fill in the blanks. Each ordered pair belongs to the red relation. One is done for you.

(2, 8)   (0, 2)   (3, 12.5)

($^-$4, 2)   ($^-$2, 0)   ($^-$7, 12.5)

For each ordered pair, draw a dot on the grid. Sketch the Cartesian graph for this relation.



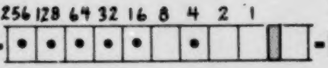
Name two more ordered pairs which belong to the red relation.

(1 , 4.5) and ($^-$6 , 8)

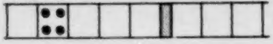
26

(Many solutions are possible.)

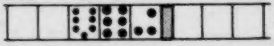
Put each number on the abacus and fill in the boxes.


Decimal Writing
256
128
64
32
16
8
4
2
1
Base Two Writing

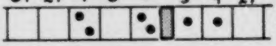
500 = = 111110100


625
125
25
5
1
Base Five Writing

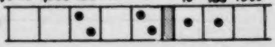
500 = = 4000


4096
512
64
8
1
Base Eight Writing

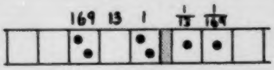
500 = = 764


81
27
9
3
1
$\frac{1}{3}$
$\frac{1}{9}$
$\frac{1}{27}$
Base Three Writing

$20\frac{4}{9}$ = = 202.11


10,000
1,000
100
10
1
$\frac{1}{10}$
$\frac{1}{100}$
$\frac{1}{1000}$
Base Ten Writing

202.11 = = 202.11


169
13
1
$\frac{1}{13}$
$\frac{1}{169}$
Base Thirteen Writing

$340\frac{14}{169}$ = = 202.11

27

$$\boxed{a * b = \frac{a}{b} + \frac{1}{2}}$$

Example: $2 * 3 = \frac{2}{3} + \frac{1}{2} = \frac{4}{6} + \frac{3}{6} = \frac{7}{6}$

Complete.

$3 * 4 = \boxed{\frac{5}{4} = 1\frac{1}{4}}$ $1 * 10 = \boxed{\frac{6}{10} = \frac{3}{5}}$

$10 * 7 = \boxed{\frac{27}{14} = 1\frac{13}{14}}$ $2 * 5 = \boxed{\frac{9}{10}}$

$4 * 3 = \boxed{\frac{11}{6} = 1\frac{5}{6}}$ $\boxed{6} * 8 = 1\frac{1}{4}$

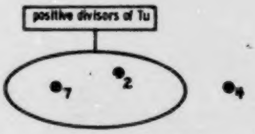
$\boxed{15} * 10 = 2$ $\boxed{14} * 5 = 3.3$

$6 * \boxed{3} = \frac{5}{2}$ $1 * \boxed{6} = \frac{2}{3}$

28

Tu is a secret whole number smaller than 100.

First clue


positive divisors of Tu
7
2
4

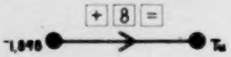
Tu could be 14 , 42 , 70 , or 98 .

Second clue

$Tu =_3 2$

Tu could be 14 or 98 .

Third clue

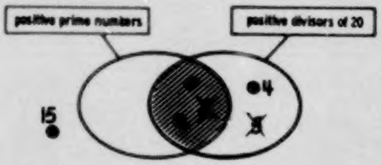


Who is Tu? 14

29

The red tag is one of these:	The blue tag is one of these:
multiples of 2	multiples of 2
multiples of 3	multiples of 3
multiples of 10	multiples of 10
odd numbers	odd numbers
positive prime numbers	positive prime numbers
positive divisors of 12	positive divisors of 12
positive divisors of 18	positive divisors of 18
positive divisors of 20	positive divisors of 20
positive divisors of 24	positive divisors of 24
positive divisors of 27	positive divisors of 27

Label the strings.



Use the spaces to briefly explain your answers.

The world population in 1979 is about 4 billion. If the present 1.7% yearly rate of net population growth continues, what will be the world's population in 1980? 4.068 billion 1981? 4.137 billion 1982? 4.207 billion

In what year will the population be 5 billion? 1993
6 billion? 2004

(Many solutions are possible depending on the method used to round off answers.)

Alice, Brenda, and Carl are three of Nabu's friends. When Nabu asked who was the oldest, they made the following statements.

Alice: I am younger than Brenda.
Carl is not the oldest.

Brenda: I am the oldest.
Carl is younger than Alice.

Carl: I am older than Brenda.
Alice is the youngest.

To confuse Nabu, each person made one true statement and one false statement. Nabu was still able to determine who was oldest and who was youngest.

Who is oldest, Alice, Brenda, or Carl? Alice

Who is youngest? Brenda

Circle each person's true statement.

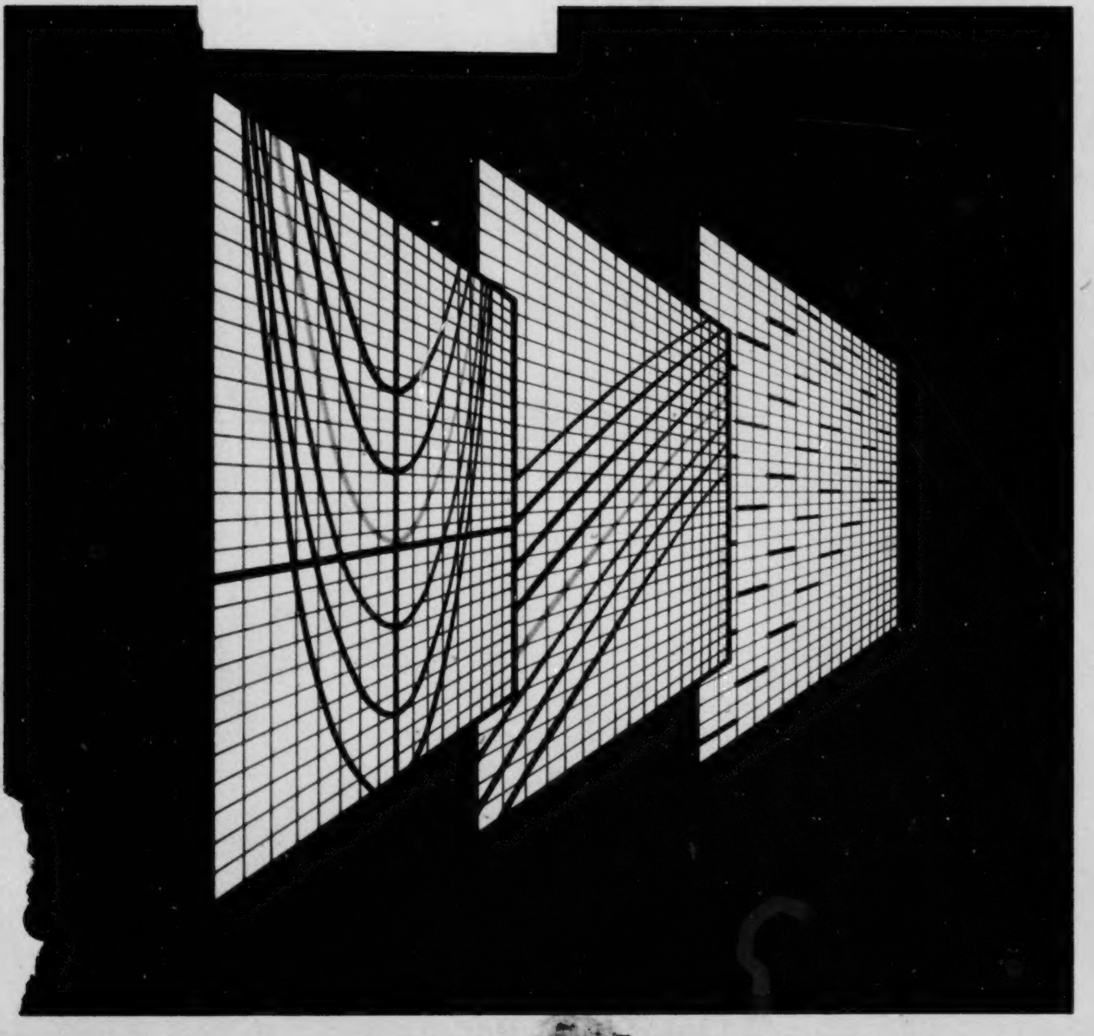
SE

Comprehensive School Mathematics Program
CEMREL, Inc.
Experimental Version
6-41101

CSMP Mathematics for the Intermediate Grades
Part VI

Teacher's Guide

'he Languages of Strings and Arrows
Geometry and Measurement
Probability and Statistics



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THE LANGUAGES OF STRINGS AND ARROWS

Comprehensive School Mathematics Program

CSMP Mathematics
for the
Intermediate Grades,
Part VI

The Languages of Strings and Arrows

CEMREL, Inc.

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THE LANGUAGES OF STRINGS AND ARROWS

INTRODUCTION

The language of strings is the nonverbal language of sets, that is, collections of objects put together in some way. The language of arrows is the nonverbal language of relations among objects. These languages permeate the CSMP content strands and are used, separately or together, to present an unending variety of numerical and nonnumerical situations.

Classification: The Language of Strings

As the word implies, classifying means putting things into classes, or as the mathematician says, into sets. If the mathematics of sets has any place in an elementary school mathematics curriculum, it is precisely because it can help students to understand and use the ideas of classification.

The basic idea is simple: Given a set S and any object x, either x belongs to S (x is in S) or x does not belong to S (x is not in S). We represent this simple act of sorting -- "in" or "out" -- neatly by using string pictures to illustrate "in" and "out" in a dramatic way. Objects to be sorted are represented by dots and the sets into which they are sorted are represented by drawing "strings" around dots. A dot inside (the region delineated by) a set's string is for an object in the set and a dot outside a set's string is for an object not in the set. (See for example the storybook WHERE'S MY NOSE?.)

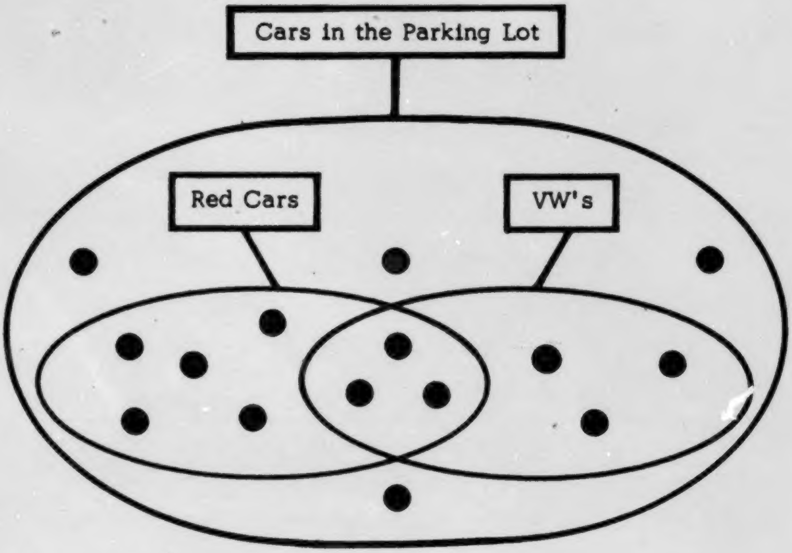
This language of strings and dots provides a precise (and nonverbal) way of recording and communicating thoughts about classification. The abilities to classify, to reason about classification, and to extract information from a classification are important skills for everyday life, for intellectual activity, in general, and for the pursuit and understanding of mathematics, in particular. Ordinarily, we are not expected to deal with such problems until our command of our natural language is fully developed and until we face classification problems in adult life. It is not that an ability to classify requires special intelligence, but that it requires a special mode of thinking. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification. It frees young minds to think logically and creatively about classes and to report their thinking long before they have extensive verbal skills.

In this strand we present situations and ask carefully phrased questions to continue to advance skills in classification, always remembering that the skills grow out of such experiences. To be able to draw strings and dots is not an objective in itself; to develop the mode of thinking involved is the objective. Thus it is important for us to construct the situations carefully. The sets into which we ask students to classify objects must be determined by well-defined attributes; otherwise, the students have the added problem of deciding whether a certain object does or does not have a certain attribute.

For example, consider this situation. A teacher announces that the blue string on the board is for all children in the class with dark hair and the red string is for all children in the class who are tall. The teacher invites John, a student in the class, to locate a dot for himself in the picture. He places a dot inside the blue string and outside the red string and labels it "J". There is a flurry of mixed agreement and disagreement in the classroom. John has brown hair and he thinks it is dark, but some say it is really light. John does not think he is tall, but someone insists that he is taller than most of the other children in the class. As a result, most of the energy in the class is directed toward making a decision about what attributes John has in this situation rather than how to classify John relative to those attributes. More care in choice of attributes would have avoided such distraction and confusion. For example, if the blue string were for all children in the class whose birthdays are in January and the red string for all the children who have a sister, John's choice of a dot for himself would be criticized strictly on its placement (i.e. his classification of himself) and not on his decision as to what attributes he has.

For this reason we make extensive use of A-Blocks (Attribute Blocks) and sets of numbers in designing classification situations. Students can immediately say whether or not a block is red, whether or not it is a square, and so on. They know whether or not a given number is less than 15, whether or not it is odd, and so on.

One reason for classifying objects is to count the objects that have a certain attribute. Suppose, for example, in a parking lot there are eight red cars and six Volkswagens. If four cars in the lot are neither red nor Volkswagens, can there be eighteen cars in the lot? Suppose we count the cars in the parking lot and find there are exactly fifteen. Is this possible? A string picture immediately settles this apparent paradox.



There are three red Volkswagens.

Relations: The Language of Arrows

Relations are interesting and important to us in our everyday lives, in our careers, in school, and in scientific pursuits. We are always trying to establish, explore, and understand relations. In mathematics it is the same; to study mathematics means to study the relations among mathematical objects like numbers or geometric shapes. The tools we use to understand the everyday world are useful to understand the world of mathematics. Conversely, the tools we develop to help us think about mathematical things often serve us in nonmathematical situations.

A serious study of anything requires a language for representing the things under investigation. The language of arrows provides an apt language for studying and talking about relations. Arrow diagrams are a handy graphical representation of a relation, somewhat the same way that a blueprint is a handy graphical representation of a house. By means of arrow diagrams, we can represent important facts about a given relation in a simple, suggestive, pictorial way -- usually more conveniently than the same information could be presented in words. (See for example the lessons on the storybook A VALENTINE MYSTERY.)

The convenience of arrow diagrams has important pedagogical consequences for introducing children to the study of relations in the early grades. A child can "read" -- and also draw -- an arrow diagram of a relation long before he or she can read or present the same information in words. The difficulty of presenting certain ideas to children lies not in their intellectual inability to grasp the ideas; rather, the limitations are often mechanical, motor problems of writing or reading about the ideas in conventional systems. Arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw. Another educational bonus occurs when arrow diagrams are used imaginatively. A story or problem may be captured by an arrow diagram, and at the same time, the resulting arrow diagram may suggest other, similar situations. This allows students to call upon previous experience or to expand their creativity.

One of the purposes of this strand is to exploit the power of the language of arrows to help children think logically about relations. Again it must be remembered that our goals concern the thinking process and not the mechanism. We resist making the ability to draw prescribed arrows an objective in itself, just as we resist viewing an arrow diagram as just another format for drill problems in arithmetic.

The general aim of The Languages of Strings and Arrows strand is to suggest situations that are inherently interesting and thought-provoking, and to give children modes of thinking and appropriate languages with which they can organize, classify, and analyze.

The content of the language strand in IG-VI centers around three games: the String Game, the Table Game, and the Relations Game. All three games have standard and analysis versions. In the standard versions, players must decide individually what information is gained from a play, whereas in the analysis versions the class as a group decides. Since collective analysis helps students to build successful strategies, playing the analysis version of a game usually precedes playing its standard version in the IG-VI language lessons.

1

THE TABLE GAME

The Table Game focuses on operations and on finding the images of ordered pairs of numbers under an operation. In the Table Game players determine which of twenty-two operations a table is for by making trial entries. Incorrect entries as well as correct entries immediately limit the possibilities for the operation. Three lessons in IG-V prepared students to play the Table Game; a fourth lesson introduced the standard version of the game. This semester (in IG-VI) we devote four lessons to the Table Game and to related activities.

To play the Table Game well, students must employ classification techniques accurately and use strategy in selecting their plays. In this chapter we introduce the analysis version of the Table Game, a game format designed to encourage systematic analysis after each play. An activity in the last of the four lessons focuses on the skill of selecting useful plays; you interrupt a game to ask for entries with the highest to lowest probability of being correct. Either extreme (a play with probability 0 or 1) gives no new information.

As a side benefit, the game inspires a new type of detective story, one that involves a secret operation rather than a secret number. The clues in such a detective story can give information about a table for the secret operation; for example, a clue might tell you how many 1's are in a given two by two table for an operation. (See Worksheets L03 and L04.)

CAPSULE LESSON SUMMARIES

Section 1.1: Introduce and play the analysis version of the Table Game. Then play the game in the usual way.

Section 1.2: Play both the analysis and standard versions of the Table Game.

Section 1.3: As an individual activity, identify an operation from selected entries in a table for the operation. Play both the analysis and standard versions of the Table Game.
(Worksheets L01, L02, L03, L04)

Section 1.4: Go over two detective stories that were assigned in Section 1.3. Given three clues, decide which of twenty-two operations a table is for. After the first clue, find entries which have a $\frac{4}{5}$ ($\frac{3}{5}$, $\frac{2}{5}$, $\frac{1}{5}$) probability of being correct. After the second clue, find entries which one can be absolutely sure are correct. Play the Table Game in the usual way.
(Worksheets L03, L04)

1.1 The Table Game #1

CAPSULE LESSON SUMMARY

Introduce and play the analysis version of the Table Game. Then play the game in the usual way.

MATERIALS

Teacher: Three copies of the Table Game Poster; felt-tip marker
Student: Table Game Analysis Pad; pencil

DESCRIPTION OF LESSON

Exercise 1: The Table Game, Analysis Version

In the analysis version of the Table Game, as in the usual version, the players make trial entries in a 9 by 9 table and try to determine which of twenty-two given operations the table is for. Whether correct or incorrect, each entry provides information[†] about the secret operation *. Good choices of trial entries are rewarded; in the analysis version, the amount of information gained from an entry determines the number of points scored. To play the analysis version of the Table Game proceed as follows.

Divide the class into two teams, Team A and Team B. On the board draw[‡] a 9 by 9 table for a secret operation and tape a copy of the Table Game Poster near the table. See the next illustration.

[†] Of course, an entry that is incorrect for all twenty-two operations would give no information.

[‡] Instead you could draw the grid on an overlay in permanent ink and then project the grid onto the board using an overhead projector.

L1

THE TABLE GAME

$+$	$+_3$	$+_1$	$+_o$
$-$	$-_3$	$-_1$	$-_o$
$\times$	$\times_3$	$\times_1$	$\times_o$
$\uparrow$	$\downarrow$	$\sqcap$	$\sqcup$
T_o	T_m	$T_<$	$T_>$
	T_{-3}	T_{-o}	

*	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7									
8									
9									

Begin the game by making one to three entries in the table as starting clues. Such an entry can be correct or incorrect (indicated by an "×" through it). Ask the class to consider collectively the twenty-two possibilities for * and to decide which of them cannot be the secret operation.

Explain to the class how to play the game after the initial analysis.

> PLAY: During a team's turn, a member suggests an entry to make in the table; for example, a player might suggest that 2 * 3 = 6. You make the entry into the table. If correct, tell the class; if incorrect, say "no" and put an "×" through the entry. Let the entire class analyze the situation. Although only the team whose turn it is can score, anyone can help with the analysis after a play.
>
> SCORE: A team gains five points each time a member's play, whether a correct entry or an incorrect one, eliminates at least one-third of the possibilities for * at that time ("one-third of" computed to the nearest whole number). The team gains an additional two points for each possibility eliminated over the required number to score. If fewer than that number are eliminated, points are neither gained nor lost.

Alternating teams, select students or ask for volunteers to make plays. Before a team makes a play, decide what is the smallest number of possibilities that must be eliminated for the team to score. After each play, compute and record the score. The class can help you with the scoring. Continue until the operation * is determined. The team with the most points wins.

Play this version of the Table Game twice. These are two of the choices for the secret operation * with appropriate starting clues. Remember that at the beginning of the game you only put starting clues in the table.

First game	Operation: $-_5$	Starting clue: $9 * 5 \neq 0$
Second game	Operation: $\sqcup$	Starting clue: $2 * 4 \neq 8$

For your convenience, completed tables for $-_5$ and $\sqcup$ are shown here.

$-_5$	1	2	3	4	5	6	7	8	9
1	0	4	3	2	1	0	4	3	2
2	1	0	4	3	2	1	0	4	3
3	2	1	0	4	3	2	1	0	4
4	3	2	1	0	4	3	2	1	0
5	4	3	2	1	0	4	3	2	1
6	0	4	3	2	1	0	4	3	2
7	1	0	4	3	2	1	0	4	3
8	2	1	0	4	3	2	1	0	4
9	3	2	1	0	4	3	2	1	0

$\sqcup$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	2	6	4	10	6	14	8	18
3	3	6	3	12	15	6	21	24	9
4	4	4	12	4	20	12	28	8	36
5	5	10	15	20	5	30	35	40	45
6	6	6	6	12	30	6	42	24	18
7	7	14	21	28	35	42	7	56	63
8	8	8	24	8	40	24	56	8	72
9	9	18	9	36	45	18	63	72	9

To help you better understand this version of the Table Game, we describe a sample playing of the first game. The operation is $-_5$ and the starting clue is $9 * 5 \neq 0$. The Table Game Poster is taped to the board and a 9 by 9 table is drawn next to it with the starting clue noted in the table. See the next illustration.

L1

THE TABLE GAME

$+$	$+_3$	$+_5$	$+_{10}$
$-$	$-_3$	$-_5$	$-_{10}$
$\times$	$\times_3$	$\times_5$	$\times_{10}$
$\uparrow$	$\downarrow$	$\sqcap$	$\sqcup$
T_D	T_M	$T_<$	$T_>$
	T_{-3}	T_{-4}	

*	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7									
8									
9					$\cancel{0}$				

T: Today we are going to play the Table Game a little differently. I put a clue in the table, $9 * 5 \neq 0$. Which of these possibilities for * (pointing to the poster) are eliminated?

S: $\times_3$, $9 \times 5 = 45$ and you can get to 0 by subtracting threes from 45.

T: 45 is a multiple of 3. So if you subtract threes from 45, you will eventually get to 0. Remember when you multiply modulo 3, you multiply in the usual way and then add or subtract threes to get 0, 1, or 2.

Cross out "$\times_3$" on the poster.

S: $\times_5$, $9 \times 5 = 45$ and 45 is a multiple of 5. So $9 \times_5 5 = 0$.

Cross out "$\times_5$" on the poster.

S: T_M, 9 is not a multiple of 5.

T: If * were T_M, $9 * 5$ would be 0; but it's not, so we can cross out "T_M".

Cross out "T_M" on the poster.

S: T_D, 9 is not a divisor of 5.

Cross out "T_D" on the poster.

S: $T_>$.

T: Is $9 > 5$?

S: Yes.

T: So if * were $T_>$, $9 * 5$ would be 1 and so 0 (pointing to "0" in the table) would not go there. But since "0" is crossed out, it doesn't go there. So we can't cross out "$T_>$", but what can we cross out?

S: $T_<$, 9 is not smaller than 5.

Cross out "$T_<$" on the poster.

T: Is there any other of these operations which we know that * cannot be?

S: No.

THE TABLE GAME

$+$	$+_3$	$+_5$	$+_{10}$
$-$	$-_3$	$-_5$	$-_{10}$
$\times$	~~$\times_3$~~	~~$\times_5$~~	$\times_{10}$
$\uparrow$	$\downarrow$	$\sqcap$	$\sqcup$
~~T_D~~	~~T_M~~	~~$T_<$~~	$T_>$
	T_{-3}	T_{-4}	

T: How many possibilities for * are left?

S: Seventeen, $22 - 5 = 17$.

L1

T: What is one-third of 17 to the nearest whole number? (6)
Which multiple of 3 is closest to 17?

S: 18.

T: What number is $\frac{1}{3} \times 18$?

S: 6.

T: So one-third of 17 to the nearest whole number is 6.
Team A, you will go first. Since the whole number closest to $\frac{1}{3} \times 17$ is 6, you must eliminate six possibilities for * by your entry. If you succeed, you score five points; otherwise, none. Also, you get two bonus points for each possibility eliminated after the required six.

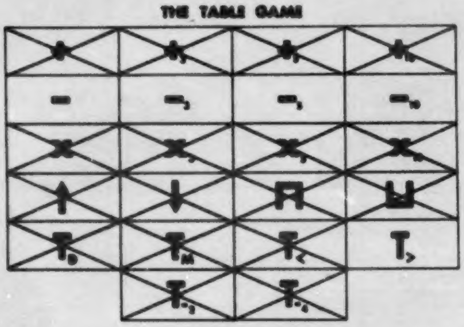
S (Team A): 1 * 1 = 0.

*	1	2	3	4	5	6	7	8	9
1	0								
2									
3									
4									
5									
6									
7									
8									
9					~~8~~				

T: "1 * 1 = 0" is correct. What can we cross out?

Players from both teams explain why * cannot be:

$+$, $+_3$, $+_5$, $+_{10}$, $\times$, $\times_{10}$, $\sqcap$, $\sqcup$, $\downarrow$, $\uparrow$, $T_{=2}$, $T_{=4}$.



T: Team A eliminated twelve possibilities for *. They needed six to score. $6 \times 2 = 12$, so Team A scores twelve bonus points plus five points for eliminating at least six.

Team A	Team B
17	

T (pointing to the poster): How many possibilities for * are left?

S: Five.

T: One-third of 5 to the nearest whole number is . . . ?

S: 2.

T: Team B needs to eliminate at least two possibilities for * in order to score.

L1

S (Team B): 2 * 8 = 1.

*	1	2	3	4	5	6	7	8	9
1	0								
2								*	
3									
4									
5									
6									
7									
8									
9					~~8~~				

T: "2 * 8 = 1" is not correct. Can we eliminate any of the five remaining possibilities?

S: No.

T: Let's check. What is 2 - 8? $(^-6)$ $2 -_3 8 = \ldots$? (0) $2 -_5 8 = \ldots$? (4) $2 -_{10} 8 = \ldots$? (4) Is "2 > 8" true or false? (False) So 2 * 8 does not equal 1 for any of the five remaining operations.

Team B does not score. There are still five possibilities for * remaining. Team A needs to eliminate two of them.

S (Team A): 3 * 3 = 0.

*	1	2	3	4	5	6	7	8	9
1	0								
2								*	
3			0						
4									
5									
6									
7									
8									
9					~~0~~				

T: "3 * 3 = 0" is correct. Can we eliminate any of the five possibilities?

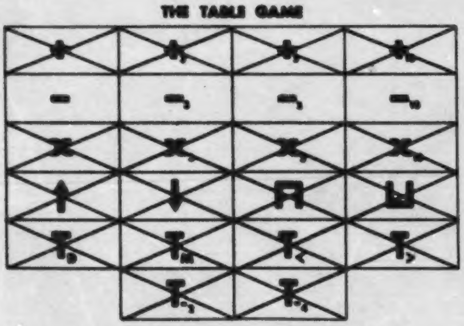
The class finds that $3 - 3$, $3 -_3 3$, $3 -_5 3$, $3 -_{10} 3$, and $3 \; T_> \; 3$ are all equal to 0. No new information is gained. Team B needs to eliminate two possibilities.

S (Team B): 6 * 7 = 0.

*	1	2	3	4	5	6	7	8	9
1	0								
2								*	
3			0						
4									
5									
6							~~0~~		
7									
8									
9					~~0~~				

T: "6 * 7 = 0" is not correct. Can we eliminate any of the five possibilities?

S: We can cross out "$T_>$" because 6 is not larger than 7.



T: Team B needs to eliminate at least two possibilities for *. Is there another we can cross out?

S: No.

Team B does not score.

T (pointing to the poster): * can be any of the four subtraction operations. One-third of 4 to the nearest whole number is . . . ?

S: 1.

T: Team A needs to eliminate at least one possibility for *.

S (Team A): 5 * 4 = 1.

*	1	2	3	4	5	6	7	8	9
1	0								
2								*	
3			0						
4									
5				1					
6							⊗		
7									
8									
9					⊗				

T: That's correct. Can we eliminate anything?

S: No, $5 - 4 = 1$, $5 -_3 4 = 1$, $5 -_8 4 = 1$, and $5 -_{10} 4 = 1$.

Team A does not score. Now Team B needs to eliminate at least one.

S (Team B): 2 * 3 = 4.

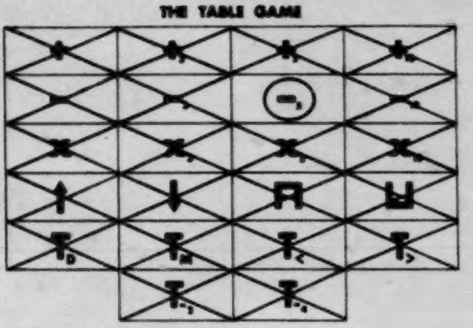
T: "2 * 3 = 4" is correct.

S: Cross off "$-_3$" because $2 -_3 3 = 2$.

S: Cross off "$-_{10}$" because $2 -_{10} 3 = 9$. Also cross off "-" because $2 - 3 = {}^-1$.

L1

S: $*$ is $-_5$!



*	1	2	3	4	5	6	7	8	9
1	0								
2			4					~~1~~	
3			0						
4									
5				1					
6							~~0~~		
7									
8									
9					~~0~~				

T: Team B scores five points plus four bonus points.

Team A	Team B
17	9

T: Team A won that game. Let's play again.

Play the second game where the secret operation $*$ is $\sqcup$ and the starting clue is $2 * 4 \neq 8$.

Exercise 2: The Table Game, Usual Rules

Play the Table Game in the usual way as described in Lesson L3.4 of IG-V. A possible game is described on the next page. At the beginning of the game, indicate only the starting clue in the table.

L 1

Operation: $\times_{10}$ Starting clue: 7 * 3 = 1

For your convenience, a completed table for $\times_{10}$ follows.

×.	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	0	2	4	6	8
3	3	6	9	2	5	8	1	4	7
4	4	8	2	6	0	4	8	2	6
5	5	0	5	0	5	0	5	0	5
6	6	2	8	4	0	6	2	8	4
7	7	4	1	8	5	2	9	6	3
8	8	6	4	2	0	8	6	4	2
9	9	8	7	6	5	4	3	2	1

We include the rules for the Table Game here for your review.

The Table Game (Usual Rules)

Play a game involving a table for an unknown operation *. Players determine which of twenty-two given operations (see poster) is *. Although all the plays in the game are made in the same way, the scoring rules divide the game into two parts with different, but related, objectives. To play the game proceed as follows.

On the board, draw (or project) a 9 by 9 table for a secret operation and tape a copy of the Table Game poster near the table. Select one student who would be a reliable scorekeeper.[†] Divide the class (except for the scorekeeper) into two teams, Team A and Team B. Distribute individual sheets from the Table Game Analysis Pad to the students.

Begin the game by making one to three entries in the table as starting clues. Such an entry can be correct or incorrect (indicated by an "×" through it).

[†] The scorekeeper could also be any adult who is available during the class time; for example, a teacher's aide or a student teacher.

Give the players time to analyze individually the situation and to mark their analysis sheets. Then call on a player from Team A to suggest an entry to the table; for example, the player might suggest that 2 * 3 = 6. You make the entry into the table. If correct, tell the class; if incorrect, say "no" and put an "X" through the entry. Allow a few minutes for the players to use the information gained from the play to help identify the operation *. Then give all of the players the option to turn in their analysis sheets with one of the operations circled. A team gains ten points for each correct answer given by a member. The scorekeeper tallies the points for each team but does not reveal them at this time.

Alternating teams, proceed in the same manner for the second, third, and fourth plays except decrease by one the number of points that can be gained each time (10-9-8-7). The last time students may turn in their analysis sheets is after the fourth play is made.

Now the scoring changes. During the remainder of the game, let each player who has not had a turn yet suggest an entry in the table (in the same manner as before). Only now players earn points for their team based on whether or not their entries are correct. The scoring for this part of the game is as follows.

- A team gains one point each time a member suggests a correct entry in a square on the diagonal (see shaded squares below).

*	1	2	3	4	5	6	7	8	9
1	▨								
2		▨							
3			▨						
4				▨					
5					▨				
6						▨			
7							▨		
8								▨	
9									▨

- A team gains two points each time a member suggests a correct entry in a square other than on the diagonal.
- A team neither gains nor loses points for incorrect entries.

The game ends after each player has had a turn. Call on a player to identify * while the scorekeeper computes the scores. We suggest that you give the scorekeeper a tally sheet similar to the one shown below. You may wish to reproduce it so that you have several copies. And, of course, fill in the correct operation for the particular game being played.

The secret operation is __________.

Points

First part of game

For correctly identifying *

- after first play 10 pts.
- after second play 9 pts.
- after third play 8 pts.
- after fourth play 7 pts.

Second part of game

For correctly making an entry

- on the diagonal 1 pt.
- other than on the diagonal 2 pts.

TEAM A	TEAM B

L 2

1.2 The Table Game #2

CAPSULE LESSON SUMMARY

Play both the analysis and standard versions of the Table Game.

MATERIALS

Teacher: Three copies of the Table Game Poster; felt-tip marker
Student : Table Game Analysis Pad; pencil

DESCRIPTION OF LESSON

Exercise 1

Play the analysis version of the Table Game twice. (The analysis version is described in Exercise 1 of Lesson L1.1.) These are some possible choices for the secret operation * with appropriate starting clues. At the beginning of a game, indicate only the starting clue in the table.

First game	Operation: ↓	Starting clue: $4 * 3 \neq 2$
Second game	Operation: $T_{=2}$	Starting clue: $8 * 4 \neq 4$

For your convenience, completed tables for ↓ and $T_{=2}$ follow.

↓	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2
3	1	2	3	3	3	3	3	3	3
4	1	2	3	4	4	4	4	4	4
5	1	2	3	4	5	5	5	5	5
6	1	2	3	4	5	6	6	6	6
7	1	2	3	4	5	6	7	7	7
8	1	2	3	4	5	6	7	8	8
9	1	2	3	4	5	6	7	8	9

$T_{=2}$	1	2	3	4	5	6	7	8	9
1	1	0	1	0	1	0	1	0	1
2	0	1	0	1	0	1	0	1	0
3	1	0	1	0	1	0	1	0	1
4	0	1	0	1	0	1	0	1	0
5	1	0	1	0	1	0	1	0	1
6	0	1	0	1	0	1	0	1	0
7	1	0	1	0	1	0	1	0	1
8	0	1	0	1	0	1	0	1	0
9	1	0	1	0	1	0	1	0	1

Erase all entries in the table before going on to Exercise 2.

Exercise 2: The Table Game, Usual Rules

Play the Table Game in the usual way. (The rules are reviewed in Exercise 2 of Lesson L1.1.) A possible game is described below. At the beginning of the game, indicate only the starting clue in the table.

Operation: $+_3$ Starting clue: $1 * 3 \neq 3$

For your convenience, a completed table for $+_3$ follows.

$+_3$	1	2	3	4	5	6	7	8	9
1	2	0	1	2	0	1	2	0	1
2	0	1	2	0	1	2	0	1	2
3	1	2	0	1	2	0	1	2	0
4	2	0	1	2	0	1	2	0	1
5	0	1	2	0	1	2	0	1	2
6	1	2	0	1	2	0	1	2	0
7	2	0	1	2	0	1	2	0	1
8	0	1	2	0	1	2	0	1	2
9	1	2	0	1	2	0	1	2	0

L 3

1.3 The Table Game #3

CAPSULE LESSON SUMMARY

As an individual activity, identify an operation from selected entries in a table for the operation. Play both analysis and standard versions of the Table Game.

MATERIALS

Teacher: Two copies of the Table Game poster; felt-tip marker
Student : Worksheets L01, L02, L03, and L04; Table Game Analysis Pad; pencil

DESCRIPTION OF LESSON

Exercise 1

Distribute copies of Worksheets L01 and L02 to the students. Allow five to ten minutes for the students to work on them individually before going over the answers. Draw each table from Worksheet L01 on the board. Also draw a 9 by 9 table and use it to discuss the two tables from Worksheet L02. When someone suggests an operation for one of the tables, label that table and check if its given entries are correct for the suggested operation. After the operation for a table is identified, a student might think that the table also could be for another operation. If so, consider that operation and let the class find an entry which rules it out as a possibility. The answer keys for these worksheets are on page 22.

Leave the 9 by 9 table on the board for use in Exercises 2 and 3.

Exercise 2: The Table Game, Analysis Version

Play the analysis version of the Table Game. A possible game is described below. At the beginning of the game, indicate only the starting clues in the table.

Operation: T_M

Starting clues: $7 * 9 \neq 16$
$2 * 7 \neq 1$

For your convenience, a completed table for T_M follows.

T_M	1	2	3	4	5	6	7	8	9
1	1	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0
3	1	0	1	0	0	0	0	0	0
4	1	1	0	1	0	0	0	0	0
5	1	0	0	0	1	0	0	0	0
6	1	1	1	0	0	1	0	0	0
7	1	0	0	0	0	0	1	0	0
8	1	1	0	1	0	0	0	1	0
9	1	0	1	0	0	0	0	0	1

Erase all entries in the table before going on to Exercise 3.

Exercise 3: The Table Game, Usual Rules

Play the Table Game in the usual way. A possible game is described below. Indicate only the starting clues in the 9 by 9 table on the board at the beginning of the game.

Operation: +

Starting clues: $1 * 1 \neq 1$
$2 * 3 \neq 0$

For your convenience, a completed table for + follows.

+	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

Worksheets L03 and L04 should be assigned to the students. They are to be discussed in the beginning of Lesson 1.4.

Name ______________________ **L 01 ***

Each table is for one of these:

THE TABLE GAME

$+$	$+_3$	$+_5$	$+_{10}$
$-$	$-_3$	$-_5$	$-_{10}$
$\times$	$\times_5$	$\times_9$	$\times_{10}$
↑	↓	⊓	⊔
$T_=$	$T_\neq$	$T_<$	$T_>$
	$T_\leq$	$T_\geq$	

Label the tables.

$+_3$	1	4
2	0	0
5	0	0

$T_=$	6	4
2	1	1
8	1	1

$-_{10}$	3	1
7	4	6
2	X	1

$\times_5$	4	8
1	4	3
2	3	1

Name ______________________ **L 02 ****

Each table is for one of these:

THE TABLE GAME

$+$	$+_3$	$+_5$	$+_{10}$
$-$	$-_3$	$-_5$	$-_{10}$
$\times$	$\times_5$	$\times_9$	$\times_{10}$
↑	↓	⊓	⊔
$T_=$	$T_\neq$	$T_<$	$T_>$
	$T_\leq$	$T_\geq$	

Label the tables.

$T_\leq$	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2		1							
3			1						
4				1					
5					1				
6						1			
7							1		
8								1	
9									1

$\times_9$	1	2	3	4	5	6	7	8	9
1	X								0
2									
3									
4									
5									
6									
7									
8									
9	0								

Name ______________________ **L 03 *****

∗ is one of the operations in the Table Game.

THE TABLE GAME

$+$	$+_3$	$+_5$	$+_{10}$
$-$	$-_3$	$-_5$	$-_{10}$
$\times$	$\times_5$	$\times_9$	$\times_{10}$
↑	↓	⊓	⊔
$T_=$	$T_\neq$	$T_<$	$T_>$
	$T_\leq$	$T_\geq$	

First clue

This table for ∗ has exactly three 0's in it.

∗	4	8
8		
4		

∗ could be $T_<$, $T_>$.

Second clue

This table for ∗ has exactly three 1's in it.

∗	6	9
5		
7		

∗ is $T_<$

Name ______________________ **L 04 ******

∗ is one of the operations in the Table Game.

THE TABLE GAME

$+$	$+_3$	$+_5$	$+_{10}$
$-$	$-_3$	$-_5$	$-_{10}$
$\times$	$\times_5$	$\times_9$	$\times_{10}$
↑	↓	⊓	⊔
$T_=$	$T_\neq$	$T_<$	$T_>$
	$T_\leq$	$T_\geq$	

First clue

No two entries in this table for ∗ are the same.

∗	3	4
2		
3		

∗ could be $\times$, ⊔ , $\times_5$, or $\times_{10}$.

Second clue

All four entries in this table for ∗ are the same.

∗	2	6
6		
3		

∗ is ⊔ .

1.4 The Table Game #4

CAPSULE LESSON SUMMARY

Go over two detective stories that were assigned in Lesson L1.3. Given three clues, decide which of twenty-two operations a table is for. After the first clue, find entries which have a $\frac{4}{5}$ ($\frac{3}{5}$, $\frac{2}{5}$, $\frac{1}{5}$) probability of being correct. After the second clue, find entries which one can be absolutely sure are correct. Play the Table Game in the usual way.

MATERIALS

Teacher:	Worksheets L03, and L04; two copies of the Table Game Poster; felt-tip marker
Student:	Worksheets L03 and L04 (assigned in Lesson 1.3); Table Game Analysis Pad; pencil

DESCRIPTION OF LESSON

Exercise 1

Distribute students' copies of Worksheet L03 and L04. Your students should have completed these worksheets and should have returned them to you before this lesson.

Collectively discuss the worksheets. Encourage students to explain how they went about determining the operation *. One way of looking at each clue is presented here. Do not expect your students to approach these problems in such a systematic way, but be able to pick up on such strategies if students begin to describe them.

On Worksheet L03 the student is asked to solve a detective story involving a secret operation *. The only limitation initially is that * is one of the twenty-two operations from the Table Game. The first clue says that this table for * has exactly three 0's in it.

*	4	8
8		
4		

L 4

The information from this clue eliminates twenty of the possibilities for *.

* CANNOT BE ANY OF THESE OPERATIONS	REASONS
$\sqcup, \sqcap$	Tables for $\sqcup$ and $\sqcap$ never have 0 as an entry.
$\uparrow, \downarrow, +, \times$	If this table[†] were for any of these four operations, none of the entries would be 0.
$+_3, +_5, +_{10}$	By examining this table, one can easily deduce that at least two of the entries would not be 0 if the table were for $+_3$ or for $+_5$ or for $+_{10}$.
$-, -_3, -_5, -_{10}$	By examining this table for any of these four subtraction operations, one sees that only two entries are 0.
$\times_3, \times_5, \times_{10}$	By examining this table, one can easily deduce that none of the entries would be 0 if this table were for $\times_3$ or for $\times_5$ or for $\times_{10}$.
$T_{=2}, T_{=4}$	All entries would be 1 if this table were for $T_{=2}$ or for $T_{=4}$.
T_D, T_M	Three entries would be 1 if this table were for T_D or for T_M.

[†] This table refers to the 4 by 4 table in the clue.

Your students should conclude that * is either $T_<$ or $T_>$.

$T_<$	4	8
8	0	0
4	0	1

$T_>$	4	8
8	1	0
4	0	0

The second clue says that this table for * has exactly three 1's in it.

*	6	9
5		
7		

Since only two possibilities for * remain, it is easy to try both and to see that * must be $T_<$.

$T_<$	6	9
5	1	1
7	0	1

On Worksheet L04 the student is asked to solve a more difficult detective story involving a secret operation *. Again the only limitation initially is that * is one of the twenty-two operations from the Table Game. The first clue says that no two entries in this table for * are the same.

*	3	4
2		
3		

L 4

The information from this clue eliminates eighteen of the possibilities for *.

* CANNOT BE ANY OF THESE OPERATIONS	REASONS
$+,\ +_3,\ +_5,\ +_{10}$	$2 + 4 = 3 + 3$, so $\left\{\begin{array}{l} 2 +_3 4 = 3 +_3 3 \\ 2 +_5 4 = 3 +_5 3 \\ 2 +_{10} 4 = 3 +_{10} 3 \end{array}\right.$
$-,\ -_3,\ -_5,\ -_{10}$	$2 - 3 = 3 - 4$, so $\left\{\begin{array}{l} 2 -_3 3 = 3 -_3 4 \\ 2 -_5 3 = 3 -_5 4 \\ 2 -_{10} 3 = 3 -_{10} 4 \end{array}\right.$
$T_0,\ T_M,\ T_<,\ T_>,\ T_{=2},\ T_{=4}$	Truth tables have only two possible entries: 0 or 1. So at least two entries in each of these tables must be the same.
$\uparrow,\ \downarrow,\ \sqcap,\ \times_3$	Can be considered individually — need only to find two entries which are the same.

Your students should conclude that * could be $\times$, $\times_5$, $\times_{10}$, or $\sqcup$.

$\times$	3	4
2	6	8
3	9	12

$\times_5$	3	4
2	1	3
3	4	2

$\times_{10}$	3	4
2	6	8
3	9	2

$\sqcup$	3	4
2	6	4
3	3	12

The second clue says that all four entries in this table for * are the same.

*	2	6
6		
3		

Since only four possibilities for * remain, it is easy to check their tables. Once two different entries are found, the operation determining those entries can be ruled out. Conclude that * is ⊔ .

⊔	2	6
6	6	6
3	6	6

Exercise 2

Prepare your board as you would to play the Table Game. Indicate in the table that 7 * 4 = 1. Distribute individual sheets from the Table Game Analysis Pad to the students.

THE TABLE GAME

$+$	$+_3$	$+_5$	$+_{10}$
$-$	$-_3$	$-_5$	$-_{10}$
$\times$	$\times_3$	$\times_5$	$\times_{10}$
$\uparrow$	$\downarrow$	$\sqcap$	$\sqcup$
T_0	T_M	$T_<$	$T_>$
	$T_{.3}$	$T_{.4}$	

*	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7				1					
8									
9									

T: The first clue to help you decide which operation this table is for is that 7 * 4 = 1. Knowing this, cross out on your analysis sheets the names of operations which * cannot be.

Allow a few minutes for individual work.

T: Did any of you eliminate all but one of the operations?

S: No.

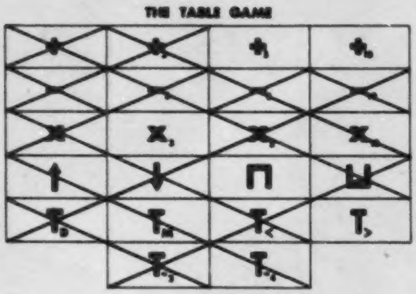
T: How many possibilities for * are left?

S: Five.

Collectively go over the possibilities for *. As students give correct reasons why * cannot be certain operations, cross out the names of those operations on the poster; for example, * cannot be $\times_{10}$ because $7 \times_{10} 4 = 8$. The twenty-two operations do not need to be considered in any order.

To one side of the poster make a list of the five remaining possibilities.

$+_5$
$+_{10}$
$\times_3$
$\sqcap$
$T_>$



*	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7				1					
8									
9									

T: Suppose I asked you to guess what number 8 * 3 is. What would you choose so that your probability of being correct is as high as possible?

S: 1, because $8 +_5 3 = 1$, $8 +_{10} 3 = 1$, $8 \sqcap 3 = 1$, and $8\ T_> 3 = 1$. The only operation which gives an answer different from 1 is $\times_3$. $8 \times_3 3 = 0$.

Record "8 * 3 = 1" on the board.

$$8 * 3 = 1$$

T: So what is the probability that "8 * 3 = 1" is correct?

S: $\frac{4}{5}$.

T: Could we do better than that with another entry?

S: If we could, the entry would have to be correct for all five of the operations.

T: So the entry's probability of being correct would be $\frac{5}{5}$, which is 1. Can you find such an entry?

Accept and check suggestions. There are no such entries but let the students freely investigate the situation for themselves.

T: If there were an entry that was correct for all five operations, what would the entry have to be?

S: Either 0 or 1, because $T_>$ has only zeros and ones in its table.

T: Could such an entry be 0?

S: No, the greatest common divisor of two integers is never 0.

T: You could check the ones in the five tables to see if there is a location in all five that has a 1 in it, but I will tell you that there is no entry that you could be absolutely sure of.

Point to the number story on the board.

L4

T: "8 * 3 = 1" has $\frac{4}{5}$ probability of being correct. Are there any others (which have $\frac{4}{5}$ probability of being correct)?

Check any suggestions; for example,

S: "1 * 9 = 0".

T: Let's check. What is $1 +_5 9$? (0) $1 +_{10} 9 = \ldots$? (0) $1 \times_3 9 = \ldots$? (0) What is the greatest common divisor of 1 and 9? (1) Is 1 larger than 9? (No) So $1 \ T_> \ 9 = 0$. For four of the five operations 1 * 9 = 0.

Record "1 * 9 = 0" under "8 * 3 = 1".

Continue in this manner until you have several number stories which have a $\frac{4}{5}$ probability of being correct. Do not dwell on this activity too long nor require that all seven possibilities be found. Your list might contain three or four of the following number stories.

8 * 3 = 1
1 * 9 = 0
9 * 2 = 1
4 * 6 = 0
3 * 7 = 0
4 * 7 = 1
6 * 5 = 1

In a similar manner find some entries which have a $\frac{3}{5}$ probability of being correct, some which have a $\frac{2}{5}$ probability of being correct, and some which have a $\frac{1}{5}$ probability of being correct. Your lists might look similar to these.

$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$
8 * 3 = 1	8 * 4 = 2	8 * 2 = 0	7 * 9 = 6
1 * 9 = 0	5 * 5 = 0	7 * 9 = 1	2 * 2 = 1
3 * 7 = 0	2 * 3 = 0	7 * 9 = 0	8 * 4 = 4
9 * 2 = 1	9 * 7 = 1		4 * 6 = 2

T: What are some integers that we could enter anywhere in the table and have probability 0 of being correct?

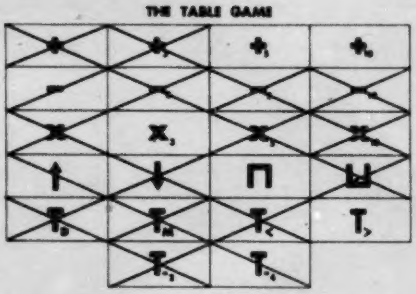
S: Any number larger than 9.

S: Any number smaller than 0.

Erase the list of number stories.

T: Your next clue is that 6 * 9 = 0.

$+_5$
$+_{10}$
$\times_3$
$\sqcap$
$T_>$



*	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									0
7				1					
8									
9									

S: * can't be $\sqcap$ because the greatest common divisor of two numbers is never 0.

S: * can't be $+_{10}$ because 6 $+_{10}$ 9 = 5.

T: What about the other three operations . . . ?

S: 6 $+_5$ 9 = 0, 6 $\times_3$ 9 = 0, and 6 is not larger than 9.
So * can be any of those three operations.

Erase " $\sqcap$ " and "$+_{10}$" and cross them out on the poster.

T: Now can you find an entry for the table which you are absolutely sure is correct?

Check suggestions made and list on the board those which are correct. For example,

S: $9 * 6 = 0$.

T: Let's check. What is $9 +_5 6$? (0) $9 \times_3 6 = \ldots$? (0) Is 9 larger than 6? (Yes) So $9\ T_> 6 = 1$. One of the three did not give 0. Does anyone else have a suggestion?

S: $4 * 6 = 0$.

T: What is $4 +_5 6$? (0) $4 \times_3 6 = \ldots$? (0) Is 4 larger than 6? (No) So $4\ T_> 6 = 0$. We know $4 * 6$ must be 0.

Continue in this manner until your class finds two or more of the following.

$4 * 6 = 0$

$2 * 3 = 0$

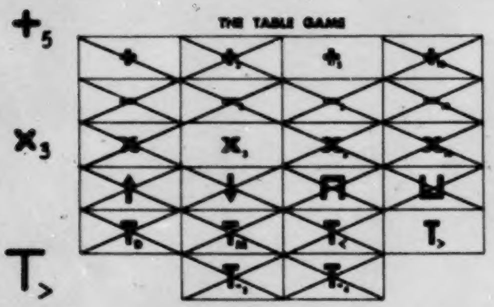
$3 * 7 = 0$

$1 * 9 = 0$

$7 * 4 = 1$

$6 * 9 = 0$

T: Your last clue is that $3 * 3 \neq 0$.



*	1	2	3	4	5	6	7	8	9
1									
2									
3			~~0~~						
4									
5									
6									0
7				1					
8									
9									

S: $*$ is $+_5$, because $3 +_5 3 = 1$, $3 \times_3 3 = 0$, and $3 \; T_> \; 3 = 0$.

If there is time remaining for the lesson, continue with Exercise 3.

Exercise 3

Play the Table Game in the usual way. A possible game is described below. At the beginning of the game, indicate only the starting clue in the table.

Operation: T_D Starting clue: $2 * 4 \neq 0$

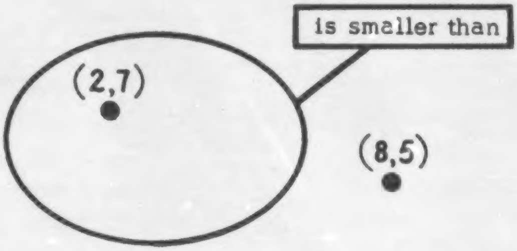
For your convenience, a completed table for T_D follows.

T_D	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	0	1	0	1	0	1	0	1	0
3	0	0	1	0	0	1	0	0	1
4	0	0	0	1	0	0	0	1	0
5	0	0	0	0	1	0	0	0	0
6	0	0	0	0	0	1	0	0	0
7	0	0	0	0	0	0	1	0	0
8	0	0	0	0	0	0	0	1	0
9	0	0	0	0	0	0	0	0	1

2

THE RELATIONS GAME

A relation is a set of ordered pairs. For example, the relation "is smaller than" is the set of ordered pairs of numbers with the first component smaller than the second component. (2, 7) belongs to that relation because 2 is smaller than 7; (8, 5) does not belong to it because 8 is not smaller than 5.



The lessons in this chapter introduce the Relations Game, a game that resembles the String Game but involves ordered pairs of numbers. In the Relations Game, players determine which of twenty-two relations a string is for by observing the placement of ordered pairs in the string picture. Unlike the String Game, the Relations Game has no specified set of playing pieces (ordered pairs to be placed in the picture). Because it is not so restricted, the game provides an opportunity for arithmetic practice with a wide range of numbers including rational numbers.

Students very quickly learn to distinguish ordered pairs with reverse order, for example (2, 7) and (7, 2), during a game for their placements in the string picture convey different information about the secret relation(s). To play the Relations Game well, students must exercise their knowledge and intuition about relations, which they have been studying throughout the CSMP curriculum with the aid of arrow representations.

CAPSULE LESSON SUMMARIES

Section 2.1: Using an arrow picture for support, locate some ordered pairs of numbers inside and some outside a string for "is a multiple of". Do a similar activity for the relation 10 x . Let students ask for the location of some ordered pairs in a string picture to help them decide which of ten relations a string is for.

Section 2.2: Using an arrow picture for support, locate some ordered pairs of numbers inside and some outside a string for $\boxed{+}\boxed{=}$. . . . Let students ask for the location of some ordered pairs in a string picture to help them decide which of eleven relations a string is for. Introduce the analysis version of the Relations Game.

Section 2.3: Considering ordered pairs of numbers, some which belong and some which do not belong to the relation "is a square of", give one component and ask what the other component could be. Play the analysis version of the Relations Game. Introduce the standard version of the Relations Game.

Section 2.4: Play the Relations Game in the usual way. Locate some ordered pairs in each of the four regions of a 2-string picture in which one string is for the relation $\boxed{+}\boxed{5}\boxed{=}$ . . . and the other is for the relation "is at least 20 smaller than".

Section 2.5: Play the analysis version of the Relations Game with a 2-string picture. Play the Relations Game according to usual rules and with 2-string pictures. Explain minor changes in the rules when two strings are involved.

Section 2.6: Play both the analysis and standard versions of the Relations Game with 2-string pictures.
(Worksheets L05, L06)

2.1 The Relations Game #1

CAPSULE LESSON SUMMARY

Using an arrow picture for support, locate some ordered pairs of numbers inside and some outside a string for "is a multiple of". Do a similar activity for the relation 10 x. Let students ask for the location of some ordered pairs in a string picture to help them decide which of ten relations a string is for.

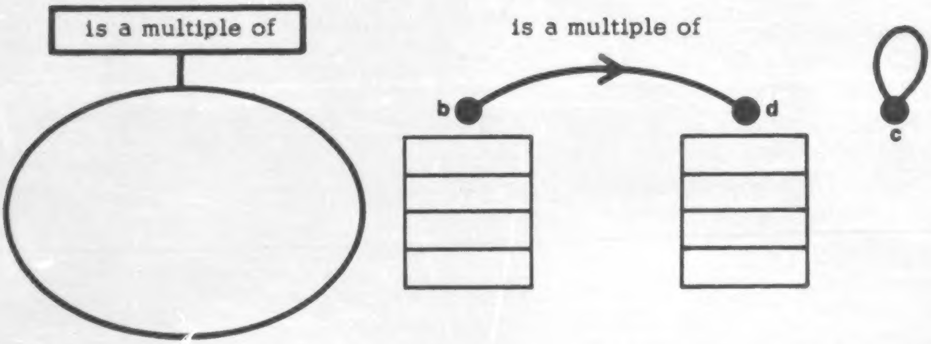
MATERIALS

Teacher: Colored chalk
Student: None

DESCRIPTION OF LESSON

Exercise 1

Draw these pictures on the board.



Do not label the dots with letters. Letters are used in the lesson description to make it easier to follow.

T: What is the red arrow for?

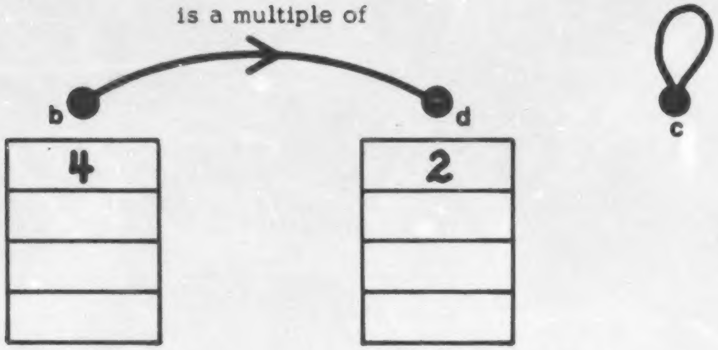
S: Is a multiple of.

T: What could these dots (point to b and d) be for?

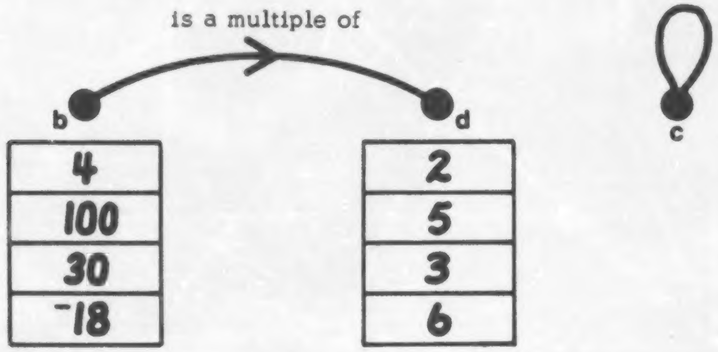
S: 4 and 2. 4 is a multiple of 2.

L 5

Record a correct response in the charts below b and d.


is a multiple of
b
d
c
4
2

Continue until four possibilities for each of the dots have been suggested. Some of the many possibilities are recorded below.


is a multiple of
b
d
c
4
100
30
⁻18
2
5
3
6

Erase the entries in the charts and ask for other possibilities for b and d satisfying particular conditions, such as:

- b is larger than 30;
- b is between 40 and 48;
- b is negative;
- b is 0 (If b is 0, d can be any integer.);
- d is larger than 50;
- d is 17;
- d is negative.

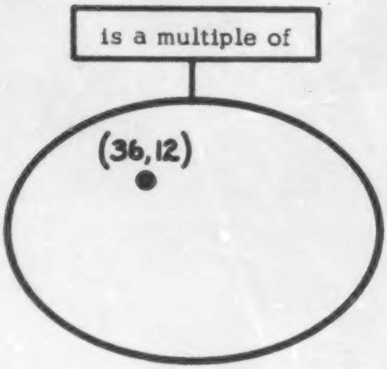
Consider the string to the left of the arrow picture.

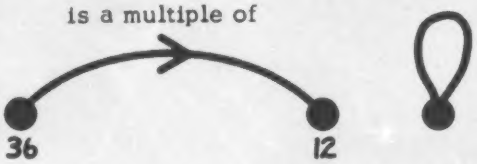
T: This string is related to this arrow (from b to d).

Label b "36" and d "12".

T: 36 is a multiple of 12. We can show this in the string picture.

Draw a dot inside the string and label it "(36, 12)".





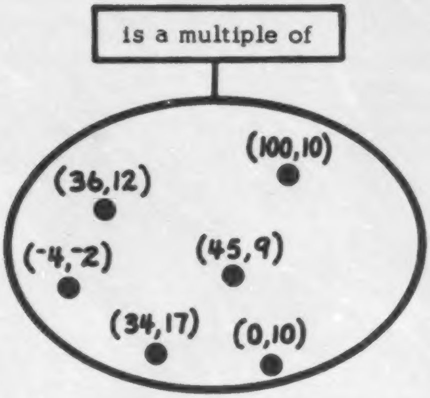
T: We have a first number (point to "36" in each picture) and a second number (point to "12" in each picture), a first component and a second component.

Point to "(36, 12)".

T: This is called an ordered pair or a couple. What are some other ordered pairs that belong inside the string?

L5

Let students offer several ordered pairs and put those belonging to the relation "is a multiple of" inside the string. Perhaps your string picture will look similar to this one.



T (tracing the loop): What could this dot (c) be for?

The dot (c) could be for any integer. A sample dialogue follows.

S: 0.

Put (0, 0) inside the string.

S: 1.

T: That's also right, 1 is a multiple of 1. So what ordered pair belongs inside the string?

S: (1, 1).

Put (1, 1) inside the string.

S: The dot (c) could be for 9 because 9 is a multiple of 9, so (9, 9) belongs inside the string.

Put (9, 9) inside the string.

T: What about $(^{-}7, {}^{-}7)$?

S: It belongs inside the string because $^{-}7$ is a multiple of $^{-}7$.

T: How do you know?

S: $1 \times {}^{-}7 = {}^{-}7$ and 1 is an integer.

Put $(^{-}7, {}^{-}7)$ inside the string.

T (pointing to c): Could any integer be here?

S: Yes.

Ask students which of these numbers are integers.

2.0	(Yes)	$^{-}3.0$	(Yes)	0.4	(No)
$\frac{1}{2}$	(No)	$\frac{2}{3}$	(No)	$\frac{4}{2}$	(Yes)
100	(Yes)	$^{-}3{,}143$	(Yes)	$\frac{^{-}5}{3}$	(No)

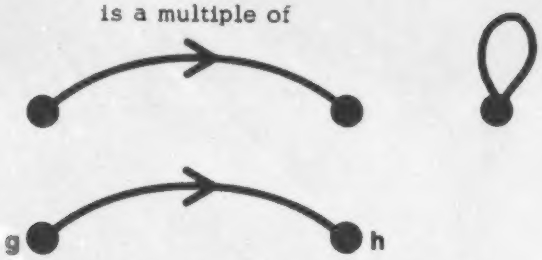
T: The components of ordered pairs in the relation "is a multiple of" are integers.

It is important that your students remember this restriction.

NOTE: We can multiply $\frac{1}{3}$ times an integer and get $\frac{4}{3}$ (multiply by 4), but we shall not consider $\frac{4}{3}$ to be a multiple of $\frac{1}{3}$ and $(\frac{4}{3}, \frac{1}{3})$ does not belong to this relation.

L 5

Draw a green arrow near the red arrow.



Do not label the dots with letters. Letters are used in the lesson description to make it easier to follow.

T: I'm thinking of a certain relation for this green arrow. What do you think this green arrow is for?

Let the students guess and then tell them it is for "is not a multiple of".

T: What could these dots (point to g and h) be for?

S: 1 and 4.

T: 1 is not a multiple of 4, so the dots could be for 1 and 4.

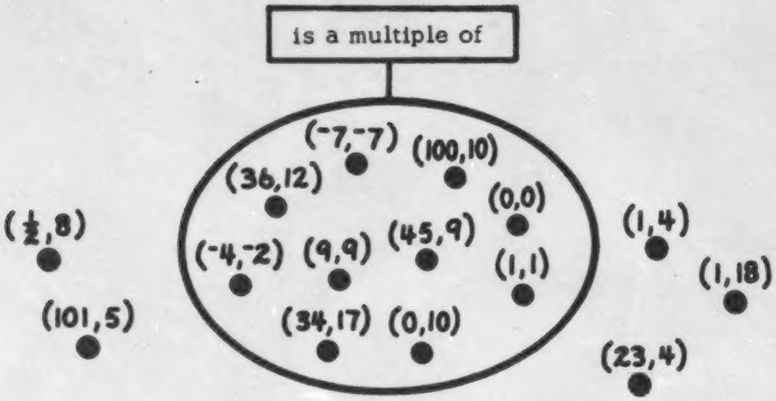
Label g and h with the student's suggestion.

T: Where does (1, 4) belong in this string picture?

S: Outside the string.

Let students suggest other ordered pairs that belong outside the string.

Your picture might look similar to this one.


is a multiple of
(⁻7,⁻7)
(100,10)
(36,12)
(0,0)
(½,8)
(⁻4,⁻2)
(9,9)
(45,9)
(1,4)
(1,1)
(1,18)
(101,5)
(34,17)
(0,10)
(23,4)

T: Give me some ordered pairs in which neither component is an integer.

S: (0.4, 96.6).

S: $(\frac{1}{2}, \frac{2}{3})$.

S: $(\frac{1}{4}, \frac{1}{8})$.

T: Where do such ordered pairs belong in the picture?

S: Outside the string.

T: What about $(\frac{1}{2}, \frac{1}{2})$?

S: Outside the string.

T: What about $(\frac{18}{3}, \frac{6}{3})$?

S: Inside, because $\frac{18}{3} = 6$, $\frac{6}{3} = 2$, and 6 is a multiple of 2.

Erase all numerals in the picture.

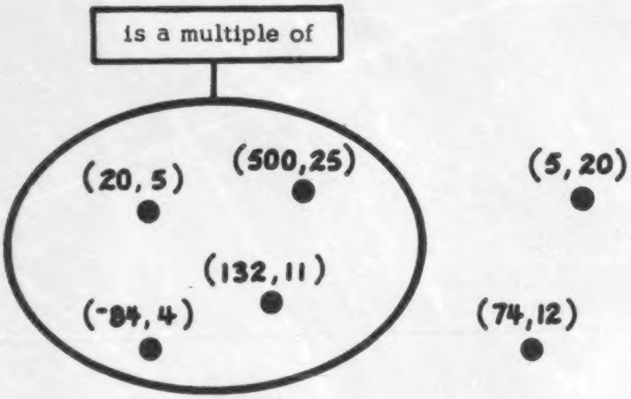
L 5

Write this list of ordered pairs near the string picture.

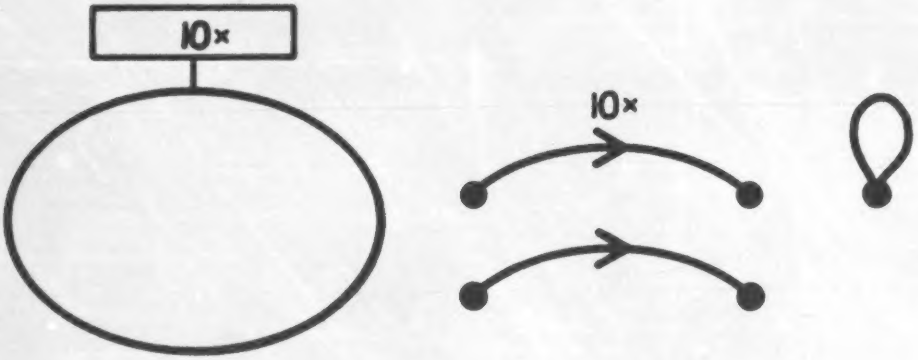
(5, 20)
(20, 5)
(500, 25)
(132, 11)
(74, 12)
$(^{-}84, 4)$

T: On a piece of paper draw a string for the relation "is a multiple of" and locate these ordered pairs (in the list) in your string picture.

After a few minutes of individual work, discuss the location of the pairs.


is a multiple of
(20, 5)
(500,25)
(132, 11)
(⁻84, 4)
(5, 20)
(74,12)

Likewise consider the relation 10 x . This relation is not restricted to integers.



Proceed in the following manner.

- Ask for ordered pairs that belong inside the string. Use the arrow picture for support. Be sure that ordered pairs with negative components are included as well as some with positive components.

- Give one of the components of an ordered pair belonging to the relation 10 x and ask for the other. Take advantage of the opportunity to review multiplication of fractions and to find equivalent names for fractions.

  $(\frac{1}{3}, \square)$ $(\frac{3}{2}, \square)$ $(\square, \frac{40}{61})$

  $(\frac{2}{5}, \square)$ $(\square, \frac{20}{3})$ $(\square, 3\frac{1}{3})$

  Use the return arrow (for $\frac{1}{10}$ x or ÷ 10) whenever it is helpful.

- Ask what number the dot with the loop is for. (0)

- Let students guess what the green arrow is for before telling them that it is for "not 10 x". Ask for ordered pairs which belong outside the string.

- List these ordered pairs on the board and ask the students to draw pictures showing whether each pair belongs inside or outside the string for 10 x. Allow a few minutes for individual work before discussing their locations.

  (7, 70) (0.74, 74) $(\frac{3}{5}, \frac{30}{5})$

  (95, 9,500) (0.732, 7.32) $(1\frac{1}{5}, 12)$

  (8.3, 830) ($^-$4, 40)

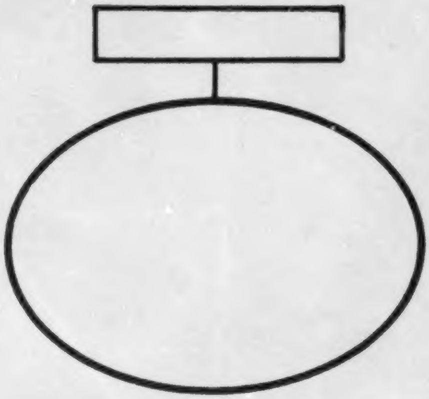
Erase everything on the board except the string and the arrows before going on to Exercise 3.

L 5

Exercise 3

List these relations on the board. Review them with your class. Mention that the relation "is a positive divisor of" is restricted to integers.

is smaller than
is at least 20 smaller than
is a positive divisor of
10x
2x
$\frac{3}{2}x$
$\boxed{+}\boxed{2}\boxed{=}\ldots$
$\boxed{+}\boxed{3}\boxed{=}\ldots$
$\boxed{+}\boxed{4}\boxed{=}\ldots$
$\boxed{+}\boxed{5}\boxed{=}\ldots$



NOTE: The string is for the relation $\boxed{+}\ \boxed{5}\ \boxed{=}$. . . . Do not reveal the relation to the students at this time. Your students will discover it during the activity.

T: This string is for one of these ten relations. I'll give you a clue as to which one.

Draw a dot inside the string and label it "(8, 88)".

T: (8, 88) belongs to the relation. Which of these relations (point to the list) can we cross off?

S: 10 x, because $10 \times 8 = 80 \neq 88$.

S: 2 x, because $2 \times 8 = 16 \neq 88$.

S: $\frac{3}{2}$ x, because $\frac{3}{2} \times 8 \neq 88$.

T: What is $\frac{3}{2} \times 8$? How do you know?

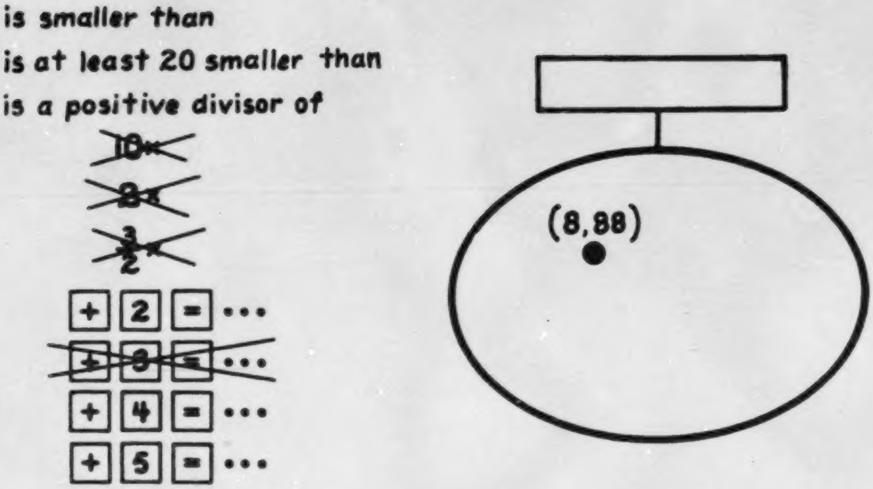
S: 12. I multiplied "3 x 8" and then divided the answer by 2.

S: 12. $\frac{1}{2} \times 8 = 4$ and $3 \times 4 = 12$.

S: Cross off " $\boxed{+}\ \boxed{3}\ \boxed{=}$ . . ." because the difference between 8 and 88 is 80 and 80 is not a multiple of 3.

S: I added threes starting at 8. I reached 87 but not 88.

With the class check that (8, 88) belongs to the remaining relations.


is smaller than
is at least 20 smaller than
is a positive divisor of
~~10×~~
~~8×~~
~~3/2×~~
+ 2 = ...
~~+ 3 = ...~~
+ 4 = ...
+ 5 = ...
(8,88)

T: Give me an ordered pair whose location, whether inside or outside the string, would eliminate some of these relations as possibilities.

Let students make suggestions one at a time until all but one of the possibilities have been eliminated. A sample dialogue follows.

S: Where does (10, 20) belong?

L 5

T: Inside the string.

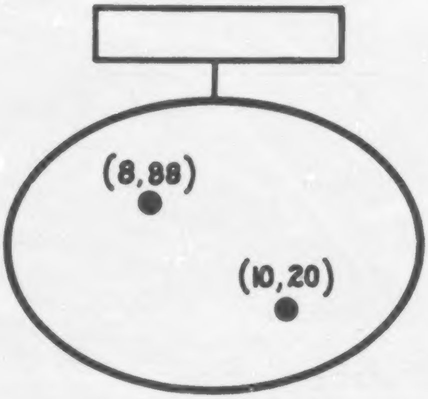
Draw a dot inside the string and label it "(10, 20)".

T: What can we cross out on the list?

S: "is at least 20 smaller than"; 10 is not 20 smaller than 20.

S: " $\boxed{+}$ $\boxed{4}$ $\boxed{=}$. . ." because 10 + 4 = 14, 14 + 4 = 18, and 18 + 4 = 22. 20 is skipped.

is smaller than
~~is at least 20 smaller than~~
is a positive divisor of
~~10×~~
~~3×~~
~~$\frac{3}{2}$×~~
$\boxed{+}$ $\boxed{2}$ $\boxed{=}$ •••
~~$\boxed{+}$ $\boxed{8}$ $\boxed{=}$ •••~~
~~$\boxed{+}$ $\boxed{4}$ $\boxed{=}$ •••~~
$\boxed{+}$ $\boxed{5}$ $\boxed{=}$ •••



T: You need some more information. What is another ordered pair for which it would be helpful to know its location?

S: Where does (1, 2) belong?

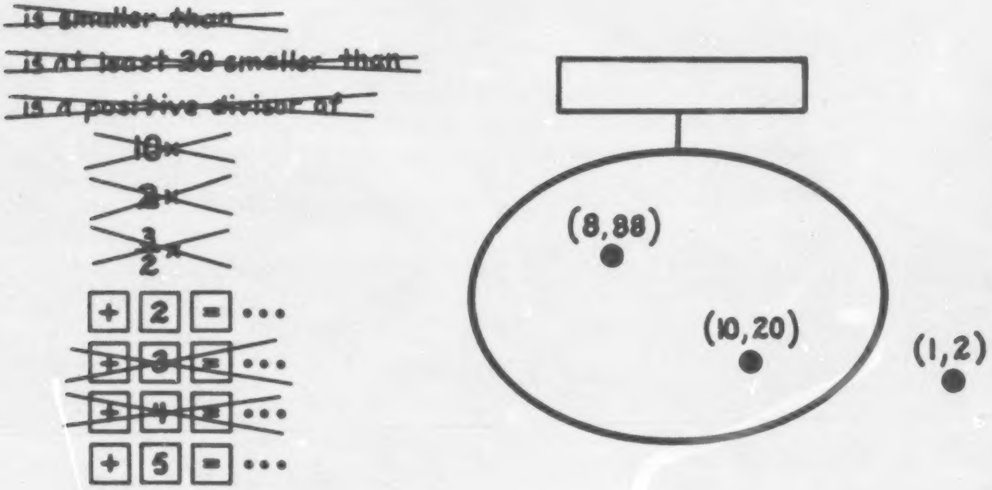
T: Outside the string.

Draw a dot outside the string and label it "(1, 2)".

T: What can we cross out on the list?

S: Cross out "is smaller than" because 1 is smaller than 2.

S: Cross out "is a positive divisor of" because 1 is a positive divisor of 2.

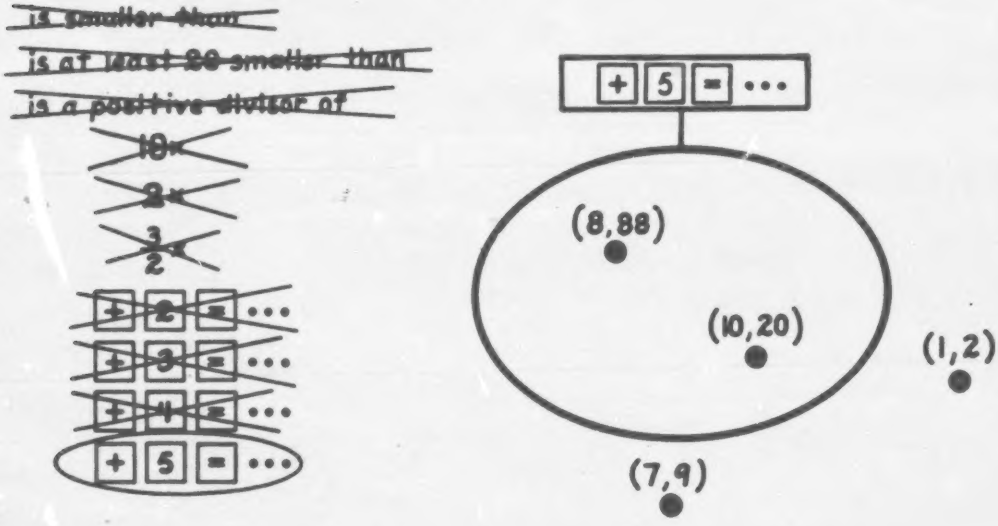

~~is smaller than~~
~~is at least 20 smaller than~~
~~is a positive divisor of~~
~~10×~~
~~2×~~
~~$\frac{3}{2}$×~~
+ 2 = ...
~~+ 3 = ...~~
~~+ 4 = ...~~
+ 5 = ...
(8,88)
(10,20)
(1,2)

T: The relation is either $\boxed{+}\ \boxed{2}\ \boxed{=}$ . . . or $\boxed{+}\ \boxed{5}\ \boxed{=}$ . . . .

S: Where does (7, 9) belong?

T: Outside the string.

S: The string is for $\boxed{+}\ \boxed{5}\ \boxed{=}$ . . . !


~~is smaller than~~
~~is at least 20 smaller than~~
~~is a positive divisor of~~
~~10×~~
~~2×~~
~~$\frac{3}{2}$×~~
~~+ 2 = ...~~
~~+ 3 = ...~~
~~+ 4 = ...~~
+ 5 = ...
+ 5 = ...
(8,88)
(10,20)
(1,2)
(7,9)

2.2 The Relations Game #2

CAPSULE LESSON SUMMARY

Using an arrow picture for support, locate some ordered pairs of numbers inside and some outside a string for $\boxed{+}$ $\boxed{=}$. . . . Let students ask for the location of some ordered pairs in a string picture to help them decide which of eleven relations a string is for. Introduce the analysis version of the Relations Game.

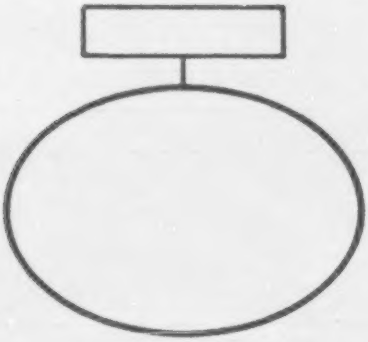
MATERIALS

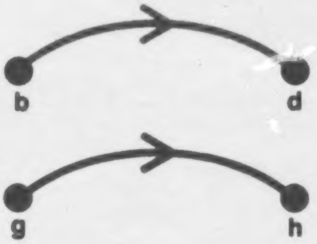
Teacher:	Colored chalk; the Relations Game Poster; red felt-tip marker
Student:	Relations Game Analysis Pad; Pencil

DESCRIPTION OF LESSON

Exercise 1

Draw these pictures on the board.





Do not label the dots with letters. Letters are used in the lesson description to make it easier to follow.

T: Last week we were looking at arrows and strings for relations. Can someone explain what the string had to do with the arrows?

S: Suppose the red arrow were for 10x. If that dot (b) were for 2, then the other dot (d) would be for 20. The pair (2, 20) would go inside the string.

T: What kind of pair . . . ?

S: Ordered pair.

T: (2, 20) belongs in the string for 10 x . Would (20, 2) also belong in the string?

S: No, because $10 \times 20 = 200 \neq 2$.

T: If the arrow were for 10 x , what would the green arrow be for?

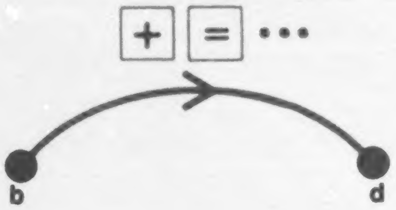
S: Not 10 x .

T: What could these dots be for (point to g and h)?

S: 3 and 40, because $10 \times 3 \neq 40$.

S: 6 and 7, because $10 \times 6 \neq 7$.

Label the red arrow "$\boxed{+}$ $\boxed{=}$. . .".



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (pointing to b and d): What could these dots be for?

S: 6 and 30.

T: Explain why.

S: Because if you put 6 on a calculator and press $\boxed{+}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$, you get 30 on the display.

T: Suppose you could not use a calculator to explain this relation. What is another way of explaining why (6, 30) belongs to the relation $\boxed{+}$ $\boxed{=}$. . . ?

S: We can add sixes to get to 30.

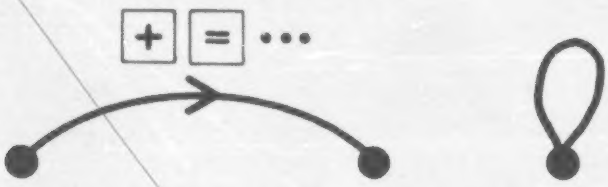
S: 5 x 6 = 30. We can multiply 6 by a whole number and get 30.

NOTE: To avoid a definition of the relation $\boxed{+}$ $\boxed{=}$. . . based on a particular kind of calculator, one can define this relation as follows: (b, d) belongs to the relation $\boxed{+}$ $\boxed{=}$. . . if and only if some positive whole number times b equals d.

Let students suggest other pairs of numbers which b and d could be. Then assign a number to one of the dots and ask what the other dot could be for. Repeat this activity several times; suggestions are given below.

b = 0.5	(d could be 1, 1.5, 2, 2.5, 3, . . .)
b = 0.25	(d could be 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2, . . .)
b = 1.5	(d could be 3, 4.5, 6, 7.5, 9, . . .)
b = $\frac{1}{3}$	(d could be $\frac{2}{3}$, 1, $\frac{4}{3}$, $\frac{5}{3}$, 2, . . .)
b = $^-7$	(d could be $^-14$, $^-21$, $^-28$, $^-35$, . . .)
d = 100	(b could be any positive divisor of 100.)
d = $^-40$	(b could be any negative divisor of $^-40$, such as $^-5$.)

Draw a red loop near the red arrow.



Point to the dot with the loop as you ask:

T: What number could this dot be for?

S: 0.

T: Yes, any whole number times 0 equals 0.

S: 7.

S: -3.

S: $\frac{1}{2}$.

S: Any number.

T: What whole number times $\frac{1}{2}$ equals $\frac{1}{2}$? (1)
What whole number times $^-3$ equals $^-3$? (1)
What whole number times 7 equals 7? (1)
So (0, 0), ($\frac{1}{2}$, $\frac{1}{2}$), ($^-3$, $^-3$), and (7, 7) all belong to the relation $\boxed{+}\ \boxed{=}$. . . . In fact, any ordered pair in which the first and second components are the same belongs to the relation $\boxed{+}\ \boxed{=}$. . . .

Erase the loop and "$\boxed{+}\ \boxed{=}$. . ." near the red arrow before going on to Exercise 2.

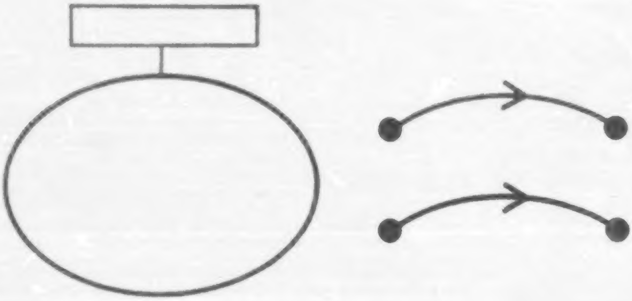
L 6

Exercise 2

List these relations on the board near the string picture.

NOTE: The string is for the relation "is at least 20 smaller than". Do not reveal the relation to the students at this time. Your students will discover it during the activity.

is smaller than
is at least 20 smaller than
is a positive divisor of
$10x$
$2x$
$\frac{3}{2}x$
$\boxed{+}\boxed{2}\boxed{=}\cdots$
$\boxed{+}\boxed{3}\boxed{=}\cdots$
$\boxed{+}\boxed{4}\boxed{=}\cdots$
$\boxed{+}\boxed{5}\boxed{=}\cdots$
$\boxed{+}\boxed{=}\cdots$



T: This string is for one of these eleven relations. I'll give you a clue as to which one.

Draw a dot inside the string and label it "$(3.5, 63\frac{1}{2})$".

T: $(3.5, 63\frac{1}{2})$ belongs to the relation. Can you find a relation (point to the list) that the string cannot be for?

S: $10x$; $10 \times 3.5 = 35 \neq 63\frac{1}{2}$.

Cross off "10 x" in the list.

S: $2x$, because $2 \times 3.5 = 7 \neq 63\frac{1}{2}$.

Cross off "2 x".

S: $\frac{3}{2}x$.

T: $\frac{3}{2} \times 3.5 = \ldots ?$

S: 5.25, because 3.5 x 3 = 10.5 and 10.5 ÷ 2 = 5.25.

Cross off " $\frac{3}{2} \times$ ".

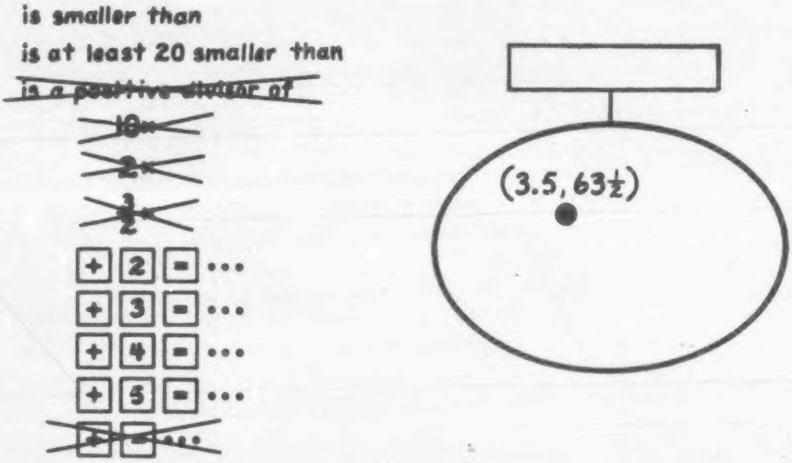
S: $\boxed{+}\ \boxed{=}$. . . , because we can get from 3.5 to 63 by adding 3.5 many times but we can't get to 63.5.

Cross off " $\boxed{+}\ \boxed{=}$ . . . ".

S: The relation "is a positive divisor of" has to do with integers, so we can cross it off.

Cross off "is a positive divisor of".

With the class check that $(3.5, 63\frac{1}{2})$ belongs to the remaining relations.

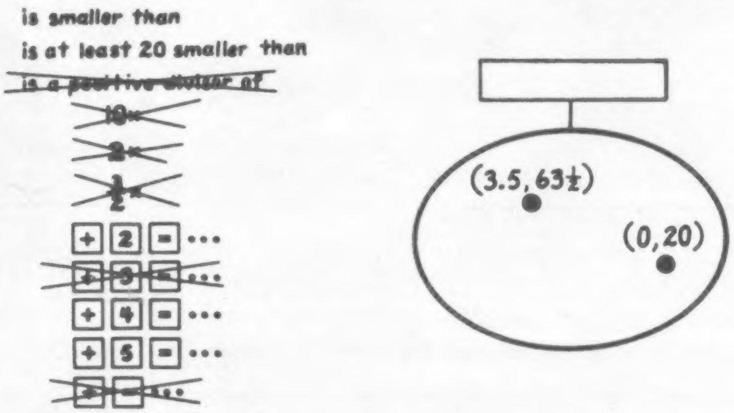


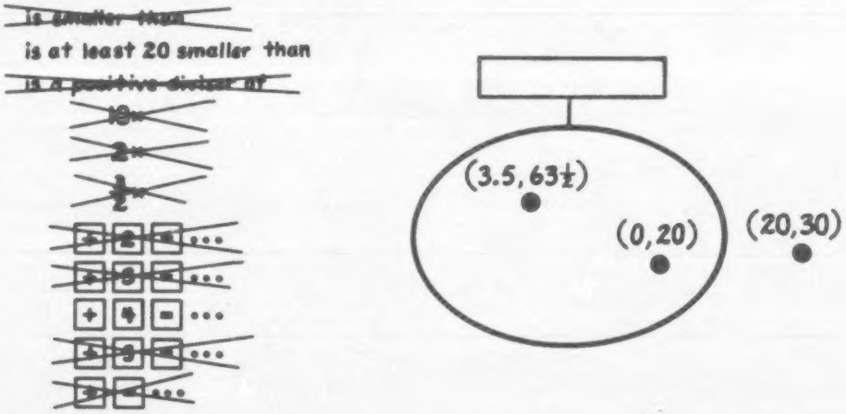
T: Give me an ordered pair whose location, whether inside or outside the string, would eliminate some of these possibilities.

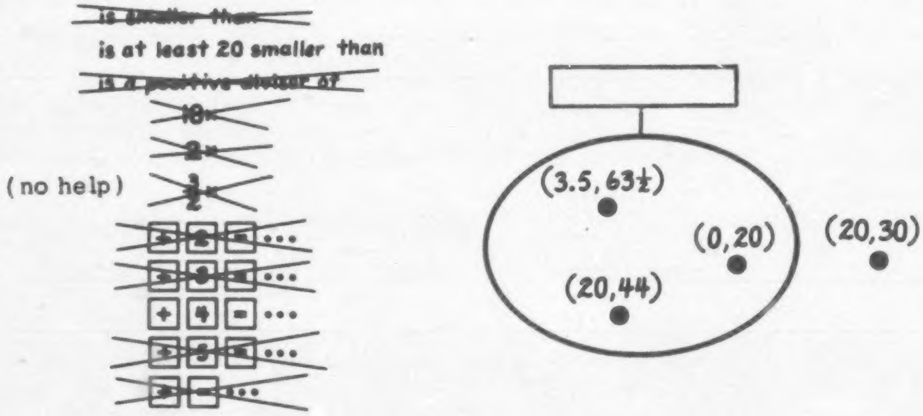
Locate in the string picture whichever ordered pair is suggested. As a class use the information of whether the ordered pair belongs inside or outside the string to see if any of the remaining possibilities are eliminated. Continue asking for ordered pairs and narrowing down possibilities until the relation is determined.

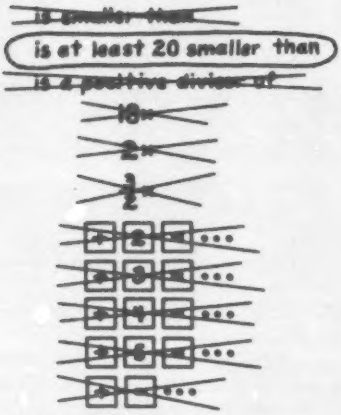
L 6

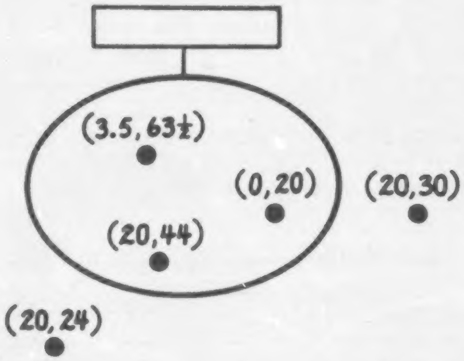
The following illustrations record the ordered pairs suggested by one of our development classes and the adjustments to the list of possible relations.


is smaller than
is at least 20 smaller than
~~is a positive divisor of~~
~~10×~~
~~2×~~
~~½×~~
+ 2 = ...
~~+ 3 = ...~~
+ 4 = ...
+ 5 = ...
(3.5, 63½)
(0,20)


~~is smaller than~~
is at least 20 smaller than
~~is a positive divisor of~~
~~10×~~
~~2×~~
~~½×~~
~~+ 2 = ...~~
~~+ 3 = ...~~
+ 4 = ...
~~+ 5 = ...~~
(3.5, 63½)
(0,20)
(20,30)


~~is smaller than~~
is at least 20 smaller than
~~is a positive divisor of~~
~~10×~~
~~2×~~
(no help)
~~½×~~
~~+ 2 = ...~~
~~+ 3 = ...~~
+ 4 = ...
~~+ 5 = ...~~
(3.5, 63½)
(0,20)
(20,30)
(20,44)





Exercise 3: The Relations Game, Analysis Version

The analysis version of the Relations Game is similar to the analysis version of the Table Game. In the Relations Game ordered pairs are put in a string picture involving a relation[†], whereas in the Table Game entries are made in a table for an operation. But in the analysis version of both games, the amount of information gained from a play determines the number of points scored. To play the analysis version of the Relations Game proceed as follows.

Divide the class into two teams, Team A and Team B. On the board draw a large string for a secret relation and near it post a list of the twenty-two possibilities for the relation (the Relations Game Poster).

Begin the game by putting one to three ordered pairs in the string picture as starting clues. Such an ordered pair can belong either inside or outside the string. Ask the class to consider collectively the twenty-two relations listed on the poster and to decide which can be eliminated as possibilities for the secret relation.

[†] The analysis version of the Relations Game can be played with more than one string. In this lesson we describe the game with one string; in Lesson L2.5 we describe the game with two strings.

Explain to your class how to play the game after the initial analysis.

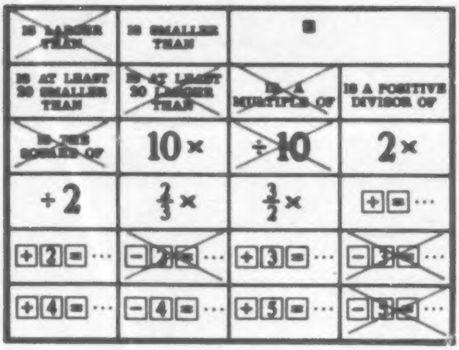
> PLAY: During a team's turn, a member selects an ordered pair. You put the ordered pair in the picture in its correct location. Let the entire class analyze the situation. Although only the team whose turn it is can score, anyone can help with the analysis after a play.
>
> SCORE: A team gains five points each time a member's play eliminates at least one-third of the remaining possible relations ("one-third of" computed to the nearest whole number). The team gains an additional two points for each possibility eliminated over the required number to score. If fewer than the required number are eliminated, points are neither gained nor lost.

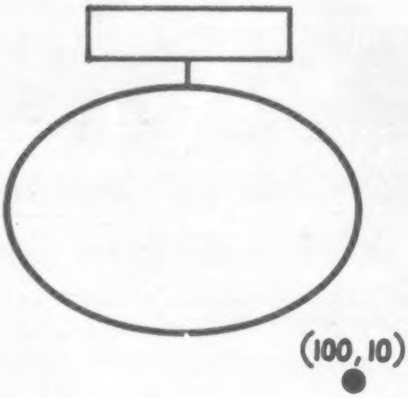
Alternating teams, select students or ask for volunteers to make plays. Before a team makes a play, decide what is the smallest number of possibilities that must be eliminated for the team to score. After each play, compute and record the score. The class can help you with the scoring. Continue until the relation is determined. The team with the most points wins.

Prepare your class to play the game by reviewing briefly the twenty-two relations involved in the game. Pay special attention to the relation "is the square of" and solicit several ordered pairs belonging to that relation. Then play the game. One choice for the secret relation is ÷ 2, and an appropriate starting clue is to locate (100, 10) outside the string.

A brief description of a sample game follows.

Starting clue


IS LARGER THAN
IS SMALLER THAN
=
IS AT LEAST 20 SMALLER THAN
IS AT LEAST 20 LARGER THAN
IS A MULTIPLE OF
IS A POSITIVE DIVISOR OF
IS THE SQUARE OF
10×
÷10
2×
÷2
$\frac{2}{3}$×
$\frac{3}{2}$×

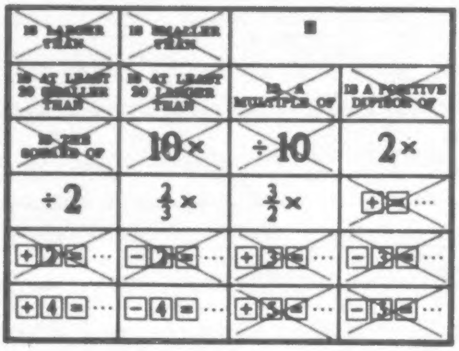


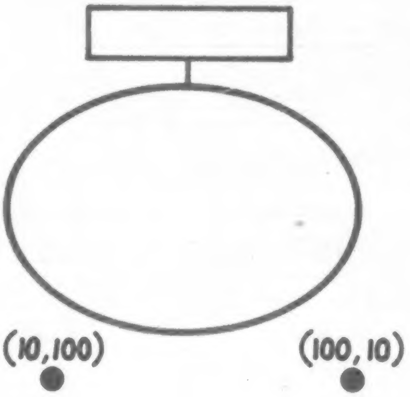
Fourteen possibilities remain.

One-third of 14 to the nearest whole number is 5.

Team A needs to eliminate at least five of the remaining possibilities to score.

S (Team A): (10, 100).


IS LARGER THAN
IS SMALLER THAN
=
IS AT LEAST 20 SMALLER THAN
IS AT LEAST 20 LARGER THAN
IS A MULTIPLE OF
IS A POSITIVE DIVISOR OF
IS THE SQUARE OF
10×
÷10
2×
÷2
$\frac{2}{3}$×
$\frac{3}{2}$×


(10,100)
(100, 10)

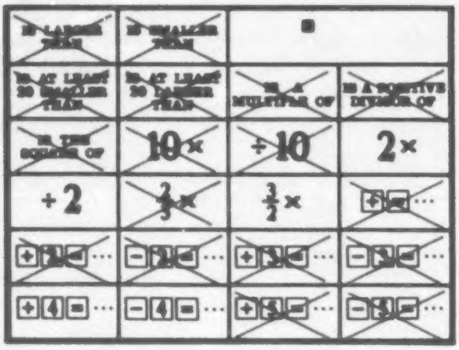
Eight more possibilities are eliminated. Team A scores five points plus six bonus points.

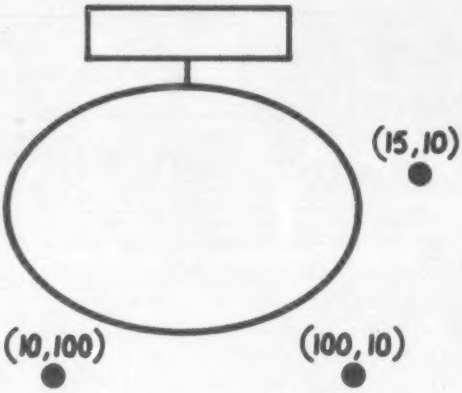
Six possibilities remain. One-third of 6 is 2.

Team B needs to eliminate at least two of the remaining possibilities to score.

L 6

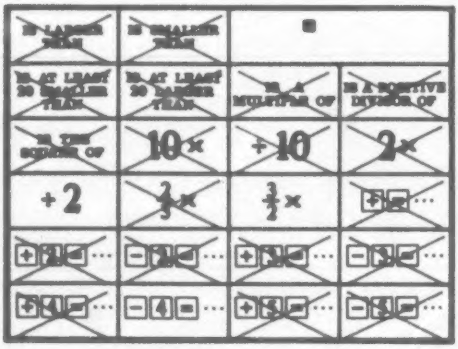
S (Team B): (15, 10).

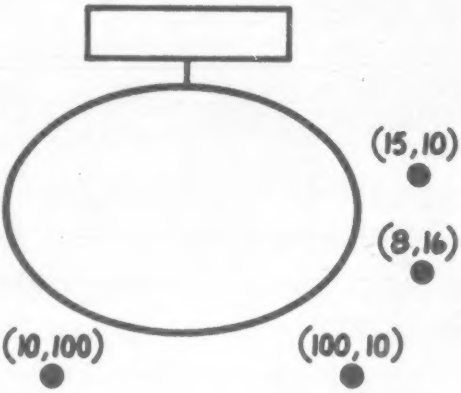

IS LARGER THAN
IS SMALLER THAN
IS AT LEAST 20 SMALLER THAN
IS AT LEAST 20 LARGER THAN
IS A MULTIPLE OF
IS A POSITIVE DIVISOR OF
IS THE SQUARE OF
10×
÷10
2×
+2
2/3×
3/2×
4
4


(15,10)
(10,100)
(100,10)

Only one possibility is eliminated. Team B does not score. Five possibilities remain. One-third of 5 to the nearest whole number is 2. Team A needs to eliminate at least two of the remaining possibilities to score.

S (Team A): (8, 16).


IS LARGER THAN
IS SMALLER THAN
IS AT LEAST 20 SMALLER THAN
IS AT LEAST 20 LARGER THAN
IS A MULTIPLE OF
IS A POSITIVE DIVISOR OF
IS THE SQUARE OF
10×
÷10
2×
+2
2/3×
3/2×
4

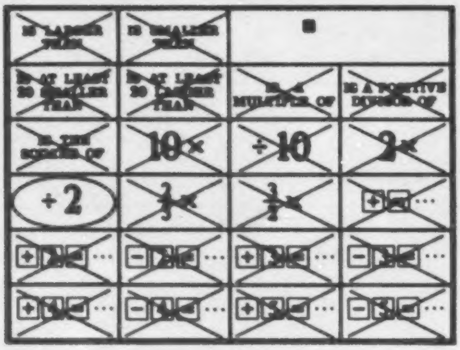

(15,10)
(8,16)
(10,100)
(100,10)

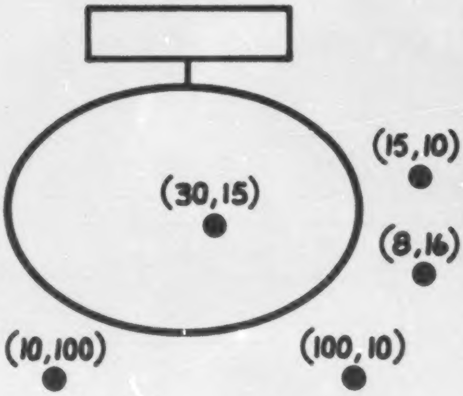
Two possibilities are eliminated. Team A scores five points but no bonus points.

Three possibilities remain. One-third of 3 is 1.

Team B needs to eliminate at least one of the remaining possibilities to score.

S (Team B): (30, 15).


IS LARGER THAN
IS SMALLER THAN
IS A MULTIPLE OF
IS A POSITIVE DIVISOR OF
10×
÷10
2×
÷2


(15,10)
(30,15)
(8,16)
(10,100)
(100,10)

Two possibilities are eliminated. Team B scores five points plus two bonus points.

Team A	Team B
11	7
5	
16	

Team A wins!

2.3 The Relations Game #3

CAPSULE LESSON SUMMARY

Considering ordered pairs of numbers, some which belong and some which do not belong to the relation "is the square of", give one component and ask what the other component could be. Play the analysis version of the Relations Game. Introduce the standard version of the Relations Game.

MATERIALS

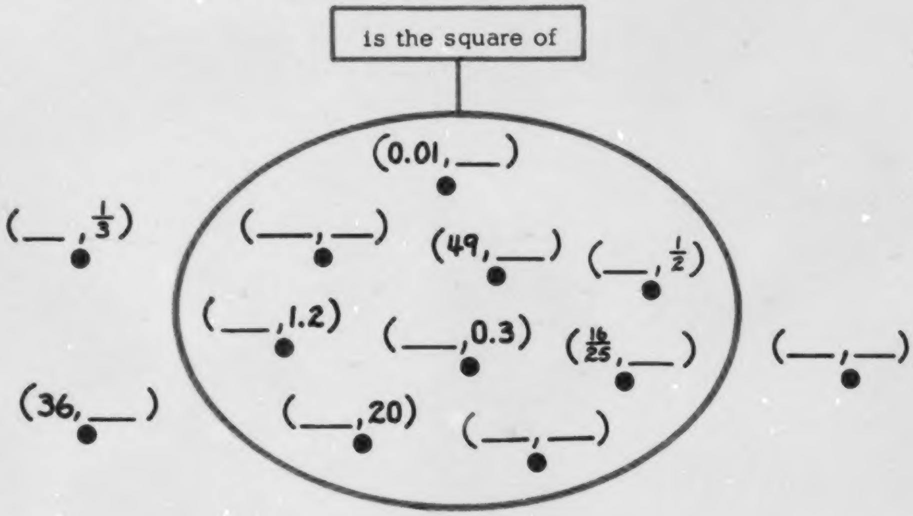
Teacher: Colored chalk; two copies of the Relations Game Poster; red felt-tip marker

Student: Relations Game Analysis Pad; pencil

DESCRIPTION OF LESSON

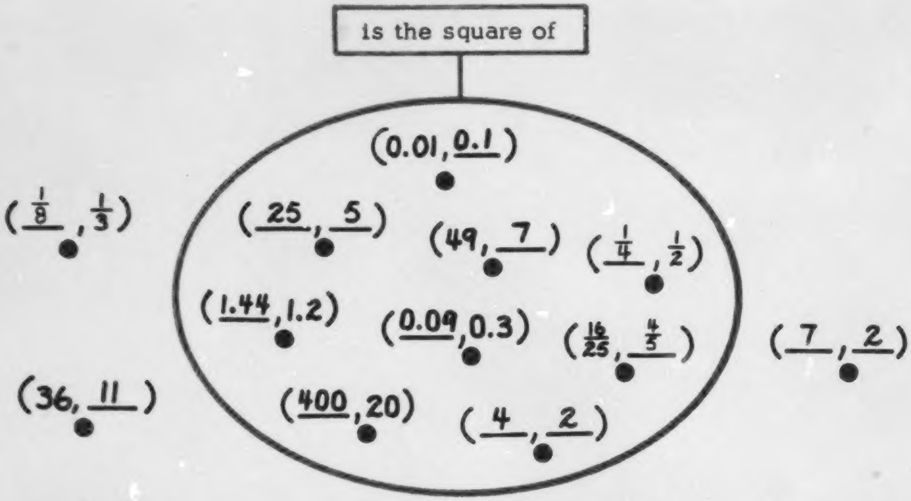
Exercise 1

Draw this picture on the board.



Ask the students to copy the picture and to fill in the blanks. Allow about five minutes for individual work and then go over possible answers collectively.

The next illustration shows only one of the many possible ways to complete the picture.


is the square of
(0.01, 0.1)
(1/8, 1/3)
(25, 5)
(49, 7)
(1/4, 1/2)
(1.44, 1.2)
(0.09, 0.3)
(16/25, 4/5)
(7, 2)
(36, 11)
(400, 20)
(4, 2)

Comments

- Be aware that there are many correct ways to fill in each blank since every number has many names. For example,

  $$"\frac{1}{4}", \quad "\frac{2}{8}", \quad "0.25", \quad "0.250", \text{ and } "\frac{25}{100}"$$

  are some of the many names for the square of $\frac{1}{2}$.

- When multiplying two decimals, students often prefer to do the calculation using fractions. For example,

  $$0.3 \times 0.3 = \frac{3}{10} \times \frac{3}{10} = \frac{9}{100} = 0.09$$

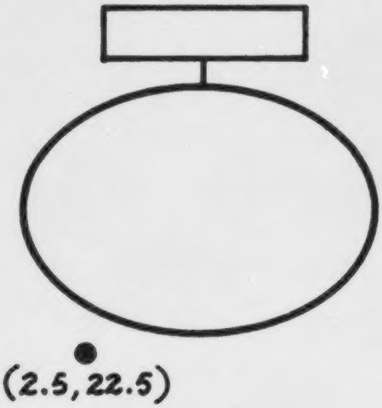
- Observe that there is only one number that is the square of another number but that a positive number is the square of two numbers. For example, in this exercise there is only one possibility for the first component of the ordered pair (___, 1.2), namely 1.44[†], but there are two possibilities for the second component of the ordered pair (49, ___), namely 7 and $^-7$.

- Your students should notice that the square of a number is not always larger than the number; for example, $(\frac{1}{2})^2 = \frac{1}{4}$ and $\frac{1}{4} < \frac{1}{2}$. For your information: the square of a number between 0 and 1 is always smaller than the number.

Exercise 2: The Relations Game, Analysis Version

Play the Relations Game as in Exercise 3 of Lesson L2.2. A possible choice for the secret relation is "is a positive divisor of". An appropriate starting clue is to locate (2.5, 22.5) outside the string. As a result of this clue, six of the twenty-two possibilities for the relation are eliminated (see the illustration below).

IS LARGER THAN	~~IS SMALLER THAN~~	=	
~~IS AT LEAST 20 SMALLER THAN~~	IS AT LEAST 20 LARGER THAN	IS A MULTIPLE OF	IS A POSITIVE DIVISOR OF
IS THE SQUARE OF	$10\times$	$\div 10$	$2\times$
$\div 2$	$\frac{2}{3}\times$	$\frac{3}{2}\times$	~~+ [illegible] = ···~~
~~+ 2 = ···~~	− 2 = ···	+ 3 = ···	− 3 = ···
~~+ 4 = ···~~	− 4 = ···	~~+ 5 = ···~~	− 5 = ···



[†] Of course, there are many names for the square of 1.2, e.g. "$1\frac{44}{100}$", "$\frac{144}{100}$", "1.440", ...

Exercise 3: The Relations Game, Usual Rules

The usual rules of the Relations Game are similar to the usual rules of the Table Game. In the Relations Game players attempt to determine which of twenty-two given relations (see poster) a string[†] is for. The game is divided into two parts. In both parts players select ordered pairs, but only in the second part do the players predict the location of ordered pairs before they are put correctly in the picture. Scoring rules change from one part to the next. To play the usual version of the Relations Game proceed as follows.

On the board draw a large string and tape a copy of the Relations Game Poster near the string. Select a student who would be a reliable scorekeeper. Divide the class (except for the scorekeeper) into two teams, Team A and Team B. Distribute individual sheets from the Relations Game Analysis Pad to the students. Tell them to use only one column of the analysis sheet, e.g. the RED column.

Begin the game by putting one to three ordered pairs in the string picture as starting clues. Such an ordered pair can belong either inside or outside the string. Give the players time to analyze the situation individually and to mark their analysis sheets. Then ask for volunteers (disregard teams here) to select ordered pairs for you to put in the picture as additional clues. Choose players who most likely will make helpful selections. When a player thinks the string is determined, he or she indicates the relation by circling it on the analysis sheet and then gives it to the scorekeeper. If a player correctly identifies the relation, the player's team gains points, how many depending upon how much information is in the picture.

no additional clue — 15 points
one additional clue — 10 points
two additional clues — 9 points
three additional clues — 8 points
four additional clues — 7 points

[†] See Lesson 2.5 for a description of how to play this game with two strings.

NOTE: You may find reasons to vary the number of additional clues (time is running short, most players have turned in their analysis sheets after two additional clues, very few players have turned in their analysis sheets after four additional clues, etc.), but in most games four will be appropriate.

Now the game changes. During the second part of the game, let each player select an ordered pair and say whether he or she thinks it belongs inside or outside the string. Tell students whether or not they are correct and put the suggested ordered pairs correctly in the picture. The scoring for this part of the game is as follows:

- a team gains two points each time a member correctly locates an ordered pair inside the string;
- a team gains one point each time a member correctly locates an ordered pair outside the string;
- a team neither gains nor loses points when a member incorrectly locates an ordered pair.

The game ends after each player has had a turn. While the scorekeeper computes the scores, call on a player to identify the secret relation.

We suggest that in the beginning you give the scorekeeper a scoring sheet similar to the one shown in the next illustration. You may wish to reproduce it so that you have several copies.

The secret relation is ______________.

Points

		Team A	Team B
First part of game			
For correctly identifying the relation			
• after no additional clue	15 pts.		
• after one additional clue	10 pts.		
• after two additional clues	9 pts.		
• after three additional clues	8 pts.		
• after four additional clues	7 pts.		
Second part of game			
For correctly locating an ordered pair			
• inside the string	2 pts.		
• outside the string	1 pt.		

OPTIONAL RULE: In the second part of the game, you can prompt students to suggest many kinds of numbers by awarding an extra point for each non-whole number component of an ordered pair belonging inside the string.

For a game played with the optional rule, modify the scoring sheet as follows.

The secret relation is ______________.

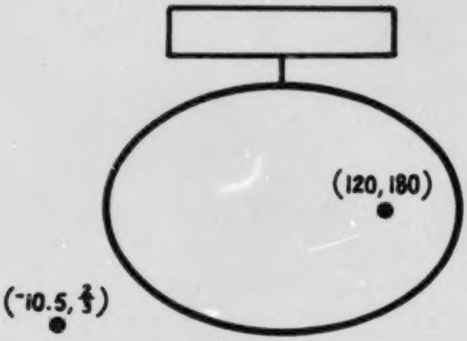
Points

		Team A	Team B
First part of game			
For correctly identifying the relation			
• after no additional clue	15 pts.		
• after one additional clue	10 pts.		
• after two additional clues	9 pts.		
• after three additional clues	8 pts.		
• after four additional clues	7 pts.		
Second part of game			
For correctly locating an ordered pair			
• inside the string	2 pts.		
(bonus for non-whole number components — 1 pt. each)			
• outside the string	1 pt.		

L7

Play this game with your class. A possible choice for the secret relation is $\frac{3}{2}x$. Appropriate starting clues are to locate (120, 180) inside the string and to locate $(^-10.5, \frac{2}{3})$ outside the string. We describe here a sample game.

starting situation



T: Knowing that (120, 180) belongs inside the string and that $(^-10.5, \frac{2}{3})$ belongs outside the string should help you to eliminate some of the possibilities for the relation. On your analysis sheet cross out relations the string cannot be for.

Allow a few minutes for players to analyze the situation individually.

NOTE: For your information only, there are six possibilities for the secret relation which remain: is at least 20 smaller than, $\frac{3}{2}x$, $\boxed{+}\ \boxed{2}\ \boxed{=}$. . . . $\boxed{+}\ \boxed{3}\ \boxed{=}$. . . , $\boxed{+}\ \boxed{4}\ \boxed{=}$. . . , and $\boxed{+}\ \boxed{5}\ \boxed{=}$. . . .

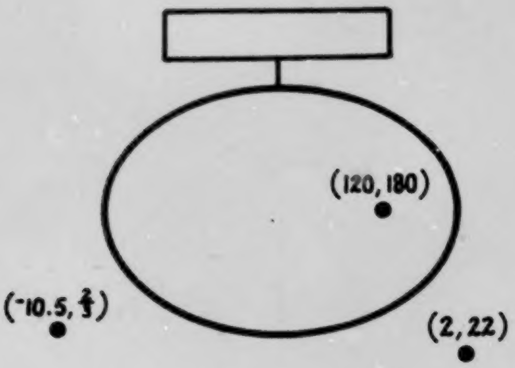
T: Do any of you want to turn in your analysis sheets for a possible fifteen points?

No one turns in an analysis sheet.

T: Give me an ordered pair whose location, whether inside or outside the string, would eliminate some of the possibilities for the relation.

S: (2, 22).

T: (2, 22) belongs outside the string.



Allow a few minutes for the players to analyze the situation.

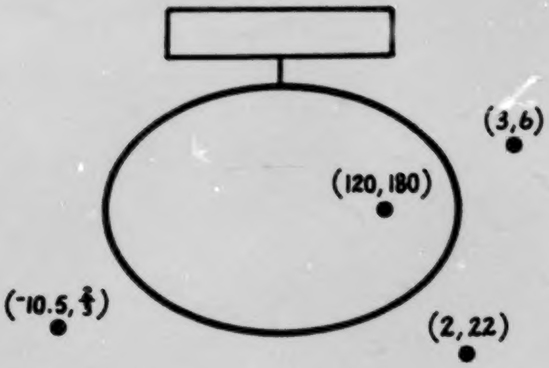
T: Do any of you want to turn in your sheets for a possible ten points?

No one turns in an analysis sheet.

T: Give me another ordered pair.

S: (3, 6).

T: (3, 6) belongs outside the string.



L 7

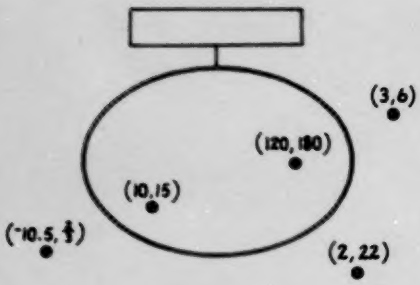
Allow a few minutes for the players to analyze the situation.

T: Do any of you want to turn in your sheets for a possible nine points?

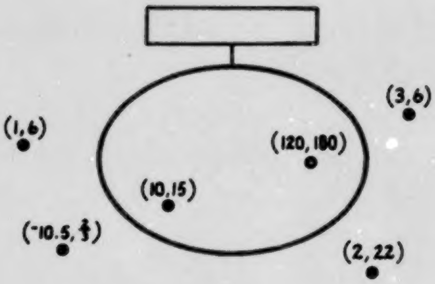
Several players turn in their analysis sheets. (The relation is actually determined at this point.)

Then two more players suggest ordered pairs and some more players turn in their analysis sheets for eight points or for seven points.

student selects (10, 15).



student selects (1, 6).



The scorekeeper tallies the points in the game so far but does not reveal the totals to the class.

Team A	Team B
34	36

Now the scoring changes. Students select ordered pairs and guess whether each belongs inside or outside the string.

S: (20, 0) belongs outside the string.

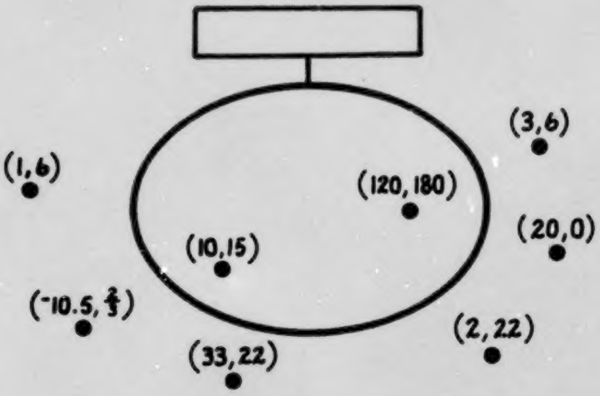
T: Yes.

Student's team gains one point.

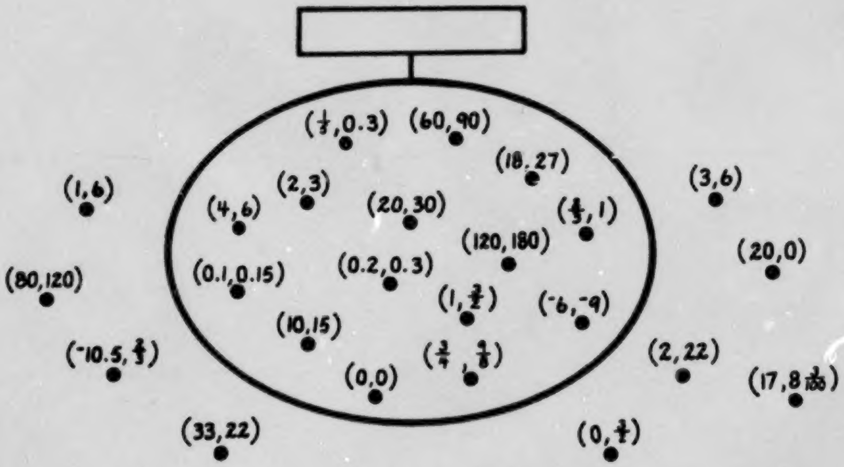
S: (33, 22) belongs inside the string.

T: No, (33, 22) belongs outside the string.

Student's team neither gains nor loses points.


(3,6)
(1,6)
(120,180)
(20,0)
(10,15)
(⁻10.5, $\frac{2}{3}$)
(2,22)
(33,22)

The game continues until every player has had a turn. The string picture at the end of the game is shown below. The optional rule was in effect.


($\frac{1}{5}$,0.3)
(60,90)
(18,27)
(1,6)
(2,3)
(3,6)
(4,6)
(20,30)
($\frac{2}{3}$,1)
(120,180)
(20,0)
(80,120)
(0.1,0.15)
(0.2,0.3)
(1,$\frac{3}{2}$)
(⁻6,⁻9)
(10,15)
(⁻10.5,$\frac{2}{3}$)
($\frac{3}{4}$,$\frac{1}{8}$)
(2,22)
(0,0)
(17,8$\frac{1}{100}$)
(33,22)
(0,$\frac{1}{2}$)

L 7

T: While the scorekeeper is computing the scores, who can tell us which relation the string is for?

S: $\frac{3}{2} x$.

The scorekeeper reveals the scores and Team B wins.

Team A	Team B
34	36
1	1
4	0
2	2
4	0
0	4
2	4
2	2
1	3
2	3
52	55

2.4 The Relations Game #4

CAPSULE LESSON SUMMARY

Play the Relations Game in the usual way. Locate some ordered pairs in each of the four regions of a 2-string picture in which one string is for the relation $\boxed{+}\ \boxed{5}\ \boxed{=}$. . . and the other is for the relation "is at least 20 smaller than".

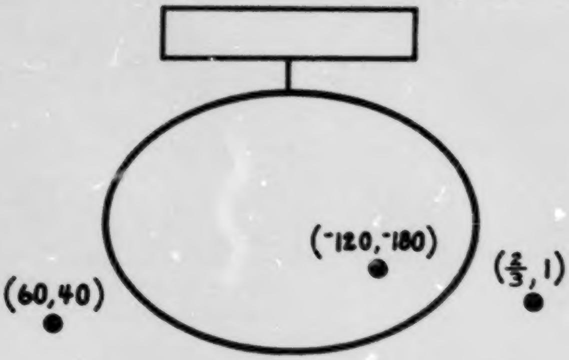
MATERIALS

Teacher:	Colored chalk; the Relations Game Poster; red felt-tip marker
Student:	Relations Game Analysis Pad; pencil

DESCRIPTION OF LESSON

Exercise 1

Play the Relations Game according to usual rules. One choice for the relation is $\boxed{-}\ \boxed{3}\ \boxed{=}$. . . . Appropriate starting clues are to locate ($^{-}120$, $^{-}180$) inside the string and to locate ($\frac{2}{3}$, 1) and (60, 40) outside the string. Tell the class that the relation is determined by these clues.



After the game, collectively analyze why the relation is determined by the starting clues.

L8

Using the information of where ($^-120$, $^-180$) is in the picture, cross off these labels.

IS LARGER THAN	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	IS AT LEAST 20 LARGER THAN	~~IS A MULTIPLE OF~~	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~10 ×~~	~~÷ 10~~	~~2 ×~~
~~÷ 2~~	~~$\frac{2}{3}$ ×~~	$\frac{3}{2}$ ×	~~+ = ⋯~~
~~+ 2 = ⋯~~	− 2 = ⋯	~~+ 3 = ⋯~~	− 3 = ⋯
~~+ 4 = ⋯~~	− 4 = ⋯	~~+ 5 = ⋯~~	− 5 = ⋯

Using the information of where (60, 40) is in the picture, cross off additional labels.

~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	~~IS AT LEAST 20 LARGER THAN~~	~~IS A MULTIPLE OF~~	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~10 ×~~	~~÷ 10~~	~~2 ×~~
~~÷ 2~~	~~$\frac{2}{3}$ ×~~	$\frac{3}{2}$ ×	~~+ = ⋯~~
~~+ 2 = ⋯~~	~~− 2 = ⋯~~	~~+ 3 = ⋯~~	− 3 = ⋯
~~+ 4 = ⋯~~	~~− 4 = ⋯~~	~~+ 5 = ⋯~~	~~− 5 = ⋯~~

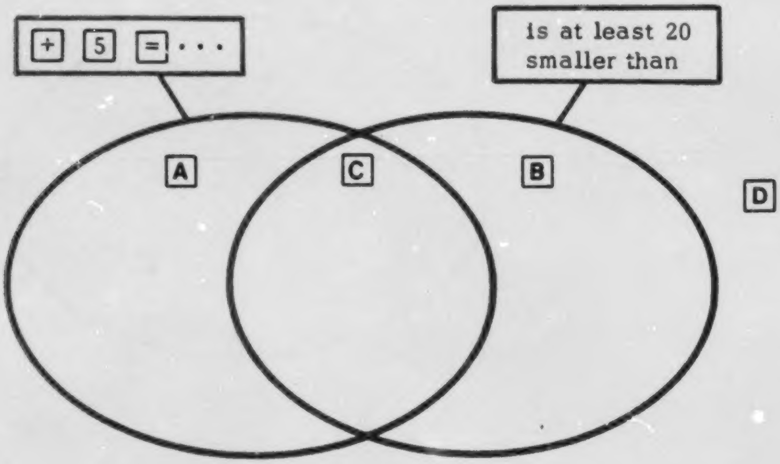
Using the information of where ($\frac{2}{3}$, 1) is in the picture, cross off one more label.

~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	~~IS AT LEAST 20 LARGER THAN~~	~~IS A MULTIPLE OF~~	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~10 ×~~	~~÷ 10~~	~~2 ×~~
~~÷ 2~~	~~$\frac{2}{3}$ ×~~	~~$\frac{3}{2}$ ×~~	~~+ = ⋯~~
~~+ 2 = ⋯~~	~~− 2 = ⋯~~	~~+ 3 = ⋯~~	− 3 = ⋯
~~+ 4 = ⋯~~	~~− 4 = ⋯~~	~~+ 5 = ⋯~~	~~− 5 = ⋯~~

NOTE: Your students might neither use the clues in any particular order nor exhaust the clues one at a time. Let the analysis flow in the style that your students initiate.

Exercise 2

Draw this string picture on the board.



Ask for ordered pairs that belong to each of the four regions. Sometimes place conditions on the components; for example,

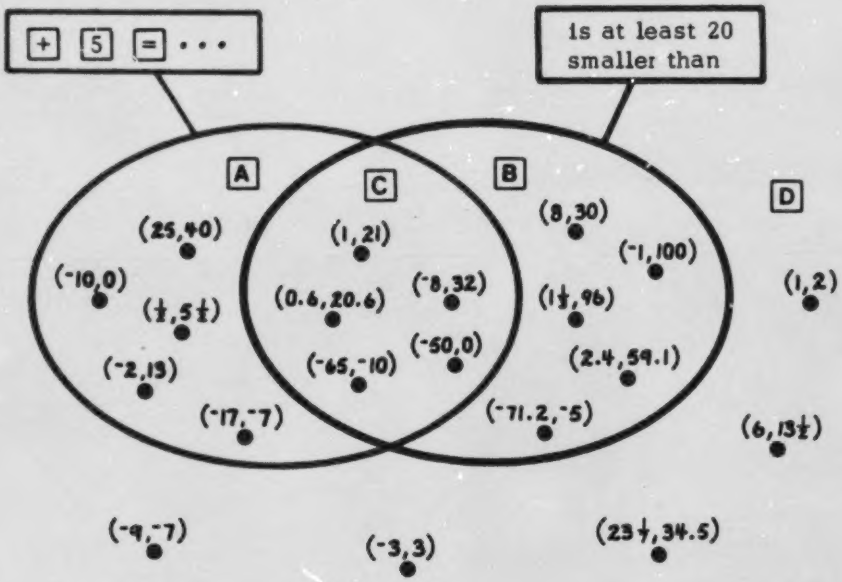
- both components are whole numbers;
- only one component is a whole number;
- neither component is a whole number;
- exactly one component is a negative number;
- both components are negative numbers.

When a student suggests an ordered pair, ask why it belongs in the region being discussed at that time. For example, a good response would be:

S: (10, 15) belongs in Region A. 10 + 5 = 15, so (10, 15) belongs in the red string. 10 is not at least 20 smaller than 15, so (10, 15) does not belong in the blue string.

L 8

Perhaps your picture will look similar to this one.



+ 5 = . . .
is at least 20 smaller than
A
C
B
D
(25,40)
(⁻10,0)
(½,5½)
(⁻2,13)
(⁻17,⁻7)
(1,21)
(0.6,20.6)
(⁻8,32)
(⁻65,⁻10)
(⁻50,0)
(8,30)
(⁻1,100)
(1⅓,96)
(2.4,59.1)
(⁻71.2,⁻5)
(1,2)
(6,13½)
(⁻9,⁻7)
(⁻3,3)
(23½,34.5)

Erase only the ordered pairs in the picture and then write this list of ordered pairs near the string picture.

$(8,103)$

$(2.7,17.7)$

$(0,0)$

$(^-4,6)$

$(^-32,^-17)$

$(\frac{2}{3},30\frac{1}{3})$

$(^-10,9.5)$

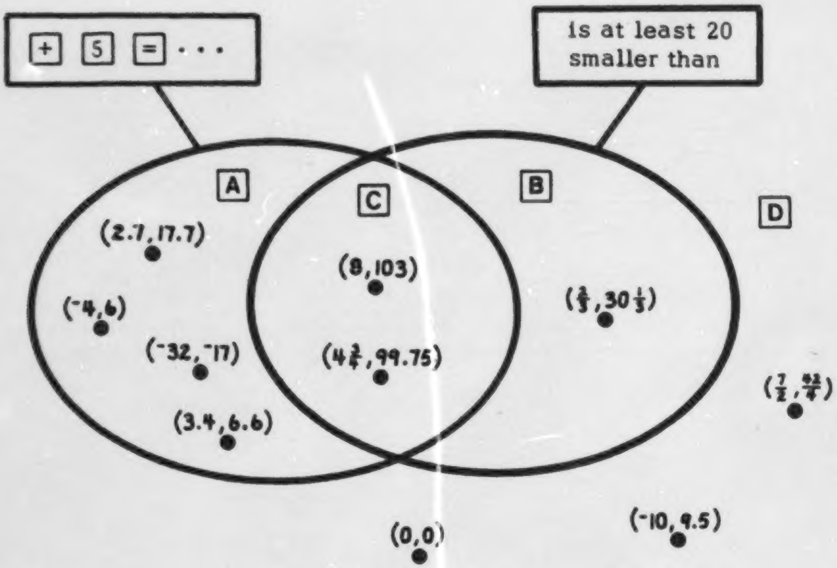
$(4\frac{3}{4},99.75)$

$(^-3.4,6.6)$

$(\frac{7}{2},\frac{42}{4})$

T: On a piece of paper copy this string picture and locate these ordered pairs (point to the list).

Allow about five minutes for individual work before discussing the location of the pairs.


+ 5 = . . .
is at least 20 smaller than
A
B
C
D
(2.7, 17.7)
(⁻4, 6)
(⁻32, ⁻17)
(3.4, 6.6)
(8, 103)
(4 3/4, 99.75)
(2/3, 30 1/3)
(7/2, 42/4)
(0, 0)
(⁻10, 9.5)

Accept any correct methods but encourage looking at the difference between the components of an ordered pair in deciding whether or not the ordered pair belongs inside the red string. For example, the difference between ⁻4 and 6 is 10, a multiple of 5, so (⁻4, 6) belongs inside the red string. Also, since ⁻4 is not at least 20 smaller than 6, (⁻4, 6) belongs outside the blue string.

2.5 The Relations Game #5

CAPSULE LESSON SUMMARY

Play the analysis version of the Relations Game with a 2-string picture. Play the Relations Game according to usual rules and with 2-string pictures. Explain minor changes in the rules when two strings are involved.

MATERIALS

Teacher: Colored chalk; three copies of the Relations Game Poster; red and blue felt-tip markers

Student: Relations Game Analysis Pad; pencil

DESCRIPTION OF LESSON

Exercise 1: The Relations Game with Two Strings, Analysis Version

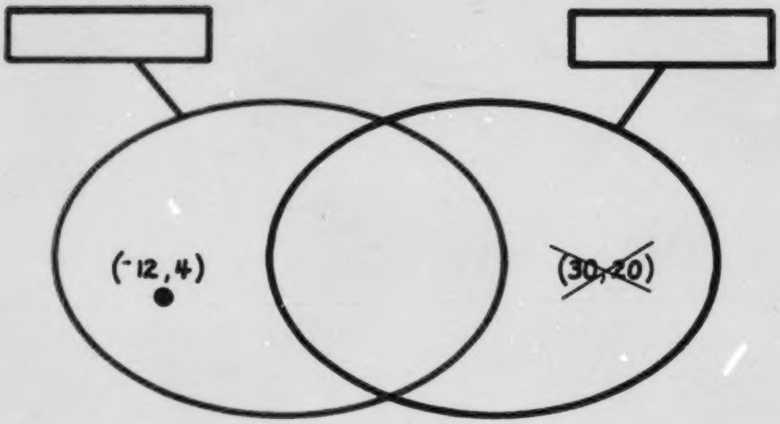
The analysis version of the Relations Game extends quite naturally from a 1-string to a 2-string game. In both games, before a team suggests an ordered pair, the class figures out the smallest number of possibilities that the team needs to eliminate in order to score. But in a 2-string game the possibilities for both strings must be considered.

Play a 2-string game with your class. Possible choices for the relations are:

the red string --- is a multiple of

the blue string --- is at least 20 larger than

Appropriate starting clues are to locate ($^-12$, 4) and to indicate one region in which (30, 20) does not belong. See the next illustration.



NOTE: In a 2-string game a starting clue can indicate a region in which a particular ordered pair does or does not belong.

We describe a sample game to clarify the rules. Two Relations Game Posters are taped to the board, one on each side of the string picture.

T:	These are your starting clues (see the preceding illustration). What does it mean to have "(30, 20)" crossed out?
S:	It means that (30, 20) does not belong in that region of the picture.
T:	Could (30, 20) belong in the blue string?
S:	Yes, but then it would be in the middle region so also belong in the red string.
T:	What can we cross out in the RED list?
S:	All the "minus" relations.
T:	Why?

S: Because 4 is larger than $^{-}12$. If you subtract a positive number from $^{-}12$, you get a smaller number.

On the RED list, cross out "$\boxed{-}\ \boxed{2}\ \boxed{=}$. . .", "$\boxed{-}\ \boxed{3}\ \boxed{=}$. . .", "$\boxed{-}\ \boxed{4}\ \boxed{=}$. . .", and "$\boxed{-}\ \boxed{5}\ \boxed{=}$. . .".

S: "$\frac{2}{3}$ x", because $\frac{2}{3} \times {}^{-}12 = {}^{-}8$.

On the RED list, cross out "$\frac{2}{3}$ x".

The analysis continues in this manner until the class agrees that only four possibilities for the red string remain.

THE RED STRING

~~IS LARGER THAN~~	IS SMALLER THAN	■	
~~IS AT LEAST 20 SMALLER THAN~~	~~IS AT LEAST 20 LARGER THAN~~	IS A MULTIPLE OF	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~10 ×~~	~~÷ 10~~	~~2 ×~~
~~÷ 2~~	~~$\frac{2}{3}$ ×~~	~~$\frac{3}{2}$ ×~~	~~$\boxed{+}\boxed{=}\cdots$~~
$\boxed{+}\boxed{2}\boxed{=}\cdots$	~~$\boxed{-}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{3}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{3}\boxed{=}\cdots$~~
$\boxed{+}\boxed{4}\boxed{=}\cdots$	~~$\boxed{-}\boxed{4}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{5}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{5}\boxed{=}\cdots$~~

NOTE: The location of $({}^{-}12, 4)$ is sufficient to eliminate these eighteen possibilities for the red string.

T: What can we cross out on the BLUE list?

After some discussion, the class realizes that if the red string cannot be for a particular relation, then the blue string can be for it, and vice versa. This is because of the special nature of the clue: $({}^{-}12, 4)$ belongs inside the red string but outside the blue string.

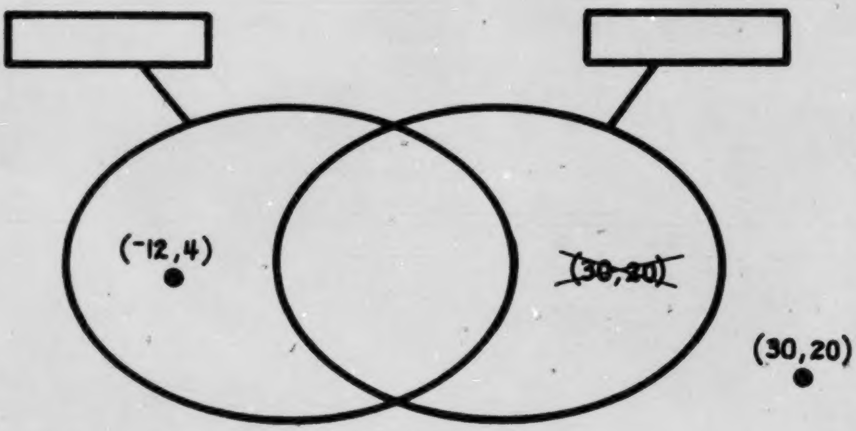
THE BLUE STRING

IS LARGER THAN	~~IS SMALLER THAN~~	■	
IS AT LEAST 20 SMALLER THAN	IS AT LEAST 20 LARGER THAN	~~IS A MULTIPLE OF~~	IS A POSITIVE DIVISOR OF
IS THE SQUARE OF	10 ×	÷ 10	2 ×
÷ 2	$\frac{2}{3}$ ×	$\frac{3}{2}$ ×	$\boxed{+}\boxed{=}\cdots$
~~$\boxed{+}\boxed{2}\boxed{=}\cdots$~~	$\boxed{-}\boxed{2}\boxed{=}\cdots$	$\boxed{+}\boxed{3}\boxed{=}\cdots$	$\boxed{-}\boxed{3}\boxed{=}\cdots$
~~$\boxed{+}\boxed{4}\boxed{=}\cdots$~~	$\boxed{-}\boxed{4}\boxed{=}\cdots$	$\boxed{+}\boxed{5}\boxed{=}\cdots$	$\boxed{-}\boxed{5}\boxed{=}\cdots$

T (pointing to the crossed out ordered pair): We haven't used this clue. Where does (30, 20) belong? Could it belong in the red string?

S: (30, 20) cannot belong in the red string because there are only four possibilities left for the red string and (30, 20) does not belong to any of them: 30 is not a multiple of 20; by adding twos or fours to 30 we never get to 20; and 30 is not smaller than 20.

S: (30, 20) belongs outside both strings.



T: Knowing that (30, 20) belongs outside both strings, what else can we cross out on the lists?

S: "is larger than" from the BLUE list, because 30 is larger than 20.

S: "$\boxed{-}\,\boxed{5}\,\boxed{=}$. . ." from the BLUE list, because 30 - 5 - 5 = 20.

S: "$\boxed{-}\,\boxed{2}\,\boxed{=}$. . ." from the BLUE list, because the difference between 30 and 20 is 10 and 10 is a multiple of 2.

S: "$\frac{2}{3}$ x" from the BLUE list, because $\frac{2}{3} \times 30 = 20$.

THE BLUE STRING

~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
IS AT LEAST 20 SMALLER THAN	IS AT LEAST 20 LARGER THAN	~~IS A MULTIPLE OF~~	IS A POSITIVE DIVISOR OF
IS THE SQUARE OF	10 ×	÷ 10	2 ×
÷ 2	~~$\frac{2}{3}$ ×~~	$\frac{3}{2}$ ×	$\boxed{+}\,\boxed{=}\cdots$
~~$\boxed{+}\,\boxed{2}\,\boxed{=}\cdots$~~	~~$\boxed{-}\,\boxed{2}\,\boxed{=}\cdots$~~	$\boxed{+}\,\boxed{3}\,\boxed{=}\cdots$	$\boxed{-}\,\boxed{3}\,\boxed{=}\cdots$
~~$\boxed{+}\,\boxed{4}\,\boxed{=}\cdots$~~	$\boxed{-}\,\boxed{4}\,\boxed{=}\cdots$	$\boxed{+}\,\boxed{5}\,\boxed{=}\cdots$	~~$\boxed{-}\,\boxed{5}\,\boxed{=}\cdots$~~

T: How many possibilities are left?

S: Eighteen, four for the red string and fourteen for the blue string.

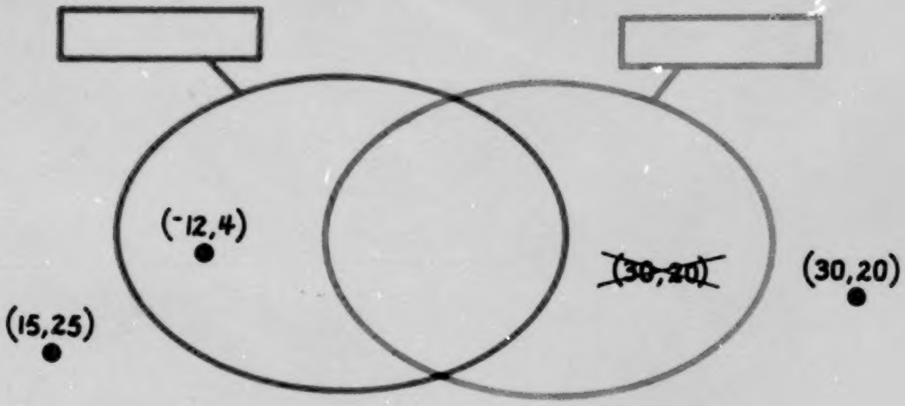
T: One-third of 18 is . . . ?

S: 6.

T: So Team A must eliminate at least six possibilities in order to score some points.

S (Team A): (15, 25).

T: (15, 25) belongs outside both strings.



The class uses the information to eliminate three possibilities, two for the red string and one for the blue string.

THE RED STRING

~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	~~IS AT LEAST 20 LARGER THAN~~	IS A MULTIPLE OF	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~10×~~	~~÷10~~	~~2×~~
~~÷2~~	~~$\frac{2}{3}\times$~~	~~$\frac{3}{2}\times$~~	~~$\boxed{+}\boxed{=}\cdots$~~
~~$\boxed{+}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{3}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{3}\boxed{=}\cdots$~~
$\boxed{+}\boxed{4}\boxed{=}\cdots$	~~$\boxed{-}\boxed{4}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{5}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{5}\boxed{=}\cdots$~~

THE BLUE STRING

~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
IS AT LEAST 20 SMALLER THAN	IS AT LEAST 20 LARGER THAN	~~IS A MULTIPLE OF~~	IS A POSITIVE DIVISOR OF
IS THE SQUARE OF	10×	÷10	2×
÷2	~~$\frac{2}{3}\times$~~	$\frac{3}{2}\times$	$\boxed{+}\boxed{=}\cdots$
~~$\boxed{+}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{2}\boxed{=}\cdots$~~	$\boxed{+}\boxed{3}\boxed{=}\cdots$	$\boxed{-}\boxed{3}\boxed{=}\cdots$
~~$\boxed{+}\boxed{4}\boxed{=}\cdots$~~	$\boxed{-}\boxed{4}\boxed{=}\cdots$	~~$\boxed{+}\boxed{5}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{5}\boxed{=}\cdots$~~

T: Team A needed to eliminate at least six possibilities and only eliminated three. They do not gain points.

How many possibilities are left?

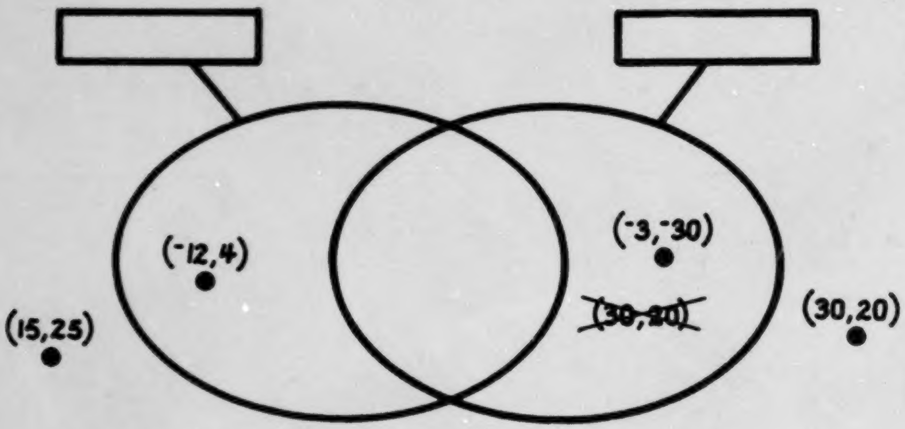
S: Fifteen, two for the red string and thirteen for the blue string.

L9

T: One-third of 15 is 5, so Team B must eliminate at least five possibilities in order to score.

S (Team B): (⁻3, ⁻30).

T: (⁻3, ⁻30) belongs inside the blue string but outside the red string.



The class uses the information to eliminate nine possibilities for the blue string.

THE RED STRING

~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	~~IS AT LEAST 20 LARGER THAN~~	IS A MULTIPLE OF	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~$10 \times$~~	~~$\div 10$~~	~~$2 \times$~~
~~$\div 2$~~	~~$\frac{2}{3} \times$~~	~~$\frac{3}{2} \times$~~	~~$\boxed{+}\boxed{=}\cdots$~~
~~$\boxed{+}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{3}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{3}\boxed{=}\cdots$~~
$\boxed{+}\boxed{4}\boxed{=}\cdots$	~~$\boxed{-}\boxed{4}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{5}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{5}\boxed{=}\cdots$~~

THE BLUE STRING

~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	IS AT LEAST 20 LARGER THAN	~~IS A MULTIPLE OF~~	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	$10 \times$	~~$\div 10$~~	~~$2 \times$~~
~~$\div 2$~~	~~$\frac{2}{3} \times$~~	~~$\frac{3}{2} \times$~~	$\boxed{+}\boxed{=}\cdots$
~~$\boxed{+}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{3}\boxed{=}\cdots$~~	$\boxed{-}\boxed{3}\boxed{=}\cdots$
~~$\boxed{+}\boxed{4}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{4}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{5}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{5}\boxed{=}\cdots$~~

T: Team B needed to eliminate at least five possibilities and eliminated nine. So Team B scores 13 points (5 + (2 x 4)).

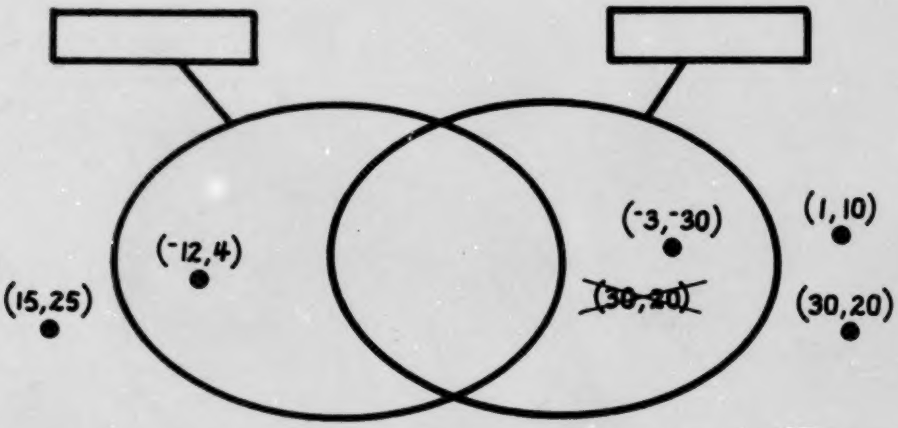
How many possibilities are left?

S: Six, two for the red string and four for the blue string.

T: One-third of 6 is 2, so Team A needs to eliminate at least two possibilities in order to score.

S (Team A): (1, 10)

T: (1, 10) belongs outside both strings.



The class uses the information to eliminate two possibilities for the blue string.

THE RED STRING

~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	~~IS AT LEAST 20 LARGER THAN~~	IS A MULTIPLE OF	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~10×~~	~~÷10~~	~~2×~~
~~÷2~~	~~$\frac{2}{3}$×~~	~~$\frac{3}{2}$×~~	~~+1= ···~~
~~+2= ···~~	~~−2= ···~~	~~+3= ···~~	~~−3= ···~~
+4= ···	~~−4= ···~~	~~+5= ···~~	~~−5= ···~~

THE BLUE STRING

~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	IS AT LEAST 20 LARGER THAN	~~IS A MULTIPLE OF~~	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~10×~~	~~÷10~~	~~2×~~
~~÷2~~	~~$\frac{2}{3}$×~~	~~$\frac{3}{2}$×~~	~~+1= ···~~
~~+2= ···~~	~~−2= ···~~	~~+3= ···~~	−3= ···
~~+4= ···~~	~~−4= ···~~	~~+5= ···~~	~~−5= ···~~

T: Team A needed to eliminate at least two possibilities and eliminated exactly two. So Team A scores 5 points.

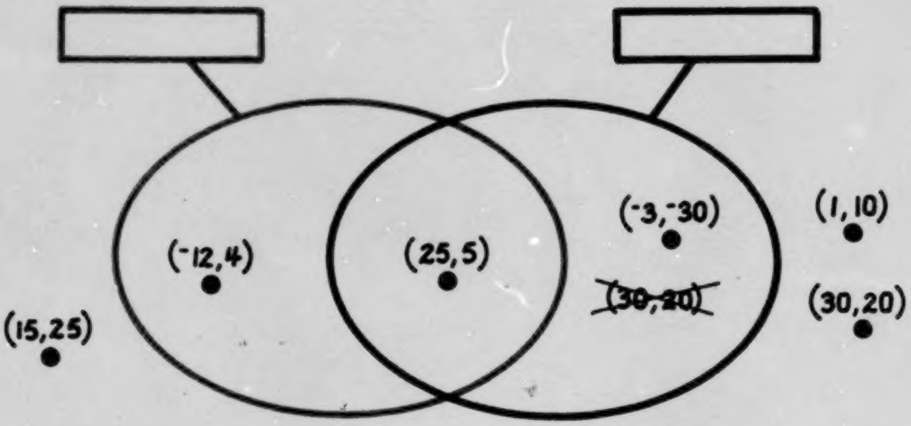
T: Four possibilities are left. One-third of 4 to the nearest whole number is . . . ?

S: 1.

T: Team B must eliminate at least one possibility in order to score.

S (Team B): (25, 5).

T: (25, 5) belongs in both strings.



S: The red string cannot be for $\boxed{+}\boxed{5}\boxed{=}$ . . . and the blue string cannot be for $\boxed{-}\boxed{3}\boxed{=}$ . . .

THE RED STRING

~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	~~IS AT LEAST 20 LARGER THAN~~	[✓ IS A MULTIPLE OF]	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~10×~~	~~÷10~~	~~2×~~
~~÷2~~	~~$\frac{2}{3}\times$~~	~~$\frac{3}{2}\times$~~	~~$\boxed{+}\boxed{=}$ ⋯~~
~~$\boxed{+}\boxed{2}\boxed{=}$ ⋯~~	~~$\boxed{-}\boxed{2}\boxed{=}$ ⋯~~	~~$\boxed{+}\boxed{3}\boxed{=}$ ⋯~~	~~$\boxed{-}\boxed{3}\boxed{=}$ ⋯~~
~~$\boxed{+}\boxed{4}\boxed{=}$ ⋯~~	~~$\boxed{-}\boxed{4}\boxed{=}$ ⋯~~	~~$\boxed{+}\boxed{5}\boxed{=}$ ⋯~~	~~$\boxed{-}\boxed{5}\boxed{=}$ ⋯~~

THE BLUE STRING

~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	[✓ IS AT LEAST 20 LARGER THAN]	~~IS A MULTIPLE OF~~	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~10×~~	~~÷10~~	~~2×~~
~~÷2~~	~~$\frac{2}{3}\times$~~	~~$\frac{3}{2}\times$~~	~~$\boxed{+}\boxed{=}$ ⋯~~
~~$\boxed{+}\boxed{2}\boxed{=}$ ⋯~~	~~$\boxed{-}\boxed{2}\boxed{=}$ ⋯~~	~~$\boxed{+}\boxed{3}\boxed{=}$ ⋯~~	~~$\boxed{-}\boxed{3}\boxed{=}$ ⋯~~
~~$\boxed{+}\boxed{4}\boxed{=}$ ⋯~~	~~$\boxed{-}\boxed{4}\boxed{=}$ ⋯~~	~~$\boxed{+}\boxed{5}\boxed{=}$ ⋯~~	~~$\boxed{-}\boxed{5}\boxed{=}$ ⋯~~

T: Team B needed to eliminate at least one possibility and eliminated two, so Team B scores 7 points (5 + 2). The strings are determined: the red string is for "is a multiple of" and the blue string is for "is at least 20 larger than".

Team A	Team B
0	13
5	7
5	20

T: Team B wins!

Exercise 2: The Relations Game with Two Strings, Usual Rules

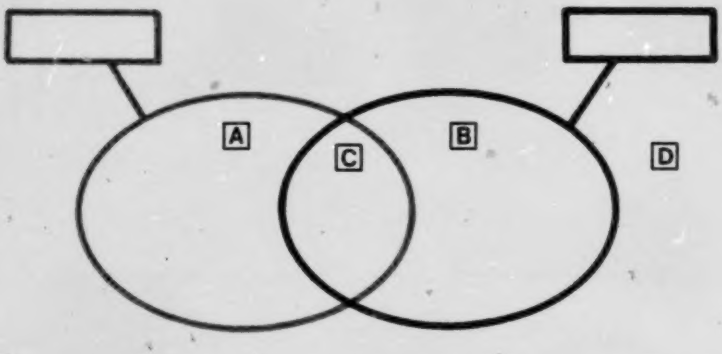
Minor modifications need to be made in the usual rules for the Relations Game (see Exercise 3 of Lesson L2.3) in order to extend it to a 2-string game.

First Part of the Game

In the first part of the game, the point system stays the same except points are awarded only when both strings are correctly identified.

Second Part of the Game

In the second part of the game, points are awarded as follows:



- A team gains four points each time a member correctly locates an ordered pair in region C.
- A team gains two points each time a member correctly locates an ordered pair in regions A or B.
- A team gains one point each time a member correctly locates an ordered pair in region D.
- A team neither gains nor loses points when a member incorrectly locates an ordered pair.

Change the scoring sheet accordingly.

The secret relations are:

red string ____________

blue string ____________

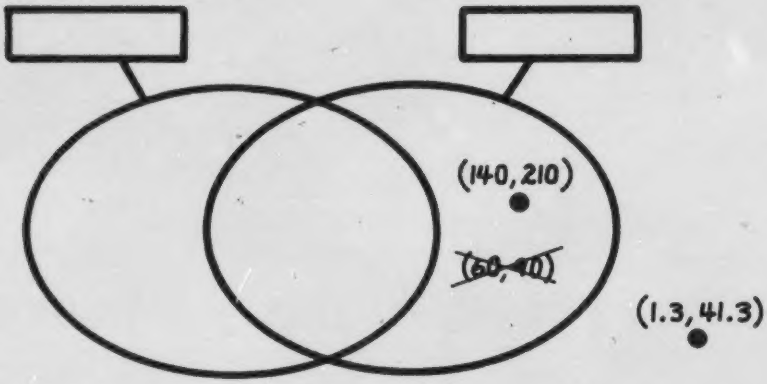
Points		TEAM A	TEAM B
First part of game			
For correctly identifying both relations			
• after no additional clue	15 pts.		
• after one additional clue	10 pts.		
• after two additional clues	9 pts.		
• after three additional clues	8 pts.		
• after four additional clues	7 pts.		
Second part of game			
For correctly locating an ordered pair			
• that belongs inside both strings	4 pts.		
• that belongs inside only one of the strings	2 pts.		
• that belongs outside both strings	1 pt.		

Play a 2-string game with your class. Possible choices for the relations are:

the red string --- $\boxed{+}\ \boxed{3}\ \boxed{=}\ldots$

the blue string --- $\frac{3}{2}\times$

Appropriate starting clues are to locate (140, 210) and (1.3, 41.2) and to indicate one region in which (60, 90) does not belong.



Tell your class that the strings are determined by these clues.

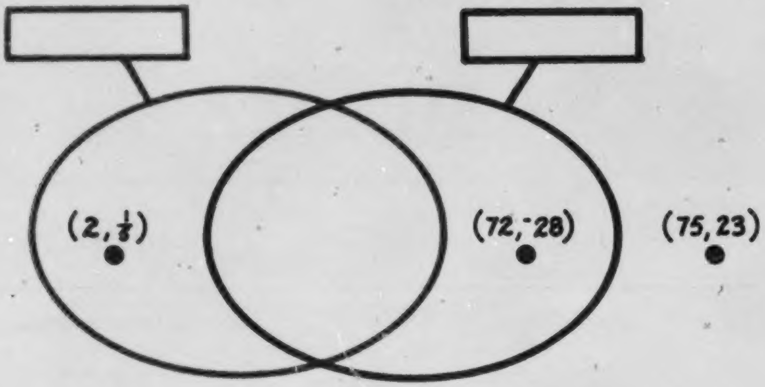
Play another 2-string game with your class. Possible choices for the relations are:

the red string --- ÷ 10

the blue string --- $\boxed{-}\ \boxed{5}\ \boxed{=}\ldots$

L 9

Appropriate starting clues are to locate the ordered pairs (72, $^-28$), (2, $\frac{1}{5}$), and (75, 23).



You may wish to tell your class that the strings are determined by these clues.

605

2.6 The Relations Game #6

CAPSULE LESSON SUMMARY

Play both the analysis and standard versions of the Relations Game with 2-string pictures.

MATERIALS

Teacher:	Colored chalk; three copies of the Relations Game Poster; red and blue felt-tip markers
Student:	Relations Game Analysis Pad; Worksheets L05 and L06; pencil

DESCRIPTION OF LESSON

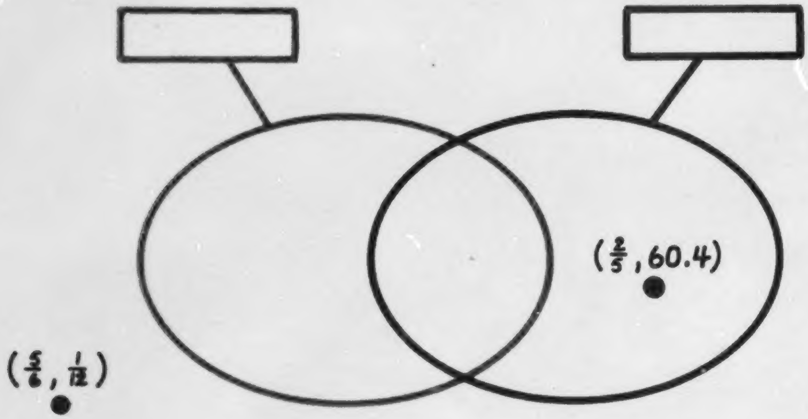
Exercise 1

Play the analysis version of the Relations Game with two strings. Possible choices for the relations are:

the red string --- is a positive divisor of

the blue string --- is smaller than

Appropriate starting clues are to locate the ordered pairs $(\frac{5}{6}, \frac{1}{12})$ and $(\frac{2}{5}, 60.4)$.



L 10

These clues eliminate twenty-four of the forty-four possibilities.

THE RED STRING

~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	IS AT LEAST 20 LARGER THAN	IS A MULTIPLE OF	IS A POSITIVE DIVISOR OF
IS THE SQUARE OF	$10\times$	~~$\div 10$~~	$2\times$
$\div 2$	$\frac{2}{3}\times$	$\frac{3}{2}\times$	~~$\boxed{+}\boxed{=}\cdots$~~
~~$\boxed{+}\boxed{2}\boxed{=}\cdots$~~	$\boxed{-}\boxed{2}\boxed{=}\cdots$	~~$\boxed{+}\boxed{3}\boxed{=}\cdots$~~	$\boxed{-}\boxed{3}\boxed{=}\cdots$
~~$\boxed{+}\boxed{4}\boxed{=}\cdots$~~	$\boxed{-}\boxed{4}\boxed{=}\cdots$	~~$\boxed{+}\boxed{5}\boxed{=}\cdots$~~	$\boxed{-}\boxed{5}\boxed{=}\cdots$

THE BLUE STRING

~~IS LARGER THAN~~	IS SMALLER THAN	■	
IS AT LEAST 20 SMALLER THAN	~~IS AT LEAST 20 LARGER THAN~~	~~IS A MULTIPLE OF~~	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~$10\times$~~	~~$\div 10$~~	~~$2\times$~~
~~$\div 2$~~	~~$\frac{2}{3}\times$~~	~~$\frac{3}{2}\times$~~	$\boxed{+}\boxed{=}\cdots$
$\boxed{+}\boxed{2}\boxed{=}\cdots$	~~$\boxed{-}\boxed{2}\boxed{=}\cdots$~~	$\boxed{+}\boxed{3}\boxed{=}\cdots$	~~$\boxed{-}\boxed{3}\boxed{=}\cdots$~~
$\boxed{+}\boxed{4}\boxed{=}\cdots$	~~$\boxed{-}\boxed{4}\boxed{=}\cdots$~~	$\boxed{+}\boxed{5}\boxed{=}\cdots$	~~$\boxed{-}\boxed{5}\boxed{=}\cdots$~~

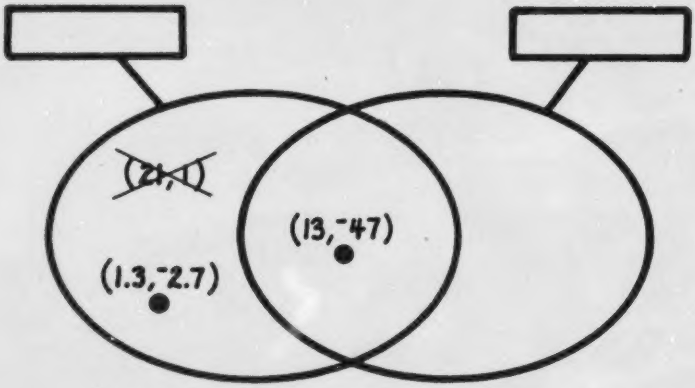
Exercise 2

Play the Relations Game with two strings according to usual rules. Possible choices for the relations are:

the red string --- $\boxed{-}\ \boxed{2}\ \boxed{=}\ldots$

the blue string --- $\boxed{-}\ \boxed{5}\ \boxed{=}\ldots$

Appropriate starting clues are to locate the ordered pairs $(1.3, {}^{-}2.7)$ and $(13, {}^{-}47)$ and to indicate one region in which $(21, 1)$ does not belong.



These clues eliminate thirty-nine of the forty-four possibilities.

THE RED STRING

IS LARGER THAN	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	~~IS AT LEAST 20 LARGER THAN~~	~~IS A MULTIPLE OF~~	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~10×~~	~~÷10~~	~~2×~~
~~÷2~~	~~$\frac{2}{3}$×~~	~~$\frac{3}{2}$×~~	~~$\boxed{+}\boxed{\ }$ ···~~
~~$\boxed{+}\boxed{2}\boxed{=}$ ···~~	$\boxed{-}\boxed{2}\boxed{=}$ ···	~~$\boxed{+}\boxed{3}\boxed{=}$ ···~~	~~$\boxed{-}\boxed{3}\boxed{=}$ ···~~
~~$\boxed{+}\boxed{4}\boxed{=}$ ···~~	$\boxed{-}\boxed{4}\boxed{=}$ ···	~~$\boxed{+}\boxed{5}\boxed{=}$ ···~~	~~$\boxed{-}\boxed{5}\boxed{=}$ ···~~

THE BLUE STRING

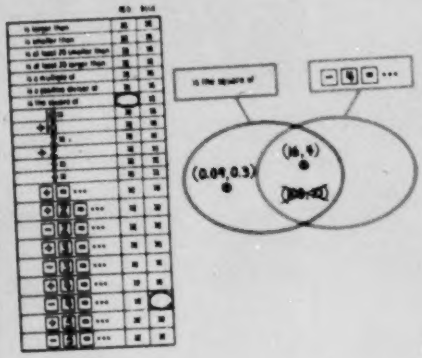
~~IS LARGER THAN~~	~~IS SMALLER THAN~~	■	
~~IS AT LEAST 20 SMALLER THAN~~	IS AT LEAST 20 LARGER THAN	~~IS A MULTIPLE OF~~	~~IS A POSITIVE DIVISOR OF~~
~~IS THE SQUARE OF~~	~~10×~~	~~÷10~~	~~2×~~
~~÷2~~	~~$\frac{2}{3}$×~~	~~$\frac{3}{2}$×~~	~~$\boxed{+}\boxed{\ }$ ···~~
~~$\boxed{+}\boxed{2}\boxed{=}$ ···~~	~~$\boxed{-}\boxed{2}\boxed{=}$ ···~~	~~$\boxed{+}\boxed{3}\boxed{=}$ ···~~	~~$\boxed{-}\boxed{3}\boxed{=}$ ···~~
~~$\boxed{+}\boxed{4}\boxed{=}$ ···~~	~~$\boxed{-}\boxed{4}\boxed{=}$ ···~~	~~$\boxed{+}\boxed{5}\boxed{=}$ ···~~	$\boxed{-}\boxed{5}\boxed{=}$ ···

Worksheets L05 and L06 are available for individual work.

L 05 *

Name ______________________

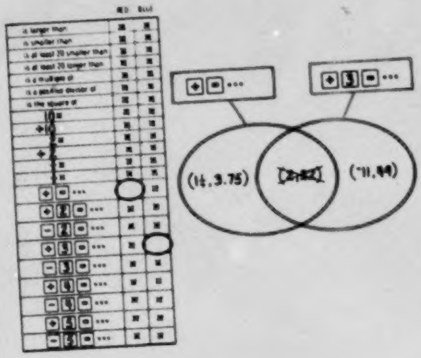
Use the clues in the picture to cross out labels the strings cannot have.
Then label the strings.


is larger than
is smaller than
is at least 20 smaller than
is at least 20 larger than
is a multiple of
is a positive divisor of
is the square of
is the square of
(0.09, 0.3)
(16, 4)

L 06 **

Name ______________________

Use the clues in the picture to cross out labels the strings cannot have.
Then label the strings.


is larger than
is smaller than
is at least 20 smaller than
is at least 20 larger than
is a multiple of
is a positive divisor of
is the square of
(1½, 3.75)
(⁻11, 44)

3

THE STRING GAME

The String Game has an analysis version as do the Table Game and the Relations Game. These analysis versions represent an approach to encouraging strategic thinking in a game that has as its goal the narrowing down of many given possibilities to one. The students' familiarity with the String Game and with the analysis versions of the other two games played this semester allows us to present the analysis version of the String Game with a 2-string picture and then very quickly to play it with even a 3-string picture!

We introduced the String Game with numbers in IG-I and have continued to use the game throughout the Intermediate Grades Program. Intuitive playing developed into strategic thinking through exercises carefully analyzing the kinds of information obtained from plays and starting clues. This semester (in IG-VI) we focus on the analysis involved in using a clue that tells how many numbers belong in one region of a string picture. To use this type of clue, all of the strings must be considered simultaneously; every combination of possible labels must be checked. The burden of this task is lightened greatly by the prior use of other clues that limit the number of possibilities for each string's label and thus limit the number of combinations.

CAPSULE LESSON SUMMARIES

Section 3.1:	Introduce and play the analysis version of the String Game.
Section 3.2:	Play the analysis version of the String Game. Then play the string Game with special scoring rules. (Worksheets L07, L08)
Section 3.3:	Play the analysis version of the String Game but this time with a 3-string picture. Play the String Game with special scoring rules and with a 3-string picture. (Worksheets L09, L10)

3.1 The String Game # 1

CAPSULE LESSON SUMMARY

Introduce and play the analysis version of the String Game.

MATERIALS

Teacher: Colored chalk; two Demonstration (3-string) Analysis Sheets; Numerical String Game kit

Student: String Game Analysis Pad; pencil

NOTE: Fold under or cross out the third column (GREEN) of the Demonstration Analysis Sheets to make them appropriate for 2-string games.

DESCRIPTION OF LESSON

The analysis version of the String Game is similar to the analysis version of the Relations Game. It may be played with two or three strings; however, in this lesson we suggest only 2-string games. To play the analysis version of the String Game proceed as follows.

Divide the class into two teams, Team A and Team B. Prepare your board for the String Game as you would to play the usual version with numbers.

Begin the game by giving several starting clues. Such a clue can indicate the region of the picture where a particular number does belong or a region where it does not, or it can indicate how many numbers belong in a particular region. Ask the class to consider collectively the sixteen labels for each string (listed on the poster) and to decide which of them the strings cannot be for.

Explain to the class how to play the game after the initial analysis.

> PLAY: During a team's turn, a member selects a game piece and places it in the string picture. If the game piece is correctly placed, say "yes" and let the piece remain in the picture. If a game piece is incorrectly placed, record the play with a crossed-out numeral in the picture and return the piece to the team's unplayed

collection. Let the entire class analyze the situation. Although only the team whose turn it is can score, anyone can help with the analysis after a play.

SCORE : A team gains five points each time a member's play eliminates at least one-fourth of the remaining possible labels for the strings ("one-fourth of " computed to the nearest whole number). The team gains an additional two points for each possibility eliminated over the required number to score. If fewer than the required number are eliminated, no points are gained and no points are lost.

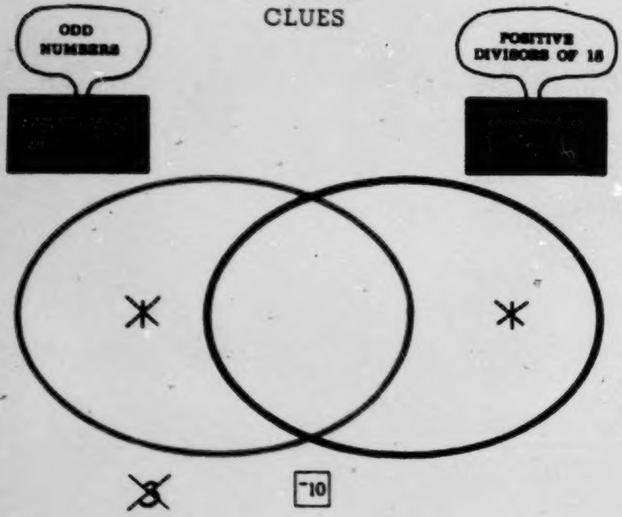
Alternating teams, select students or ask for volunteers to make plays. Before a team makes a play, decide what is the smallest number of possibilities that must be eliminated for the team to score. After each play, compute and record the score. The class can help you with the scoring. Continue until the strings are determined. The team with the most points wins.

Play the analysis version of the String Game with your class. Two games are suggested on the next two pages. String labels, appropriate starting clues, lists showing which labels are eliminated by the starting clues, and a crib-sheet for each game are given.

L 11

Game #1

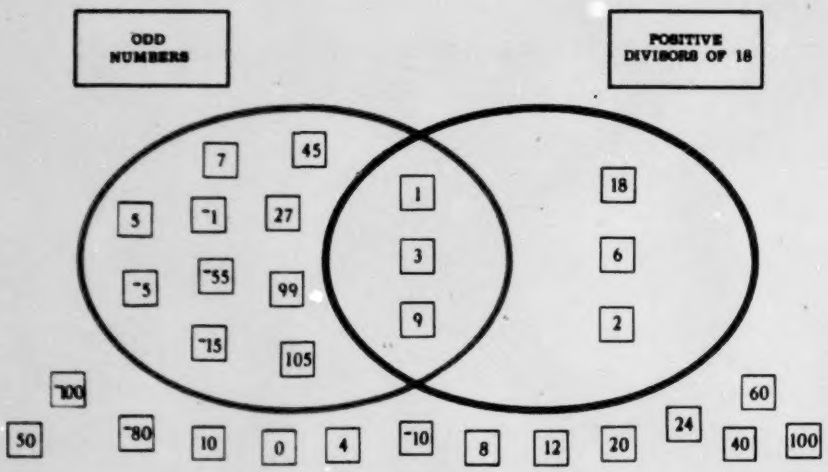
STARTING CLUES



ODD NUMBERS
POSITIVE DIVISORS OF 18
*
*
3
-10

RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
LARGER THAN 50	LARGER THAN 50
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
LARGER THAN -10	LARGER THAN -10
SMALLER THAN -10	SMALLER THAN -10
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27

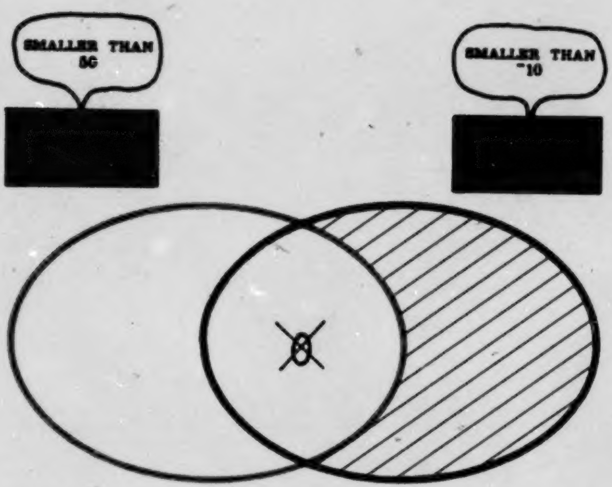
CORRECT PLACEMENT
OF GAME PIECES



ODD NUMBERS
POSITIVE DIVISORS OF 18
7
45
5
-1
27
-5
-55
99
-15
105
1
3
9
18
6
2
-100
60
50
-80
10
0
4
-10
8
12
20
24
40
100

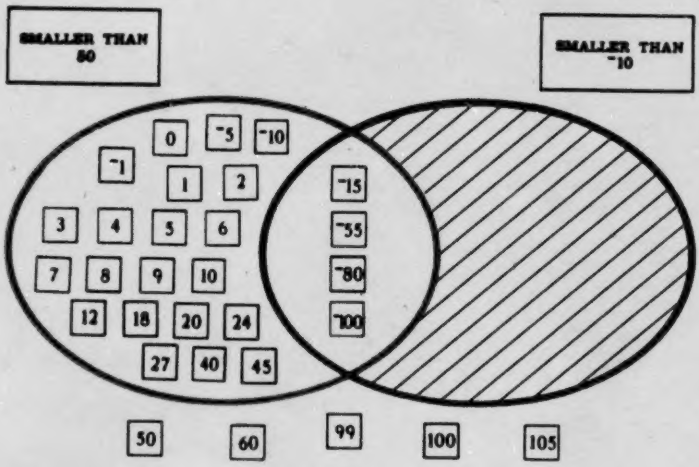
Game #2

STARTING CLUES


SMALLER THAN 50
SMALLER THAN -10
0

RED	BLUE
MULTIPLES OF 2	~~MULTIPLES OF 2~~
MULTIPLES OF 3	~~MULTIPLES OF 3~~
MULTIPLES OF 4	~~MULTIPLES OF 4~~
MULTIPLES OF 5	~~MULTIPLES OF 5~~
MULTIPLES OF 10	~~MULTIPLES OF 10~~
ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
LARGER THAN 50	LARGER THAN 50
SMALLER THAN 50	~~SMALLER THAN 50~~
LARGER THAN -10	~~LARGER THAN -10~~
SMALLER THAN -10	SMALLER THAN -10
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27

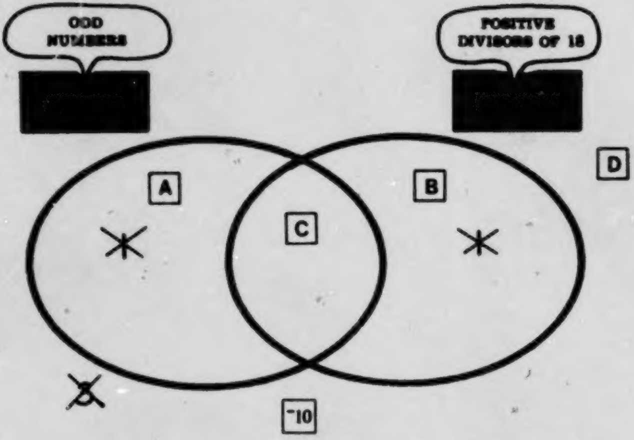
CORRECT PLACEMENT OF GAME PIECES


SMALLER THAN 50
SMALLER THAN -10
0
-5
-10
-1
1
2
3
4
5
6
7
8
9
10
12
18
20
24
27
40
45
-15
-55
-80
-100
50
60
99
100
105

L 11

A brief description of a sample playing of game #1 follows.

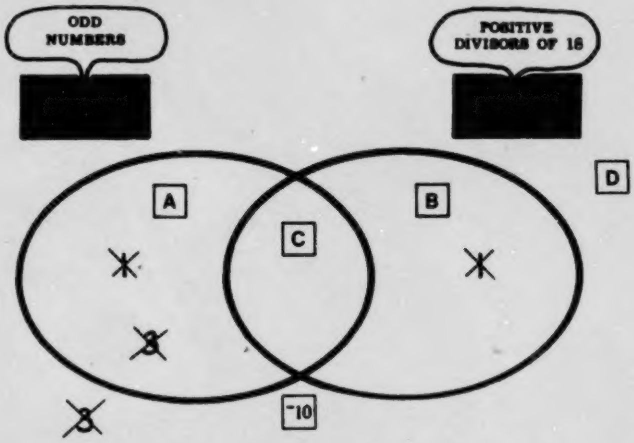
RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
LARGER THAN 50	LARGER THAN 50
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
LARGER THAN $^{-}10$	LARGER THAN $^{-}10$
SMALLER THAN $^{-}10$	SMALLER THAN $^{-}10$
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27



Twenty-four labels remain as possibilities for the strings. One-fourth of 24 is 6. Team A needs to eliminate at least six of the remaining possible labels to score.

A member of Team A puts $\boxed{3}$ in region A. (Incorrect)

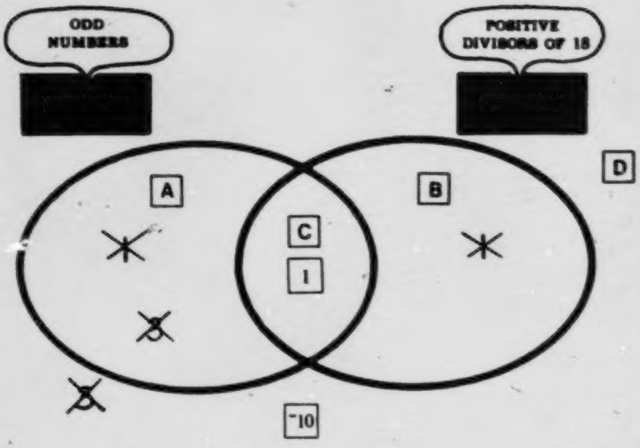
RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
LARGER THAN 50	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
LARGER THAN $^{-}10$	LARGER THAN $^{-}10$
SMALLER THAN $^{-}10$	~~SMALLER THAN $^{-}10$~~
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27



Only four more possibilities are eliminated. Team A does not score. Twenty labels remain as possibilities for the strings. One-fourth of 20 is 5. Team B needs to eliminate at least five of the remaining possible labels to score.

A member of Team B puts 1 in region C. (Correct)

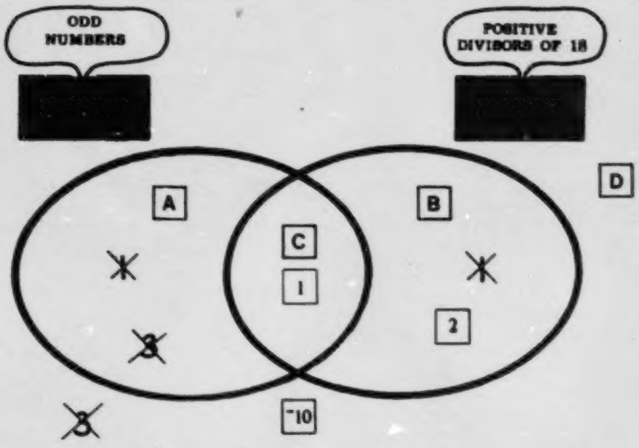
RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	ODD NUMBERS
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
LARGER THAN $^-10$	LARGER THAN $^-10$
~~SMALLER THAN $^-10$~~	~~SMALLER THAN $^-10$~~
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27



Seven more possibilities are eliminated. Team B scores five points plus four bonus points. Thirteen labels remain as possibilities for the strings. One-fourth of 13 to the nearest whole number is 3. Team A needs to eliminate at least three of the remaining possible labels to score.

A member of Team A puts 2 in region B. (Correct)

RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN $^-10$~~	LARGER THAN $^-10$
~~SMALLER THAN $^-10$~~	~~SMALLER THAN $^-10$~~
~~POSITIVE DIVISORS OF 12~~	POSITIVE DIVISORS OF 12
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 18
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~

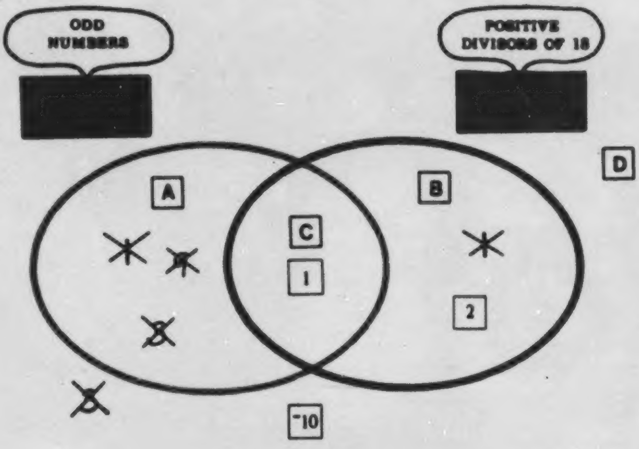


Seven more possibilities are eliminated. Team A scores five points plus eight bonus points. Six labels remain as possibilities for the strings. One-fourth of 6 to the nearest whole number is 2. Team B needs to eliminate at least two of the remaining possible labels to score.

L 11

A member of Team B puts $\boxed{9}$ in region A. (Incorrect)

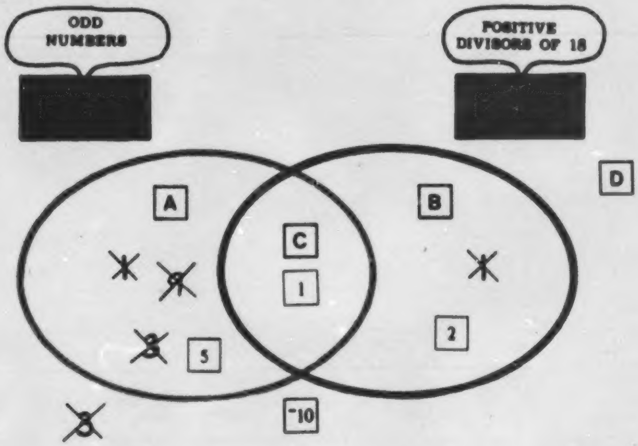
RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN ⁻10~~	LARGER THAN ⁻10
~~SMALLER THAN ⁻10~~	~~SMALLER THAN ⁻10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 18
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~



Two more possibilities are eliminated. Team B scores five points. Four labels remain as possibilities for the strings. One-fourth of 4 is 1. Team A needs to eliminate at least one of the remaining possible labels to score.

A member of Team A puts $\boxed{5}$ in region A. (Correct)

RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN ⁻10~~	~~LARGER THAN ⁻10~~
~~SMALLER THAN ⁻10~~	~~SMALLER THAN ⁻10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 18
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~



Two more possibilities are eliminated. Team A scores five points plus two bonus points. The strings are determined. Team A wins 20 to 14.

3.2 The String Game #2

CAPSULE LESSON SUMMARY

Play the analysis version of the String Game. Then play the String Game with special scoring rules.

MATERIALS

Teacher: Colored chalk; two Demonstration (3-string) Analysis Sheets; Numerical String Game kit

Student: String Game Analysis Pad; pencil; Worksheets L07 and L08

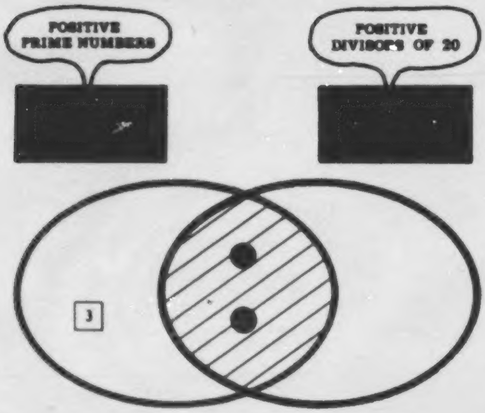
NOTE: Fold under or cross out the third column (GREEN) of the Demonstration Analysis Sheets to make them appropriate for 2-string pictures.

DESCRIPTION OF LESSON

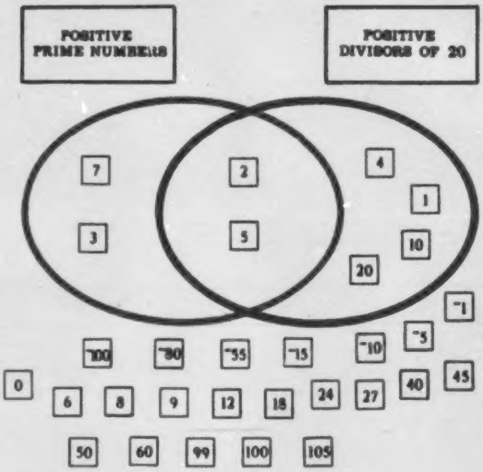
Exercise 1: The String Game, Analysis Version

Play the analysis version of the String Game. A possible game is suggested below.

STARTING CLUES


POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 20
3

CORRECT PLACEMENT OF GAME PIECES


POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 20
7
3
2
5
4
1
10
20
⁻1
⁻5
⁻100
⁻80
⁻55
⁻15
⁻10
0
6
8
9
12
18
24
27
40
45
50
60
99
100
105

L 12

To analyze the starting situation, first consider the location of 3 in the picture and then use the information that there are only two numbers in the intersection of the strings. Three types of argument contribute to the analysis of the starting situation. See the lists on the next page for successive eliminations.

1) Eliminate possible labels for the strings using the information that 3 belongs to the red string but not to the blue string.

2) For each of the remaining possibilities for the blue string, check that there is at least one of the remaining possibilities for the red string which it could be matched with to give exactly two numbers in the intersection. If there is not, cross that label out on the BLUE list. These labels can be crossed out from the BLUE list using this type of argument.

Multiples of 2
Multiples of 5
Multiples of 10
Larger than 50
Smaller than $^-10$

3) For each of the remaining possibilities for the red string, check that there is at least one of the remaining possibilities for the blue string which it could be matched with to give exactly two numbers in the intersection. If there is not, cross that label out on the Red list. These labels can be crossed out from the RED list using this type of argument.

Multiples of 3
Smaller than 50
Larger than $^-10$
Positive Divisors of 24
Positive Divisors of 27

The lists after each of these three stages of analysis are shown here.

RED	BLUE
~~MULTIPLES OF 2~~	MULTIPLES OF 2
MULTIPLES OF 3	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	MULTIPLES OF 4
~~MULTIPLES OF 5~~	MULTIPLES OF 5
~~MULTIPLES OF 10~~	MULTIPLES OF 10
ODD NUMBERS	~~ODD NUMBERS~~
POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	LARGER THAN 50
SMALLER THAN 50	~~SMALLER THAN 50~~
LARGER THAN ¯10	~~LARGER THAN ¯10~~
~~SMALLER THAN ¯10~~	SMALLER THAN ¯10
POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~

→

RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
MULTIPLES OF 3	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~
SMALLER THAN 50	~~SMALLER THAN 50~~
LARGER THAN ¯10	~~LARGER THAN ¯10~~
~~SMALLER THAN ¯10~~	~~SMALLER THAN ¯10~~
POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~

→

RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN ¯10~~	~~LARGER THAN ¯10~~
~~SMALLER THAN ¯10~~	~~SMALLER THAN ¯10~~
POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

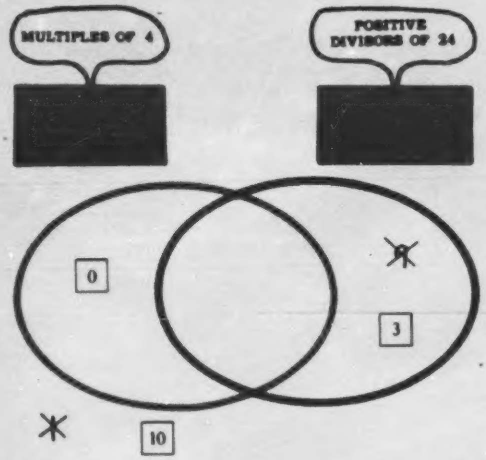
NOTE: Do not be concerned if your class does not reduce the lists to six possibilities: four for the red and two for the blue. The game simply proceeds with whichever labels are remaining on the lists.

L 12

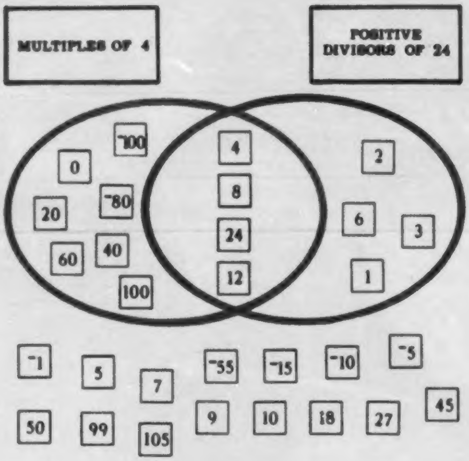
Exercise 2: The String Game with Special Scoring Rules

Play Version E of the String Game (see pages 128-130). You may need to review the rules for your class. A possible game is suggested below.

STARTING CLUES


MULTIPLES OF 4
POSITIVE DIVISORS OF 24
0
3
10

CORRECT PLACEMENT OF GAME PIECES

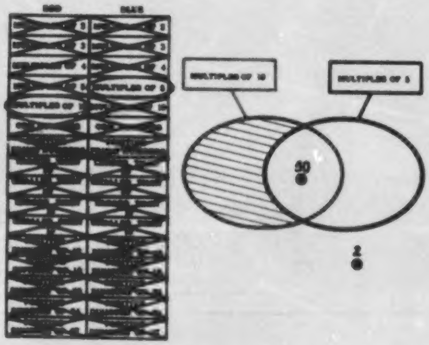

MULTIPLES OF 4
POSITIVE DIVISORS OF 24
⁻100
0
⁻80
20
40
60
100
4
8
24
12
2
6
3
1
⁻1
5
7
⁻55
⁻15
⁻10
⁻5
50
99
105
9
10
18
27
45

Worksheets L07 and L08 are available for individual work.

Name ____________________

L 07 *

Use the clues in the picture to cross out labels the strings cannot have.
The two strings have different labels. Label the strings.

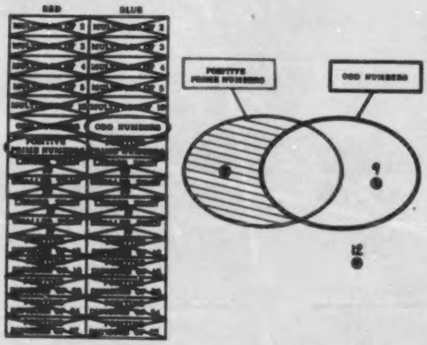


RED
BLUE
MULTIPLES OF 5
MULTIPLES OF
MULTIPLES OF 10
MULTIPLES OF 5
30
2

Name ____________________

L 08 **

Use the clues in the picture to cross out labels the strings cannot have.
Then label the strings.



RED
BLUE
ODD NUMBERS
POSITIVE PRIME NUMBERS
POSITIVE PRIME NUMBERS
ODD NUMBERS
9
12

L 13

3.3 The String Game #3

CAPSULE LESSON SUMMARY

Play the analysis version of the String Game but this time with a 3-string picture. Play the String Game with special scoring rules and with a 3-string picture.

MATERIALS

Teacher: Colored chalk; two Demonstration (3-string) Analysis Sheets; Numerical String Game kit

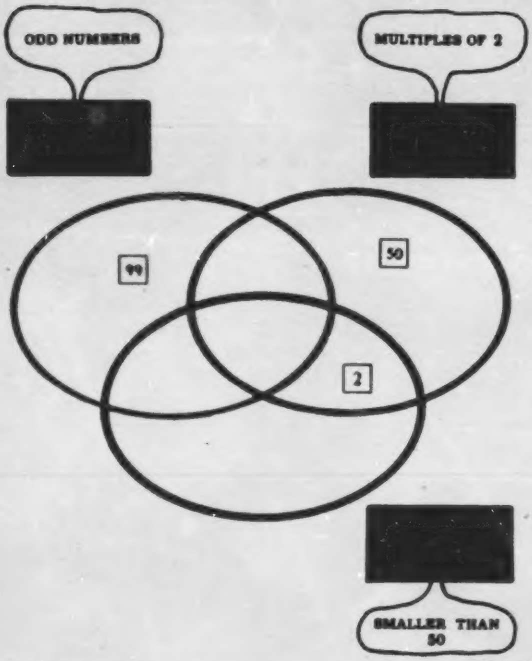
Student : String Game Analysis Pad; pencil; Worksheets L09 and L10

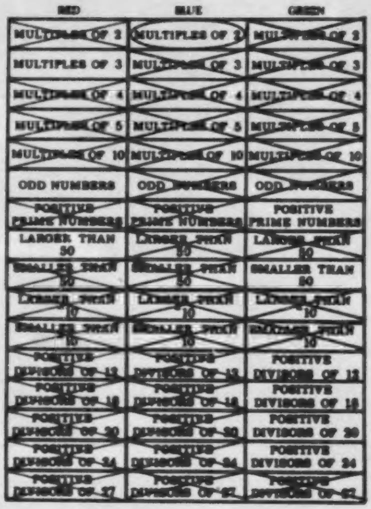
DESCRIPTION OF LESSON

Exercise 1: The String Game, Analysis Version

Play the analysis version of the String Game with three strings. A game is suggested below on the left. The labels which are eliminated by the starting clues are crossed off the lists on the right. A crib-sheet is provided on the next page.

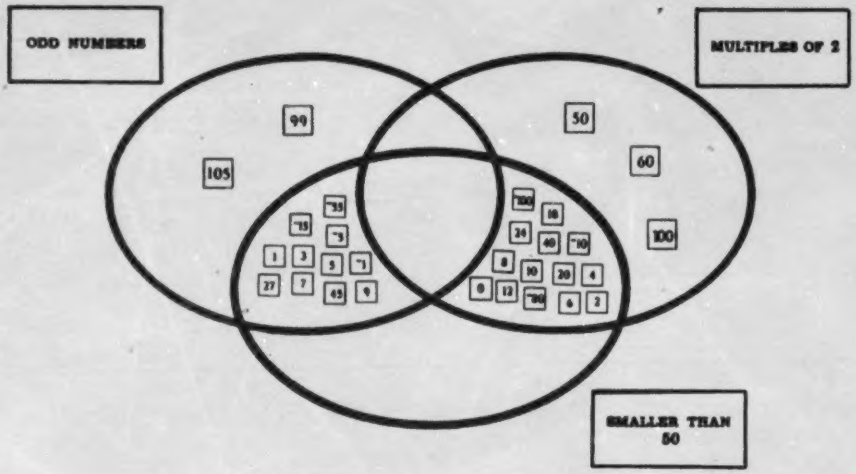
STARTING CLUES





RED	BLUE	GREEN
~~MULTIPLES OF 2~~	MULTIPLES OF 2	~~MULTIPLES OF 2~~
MULTIPLES OF 3	~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~	POSITIVE PRIME NUMBERS
LARGER THAN 50	~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~	SMALLER THAN 50
~~LARGER THAN 10~~	~~LARGER THAN 10~~	~~LARGER THAN 10~~
~~SMALLER THAN 10~~	~~SMALLER THAN 10~~	~~SMALLER THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~	POSITIVE DIVISORS OF 12
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 18
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 24
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

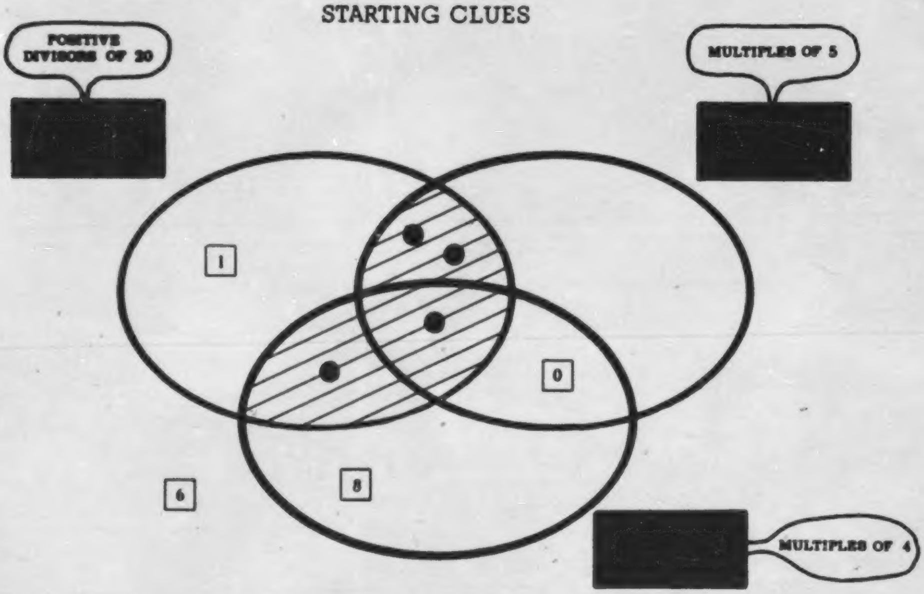
CORRECT PLACEMENT
OF GAME PIECES


ODD NUMBERS
MULTIPLES OF 2
SMALLER THAN 50
99
105
50
60
100
-35
-15
-3
1
3
5
-1
27
7
45
9
-100
18
24
40
-10
8
10
20
4
0
12
-30
6
2

Exercise 2: The String Game with Special Scoring Rules

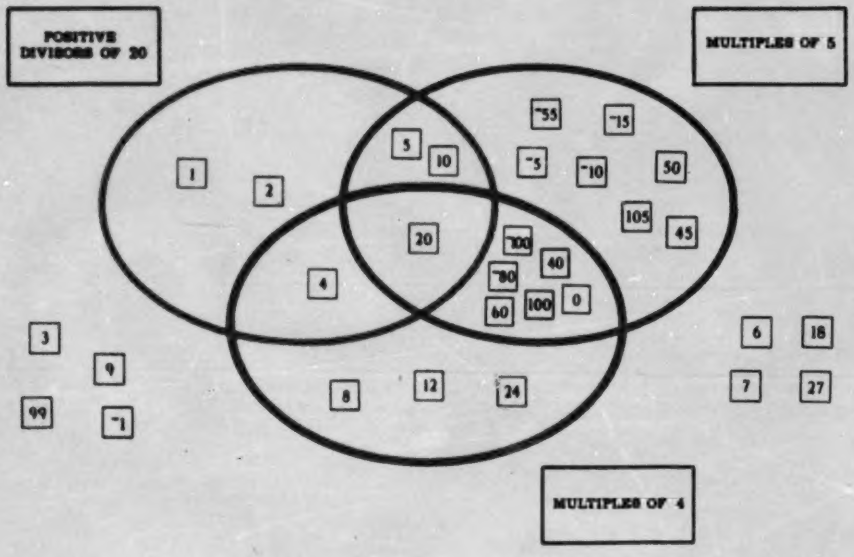
Play Version E of the String Game (see pages 128-130). A game is suggested below. The strings are determined by the starting clues. A crib-sheet is provided on the next page.

STARTING CLUES


POSITIVE DIVISORS OF 20
MULTIPLES OF 5
MULTIPLES OF 4
1
0
6
8

L 13

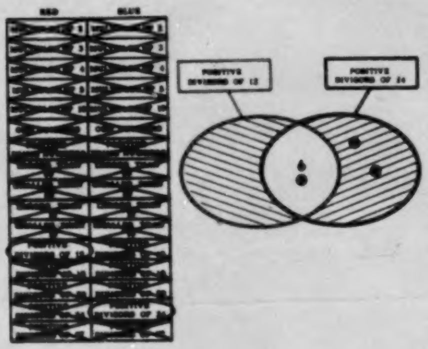
CORRECT PLACEMENT OF GAME PIECES


POSITIVE DIVISORS OF 20
MULTIPLES OF 5
MULTIPLES OF 4
1
2
5
10
20
4
⁻55
⁻15
⁻5
⁻10
50
105
45
⁻100
40
⁻80
0
60
100
3
9
99
⁻1
8
12
24
6
18
7
27

Worksheets L09 and L10 are available for individual work or for collective analysis. You may wish to discuss what the hatching implies on Worksheet L09, namely that the two sets are identical except for one element. Both worksheets are difficult.

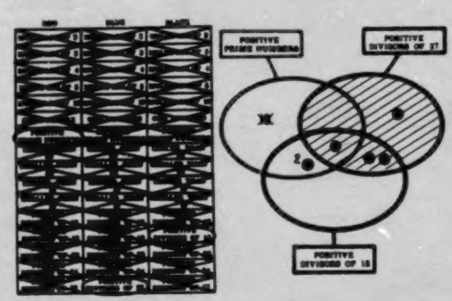
Name ______________________ **L 09** *

Use the clues in the picture to cross out labels the strings cannot have.
Then label the strings.


RED
BLUE
POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 24

Name ______________________ **L 10** **

Use the clues in the picture to cross out labels the strings cannot have.
Then label the strings.


RED
BLUE
BLACK
POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 27
2
POSITIVE DIVISORS OF 18

Hint: How many numbers belong in the blue string?

APPENDIX: The String Game

The String Game is introduced in Part I (Grade Two) of the CSMP Upper Primary Grades Program and is used in many versions throughout the CSMP Elementary School Program. It gives students an opportunity, in a game-like atmosphere, to become familiar with the language of strings, while at the same time it involves them in the kind of reasoning that will be developed and reinforced in various contexts throughout the program.

The String Game is first played using A-Blocks, but later on variations which make use of numbers make their appearance. In this appendix we provide you with the necessary information and examples to enable you to play the game using A-Blocks or using numbers, and we begin the process of detailing the various possibilities open to you when you play the game. The different variations of the game described in this appendix are referred to in several of the lessons in this chapter. However, we suggest that you do not restrict your playing of the String Game just to the few lessons in this semester when it has been scheduled. It has been found that is is most beneficial (and enjoyable) for students if you make a regular practice of playing the game about once every two weeks, or whenever you have an extra 10-15 minutes during the course of the day.

Equipment

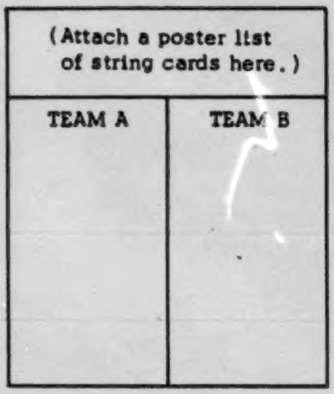
PLAYING BOARD

The equipment can be most easily managed during the game if you have a magnetic[†] chalkboard available. Many of the permanently mounted chalkboards in classrooms are magnetic, and you should test yours using one of the demonstration Minicomputer checkers. If your permanent chalkboards are not magnetic, try any portable chalkboard (grid board, etc.) that the school has available. If you do not have a magnetic chalkboard available, you can use your regular chalkboard.

[†] Strictly speaking the chalkboard itself is not magnetic, but rather magnet-sensitive; i.e., magnets adhere to it. However, for simplicity we call such boards "magnetic."

GAME BOARD

The Game Board is divided into three regions all illustrated below.



a) Magnetic: If you do locate a magnetic chalkboard, you will also need a magnetic or metallic Game Board. With a large magnetic chalkboard, say at least three chalkboard squares, the Game Board could be drawn on the board. However, if you have a relatively small (portable) magnetic chalkboard, then you will need to obtain a sheet of metal (minimum size: 60 cm by 80 cm) or to locate a convenient metallic surface in the classroom such as the side of a file cabinet on which to put the Game Board. In this case draw the markings for the three regions of the Game Board on a large sheet of (easel pad) paper and tape this paper to your metallic surface.

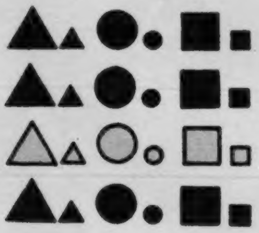
b) Non-magnetic: If you do not have a magnetic chalkboard available for the playing board, then your Game Board could be a large piece of poster board (minimum size: 60 cm by 80 cm). Make the markings for the three regions of the Game Board with a heavy felt-tip pen.

GAME PIECES and STRING CARDS

One set of game pieces and two sets of string cards are needed for each version of the game. Tape a poster list of the string cards to the upper section of the Game Board — it is a constant reminder during the game of the possible labels for the strings.

Game Pieces | **String Cards**

String Game with A-Blocks



RED	YELLOW	GREEN	BLUE
NOT RED	NOT YELLOW	NOT GREEN	NOT BLUE
○	△	□	BIG
NOT ○	NOT △	NOT □	LITTLE

String Game with Numbers: Multiples and Order

$^{-}100$	$^{-}80$	$^{-}55$	$^{-}20$	$^{-}15$	$^{-}12$
$^{-}6$	$^{-}5$	$^{-}2$	$^{-}1$	0	1
7	10	15	24	27	30
40	45	60	99	100	105

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5	MULTIPLES OF 10
ODD NUMBERS	LARGER THAN 30	SMALLER THAN 50	LARGER THAN $^{-}10$	SMALLER THAN $^{-}10$

String Game with Numbers: Multiples, Divisors, and Order

$^{-}100$	$^{-}80$	$^{-}55$	$^{-}15$	$^{-}10$	$^{-}5$
$^{-}1$	0	1	2	3	4
5	6	7	8	9	10
12	18	20	24	27	40
45	50	60	99	100	105

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

Game pieces and string cards for the String Game with Numbers, all versions, can be found in the Numerical String Game Kit. You will have to note which number cards and string cards to use with the particular version you are playing.

a) Magnetic: The game pieces (A-Blocks or number cards) can be magnetized by sticking to the back of each one a small piece of magnetic material. (Magnetic material is included in the Numerical String Game Kit, and is also available in many hobby and notions departments of stores.) To each string card, stick a small piece of magnetic material to the front of the card, taking care not to obscure what is written thereon.

b) Non-magnetic: Game pieces and string cards can be attached to the Game Board by means of loops of masking tape stuck to the backs. A string card should have a loop of masking tape stuck to the front so that what is written on the card is not obscured. With this type of equipment, you should prepare yourself for running repairs by having sufficient masking tape available so that if a loop of tape loses its stickiness, it can be replaced on the spot. An alternative is to use a small wad of a plastic caulking compound (Rope Caulk or Mortite, for example) in place of the loop of masking tape. This material will maintain its stickiness over a much longer period of time.

Preparation for the Game

Draw two (or three, depending on which variation you are using) large, overlapping strings on the playing board using two (or three) different colors. Next to each of these strings attach one string card face-down. Stand the Game Board on the chalk tray, or have it near and convenient to the playing board. Share the game pieces between the two sections of the Game Board labeled "TEAM A" and "TEAM B" by randomly placing one-half of the pieces in one section and the remainder in the other section. Divide the class into two teams using whatever method is acceptable to your class; e.g., boys vs. girls; rows 1, 2, 3 vs. rows 4, 5, 6; etc. Call one team "Team A" and the other "Team B". Before any student takes a turn, correctly place an equal number (at least one) of each team's game pieces in the string picture. This removes from the first players the necessity of operating on the basis of pure guesswork. You can influence how long the game will take by the number of pieces you place on the game board before the game begins.

Object of the Game

Each team tries to place all of its game pieces correctly (according to the face-down string cards) in the string picture. The winning team is the one which is first to place all of its game pieces correctly and which identifies the face-down cards correctly, after playing according to the rules.

Rules of the Game

1) The game is played in silence. This is to avoid one student telling another where to play a certain game piece, or what the strings are for. Each student should have the opportunity to analyze the game alone. Infringement of this rule by anyone is penalized by the talker's team losing its next turn.

2) The teams alternate, and the members take turns within each team. A player comes to the board and selects a piece from his or her team's collection to place in one of the regions of the string picture.

3) You are the judge. If the piece is correctly placed, say "yes." The piece remains in the string picture and the player immediately has a second turn (no player may have more than two consecutive turns). If the piece is incorrectly placed, say "no." The player returns the piece to the team's unplayed collection and play passes to the other team.

   As an aid in judging, prepare a "crib-sheet" showing the correct position of each game piece or at least reminding you of what is on the face-down cards. If at any time you discover that you have made an error, say so immediately and rectify the mistake. Either move an incorrectly placed piece whose position you had approved to its correct region or replace in the string picture a correctly placed piece that had been rejected because you had disapproved its position.

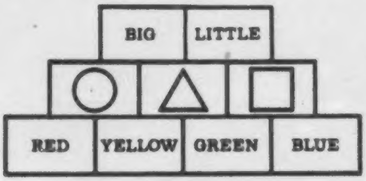
4) When a team has correctly placed all of its pieces, the player who placed the last piece may thereupon attempt to identify each of the string cards.

If these are all correct, the team has won. If a mistake is made (even if it is only in the case of one of the string cards), simply indicate that the identification is incorrect and let the game continue.

5) If a team has exhausted its stock of game pieces, and the strings have not been identified, that team continues on its turn to attempt to identify the strings, while the other team works to place its game pieces.

Version A: The String Game with A-Blocks

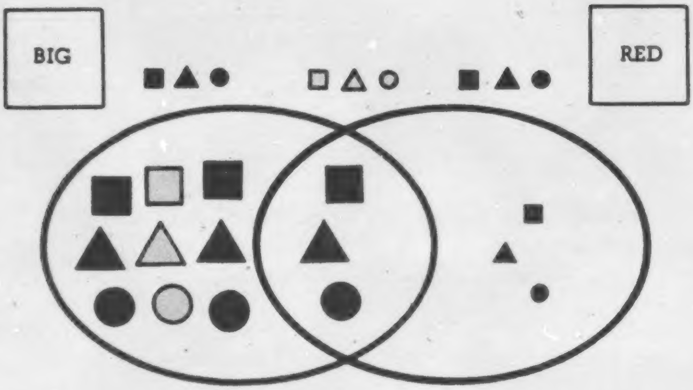
This simplest version of the game uses twenty-four A-Blocks as game pieces and only nine string cards.



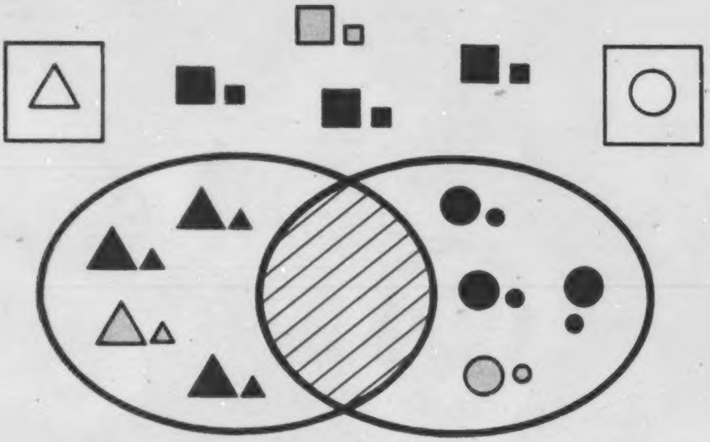
We provide here several example "crib-sheets" for variations of the game with two and with three strings.

Variation #1: Two Strings

Example 1: No Empty Regions

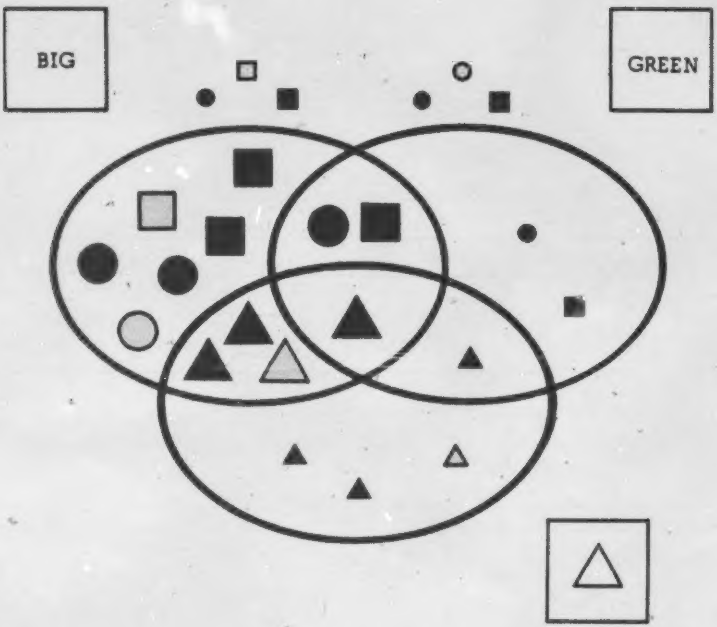


Example 2: One Empty Region

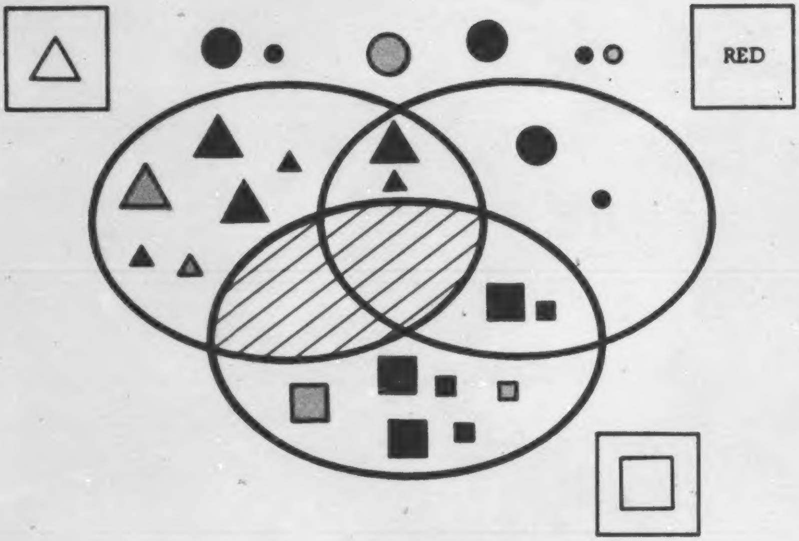


Variation #2: Three Strings

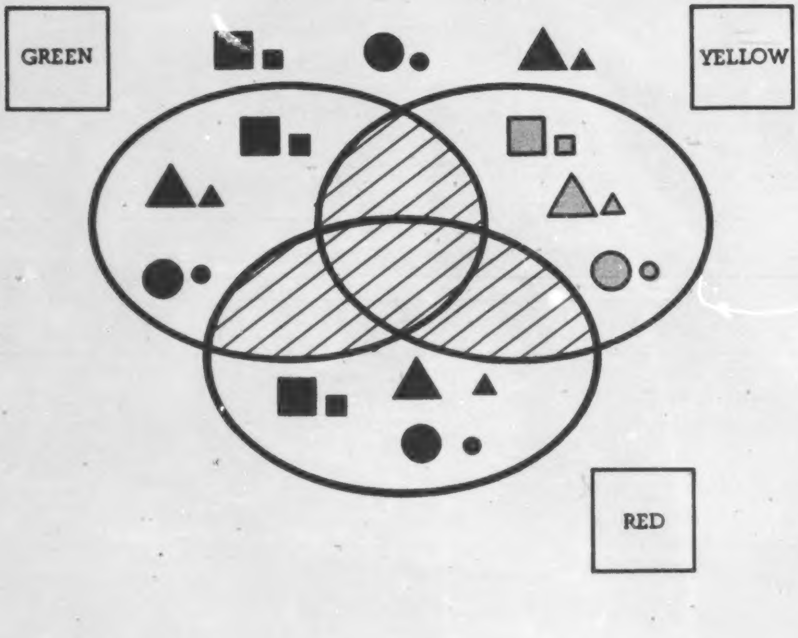
Example 1: No Empty Regions



Example 2: Two Empty Regions



Example 3: Four Empty Regions


GREEN
YELLOW
RED

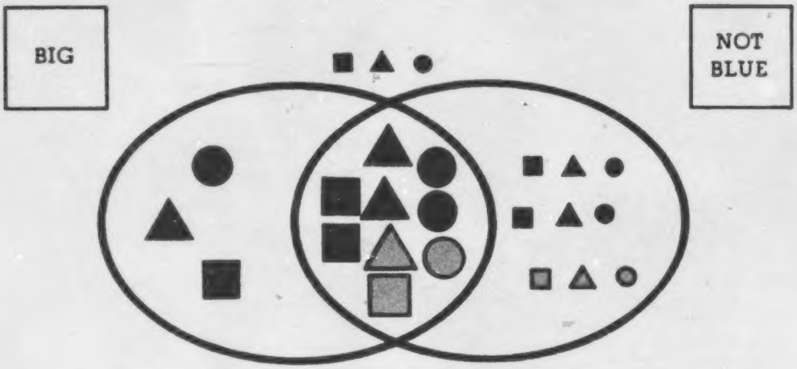
Version B: The String Game with A-Blocks, using "Not" Cards

A more complicated version of the game with A-Blocks uses all sixteen of the string cards. (See the description under "Equipment" of game pieces and the string cards needed for the game with A-Blocks.)

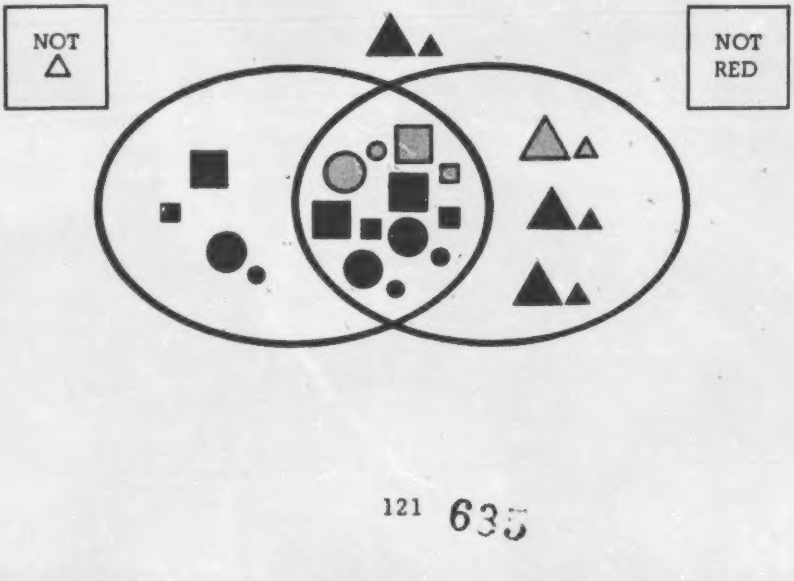
NOTE: Since "NOT BIG" and "LITTLE" describe exactly the same A-Blocks (and similarly for "NOT LITTLE" and "BIG"), such a "not" card is redundant. If a student questions why we do not include these possibilities, let the class discuss that it is unnecessary.

We provide here several example "crib-sheets" for this version of the String Game played with two strings.

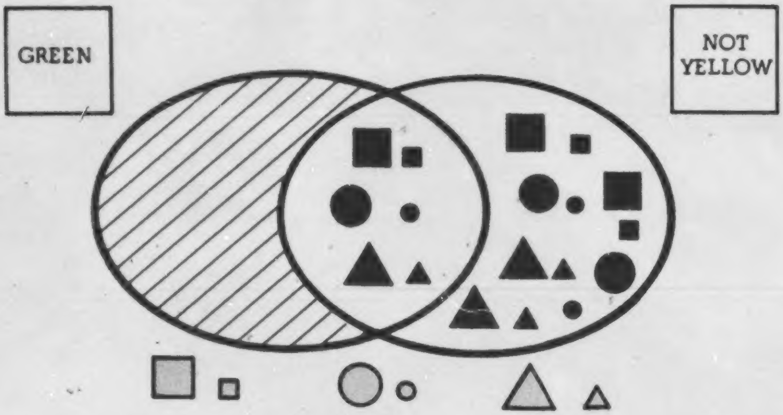
Example 1: No Empty Regions



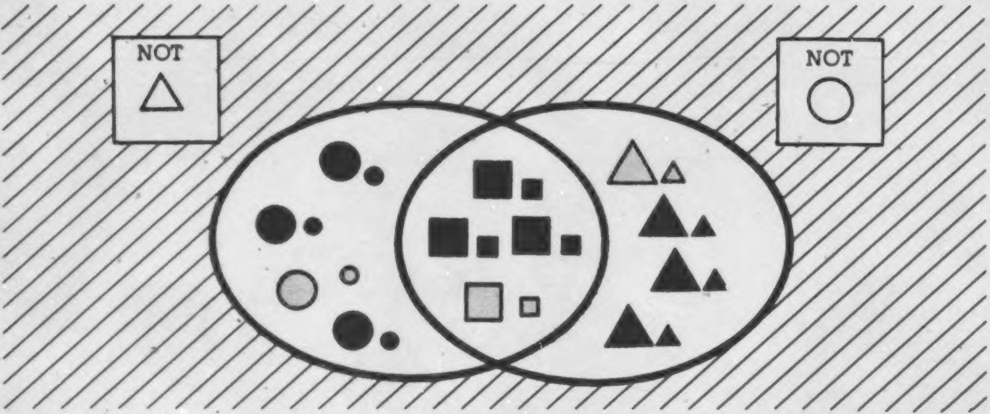
Example 2: No Empty Regions



Example 3 : One Empty Region



Example 4 : One Empty Region



Of course, by hatching the "outside" region of this diagram we mean to indicate that no game pieces can be placed there correctly. Strictly speaking, that region is not empty because, for example, the number 50 is in the outside region.

You should be warned that the "NOT" version of the game played with three strings is almost impossible to judge without a "crib-sheet" and is correspondingly difficult to play. Hence you would be well-advised to avoid it until such time as you think the two-string version is no longer challenging enough for the majority of your students.

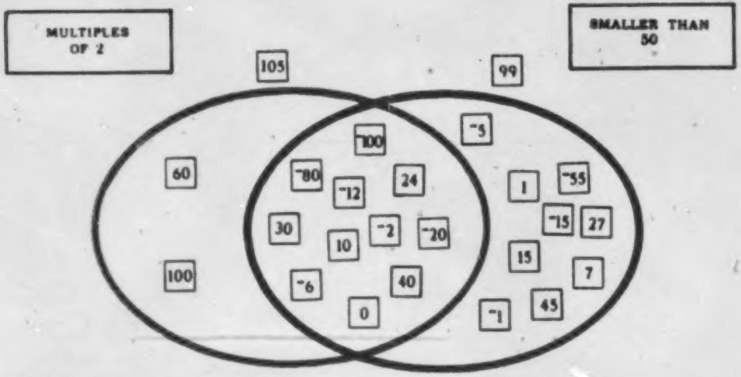
Version C: The String Game with Numbers: Multiples and Order

This version of the game uses twenty-four number cards as game pieces and ten string cards as described under "Equipment". The equipment for this game is included in the Numerical String Game Kit. Note that in the Kit three sets of string cards are included. This allows you to give the same label to more than one string.

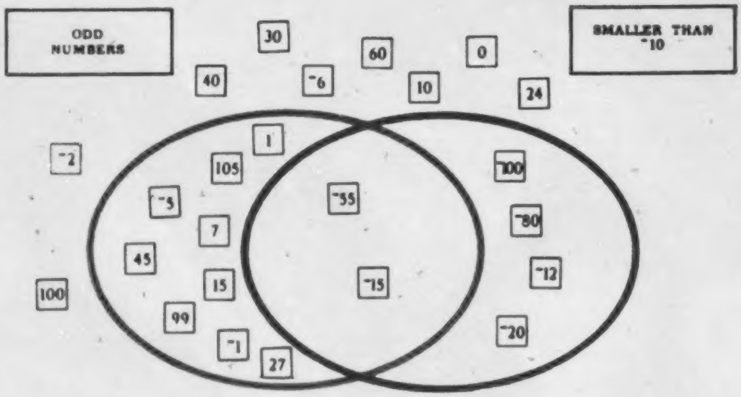
We provide here several example "crib-sheets" for variations of this game using two and three strings.

Variation #1: Two Strings

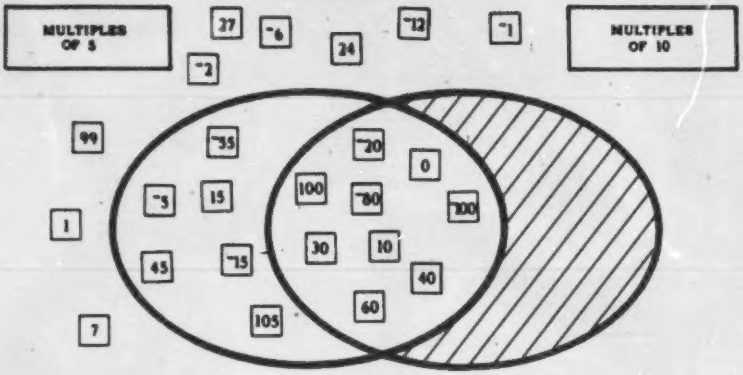
Example 1: No Empty Regions


MULTIPLES OF 2
SMALLER THAN 50
105
99
-5
-100
60
-80
-12
24
1
-55
30
10
-2
-20
-15
27
15
100
-6
40
7
0
-1
45

Example 2: No Empty Regions


ODD NUMBERS
SMALLER THAN -10
30
60
0
40
-6
10
24
1
-2
105
-100
-5
-55
7
-80
45
-12
-15
15
100
99
-20
-1
27

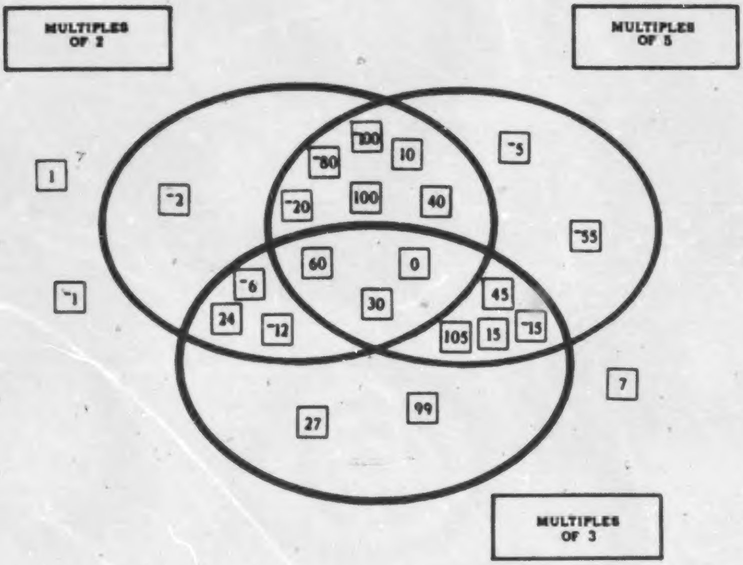
Example 3 : One Empty Region


MULTIPLES OF 5
MULTIPLES OF 10
27
-6
24
-12
-1
-2
99
-35
-20
0
-5
15
100
-80
-100
1
45
-15
30
10
40
60
105
7

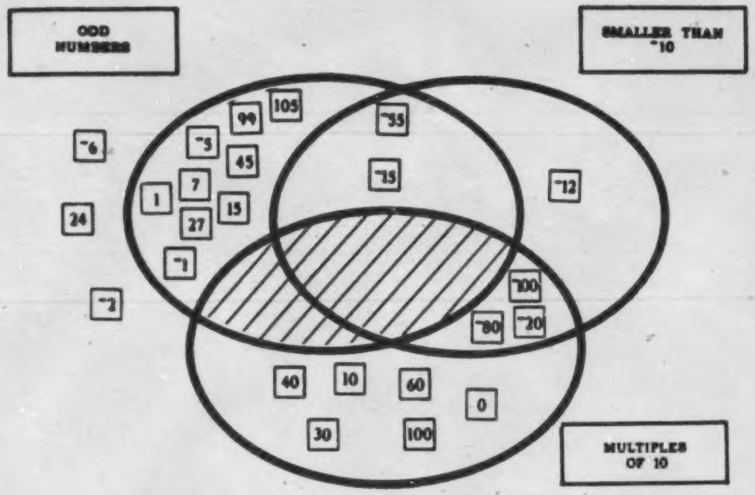
Variation #2 : Three Strings

CAUTION : The jump in difficulty from the two-string variation to the three-string variation is much greater in the case of Version C of the String Game than it does in the case of Version A. Hence you should not expect to proceed to Variation #2 in this case as rapidly as you might have done with Version A.

Example 1 : No Empty Regions


MULTIPLES OF 2
MULTIPLES OF 5
1
-2
-100
10
-80
-20
100
40
-5
-35
-1
60
0
-6
24
-12
30
45
105
15
-15
7
27
99
MULTIPLES OF 3

Example 2: Two Empty Regions

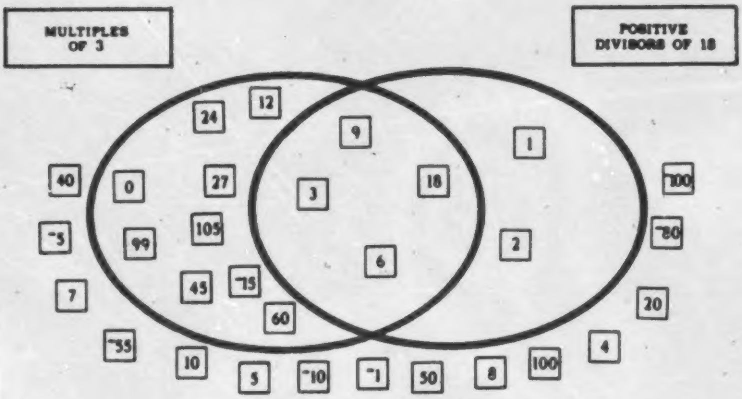

ODD NUMBERS
SMALLER THAN ¯10
MULTIPLES OF 10
¯6
24
¯2
99
105
¯5
45
7
1
27
15
¯1
¯35
¯15
¯12
¯100
¯80
¯20
40
10
60
0
30
100

Version D: The String Game with Numbers: Multiples, Divisors, and Order

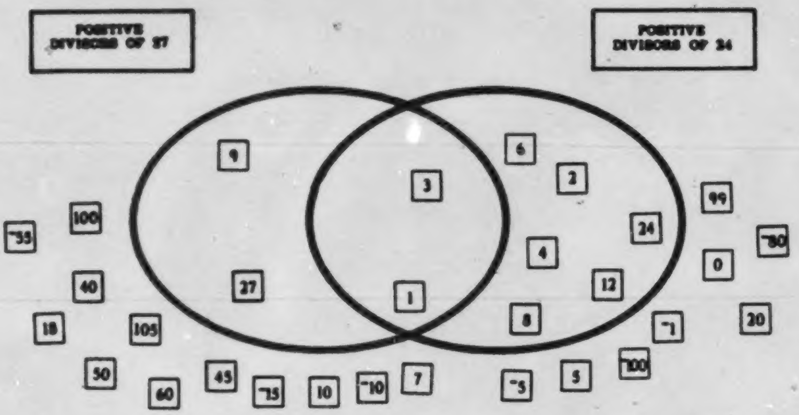
This version of the game uses thrity number cards as game pieces and sixteen string cards as described under "Equipment". The equipment for this game is included in the Numerical String Game Kit. Note that in the Kit three sets of string cards are included. This allows you to give the same label to more than one string. Two examples of this are shown in "crib-sheet" examples provided here; however, before using such a game the players should have previously understood that this was a possibility.

Variation #1: Two Strings

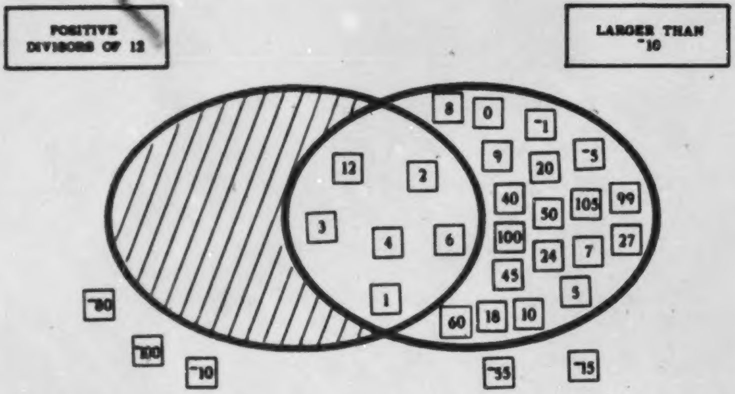
Example 1: No Empty Regions


MULTIPLES OF 3
POSITIVE DIVISORS OF 18
24
12
9
1
40
0
27
3
18
¯100
¯5
99
105
6
2
¯80
7
45
¯15
60
20
¯35
10
5
¯10
¯1
50
8
100
4

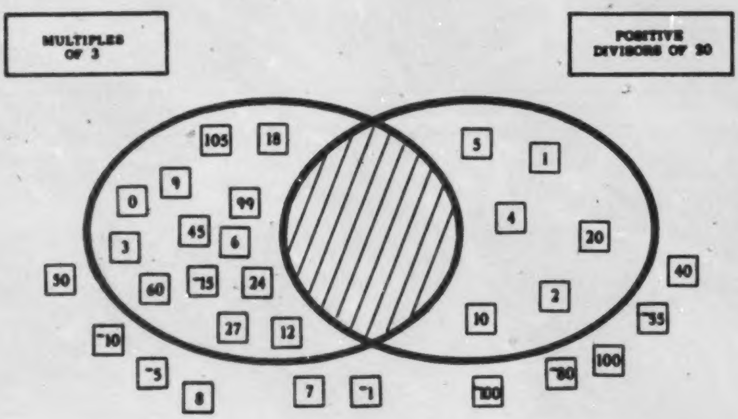
Example 2: No Empty Regions


POSITIVE DIVISORS OF 27
POSITIVE DIVISORS OF 24
9
3
6
2
24
4
12
27
1
8
100
-35
40
18
105
50
60
45
-15
10
-10
7
-5
5
-100
-1
20
99
-80
0

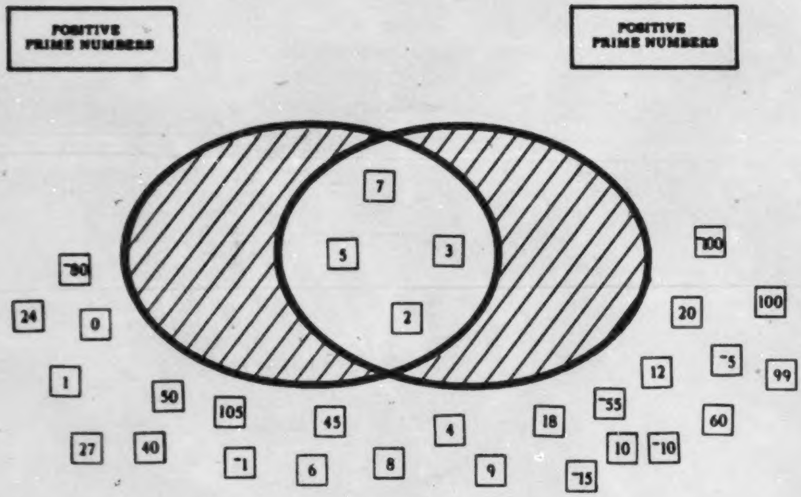
Example 3: One Empty Region


POSITIVE DIVISORS OF 12
LARGER THAN -10
8
0
-1
9
20
-5
12
2
40
105
99
3
50
4
6
100
24
7
27
45
5
1
60
18
10
-80
-100
-10
-35
-15

Example 4: One Empty Region


MULTIPLES OF 3
POSITIVE DIVISORS OF 20
105
18
5
1
9
0
99
4
20
45
3
6
50
60
-15
24
40
2
10
-35
27
12
-10
-5
8
7
-1
-100
-80
100

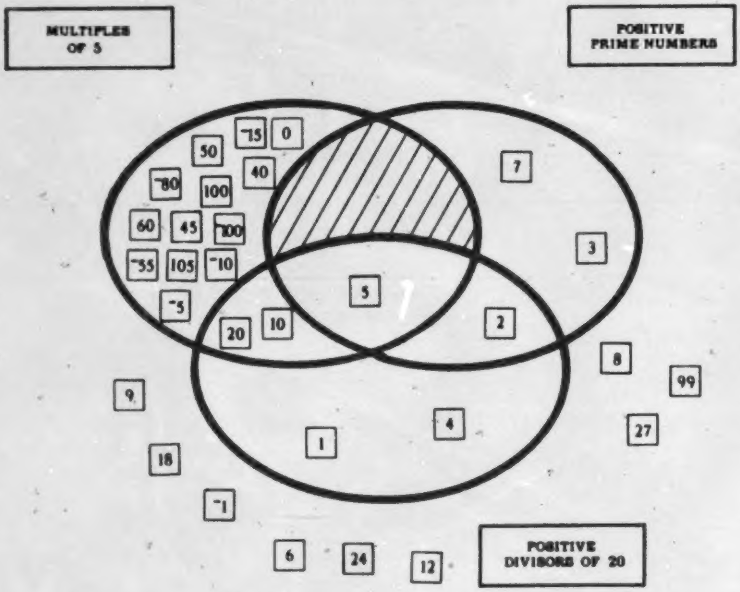
Example 5: Two Empty Regions


POSITIVE PRIME NUMBERS
POSITIVE PRIME NUMBERS
7
5
3
2
-80
24
0
1
50
105
45
4
18
-55
27
40
-1
6
8
9
-15
10
-10
12
20
-100
100
-5
99
60

Variation #2: Three Strings

The warning concerning the jump from the two-string variation to the three-string variation on page 124 applies even more so to Version D.

Example 1: One Empty Region


MULTIPLES OF 5
POSITIVE PRIME NUMBERS
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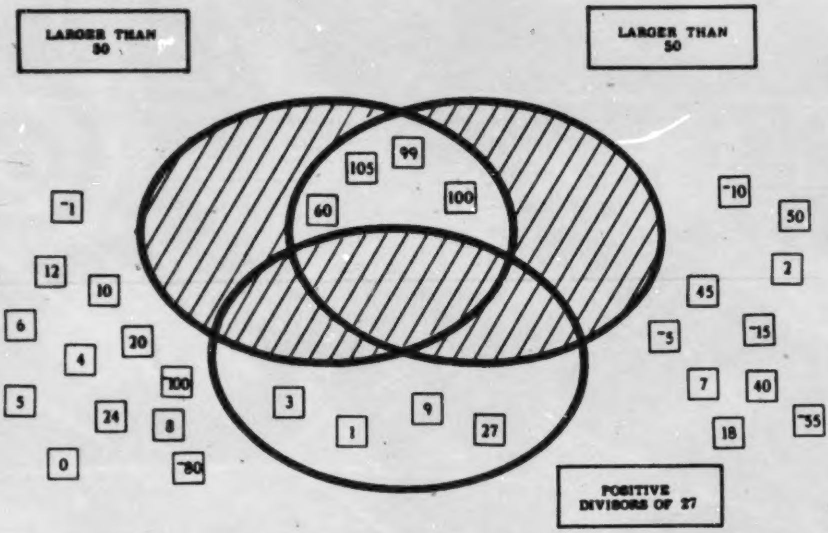
0
50
40
-80
100
60
45
-100
-55
105
-10
-5
20
10
5
7
3
2
8
99
27
9
4
1
18
-1
6
24
12
POSITIVE DIVISORS OF 20

Example 2: Five Empty Regions



Version E: The String Game with Special Scoring Rules

In this version of the game, we introduce special scoring rules to determine the winning team. Both the String Game with numbers and with shapes can be played with these scoring rules; however, in the lessons of this book only the Numerical String Game is played this way.

Scoring

Each student is given a list of the possible string labels on a sheet from the individual String Game Analysis Pad. One round of the game is completed by having one player from each team take a turn. After each round the students have an opportunity to privately identify the face-down string cards. They do this by circling on their RED and BLUE lists what they believe is on the hidden tags and submitting it to the judge (teacher). However, each student has just one chance to make a private identification. If a player correctly identifies the string cards after a particular round, then points (according to the following schedule) are added to the player's team score.

Round 1 - 200 points
Round 2 - 150 points
Round 3 - 100 points
Round 4 - 75 points
Round 5 - 50 points
Round 6 - 25 points

After Round 6, play out the game without further private identifications. Additional points cannot be scored by a team until the end of the game. The team that correctly identifies the string cards at the end of the game scores an additional 100 points plus 5 points for each unplayed game piece the opposing team has left on the Game Board. The winning team is the one with the greater total score.

Comments

1) Allow about two to three minutes before the start of Round 1 for students to analyze the starting clues (correctly placed game pieces).

2) Keep count of the rounds played so that both you and the students know which round has just been completed.

3) After each round give the students time (perhaps 15-30 seconds) to study the picture, to mark their lists, and if desired to submit them.

4) When the students submit their lists, write on each one the round after which it was turned in and sort them by teams. Do not return them to the students. You can total the team scores at the end of the game. Do not tell the students that their identifications are correct or incorrect, but encourage them to write their findings on a separate sheet of paper for recollection when they play. They should continue to check their identifications during the remainder of the game.

5) It may be even more important to insist on silence during this kind of game because an individual can have more influence on a team's winning or losing. We suggest you be strict about penalties for talking during the game.

Purpose

Scoring rules should not be imposed on the String Game until students have become very familiar and experienced in the play of the game. Many students already will be playing strategically, analyzing the clues given by correctly played game pieces. Others may still be playing in a random way. The introduction of scoring rules obviously will favor the better players and perhaps challenge them in a way the usual game is no longer doing. It provides them the opportunity to more dramatically affect their team's chances of winning. For many students the scoring rules may provide incentive to begin analyzing the game and thinking about what information is given by a correctly played game piece.

We do not suggest that you play the String Game very often with these scoring rules. They do impose complication and give perhaps too much advantage to the better players. After playing the game with scoring rules a couple of times, you should find that many students have become significantly better players. Returning to the usual way of playing the game, most students will continue to play more thoughtfully.

One advantage of using scoring rules when you play the String Game with only a small amount of time available is that you can stop the game before the end and have a reasonable way to determine the winning team. The winning team would be the one with the larger total number of points from identification of string cards.

GEOMETRY
AND
MEASUREMENT

Comprehensive School Mathematics Program

CSMP Mathematics for the Intermediate Grades, Part VI

Geometry and Measurement

CEMREL, Inc.

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SUGGESTED CHRONOLOGICAL ORDER OF G-LESSONS

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GEOMETRY AND MEASUREMENT

INTRODUCTION

Geometry is useful and interesting, an integral part of mathematics. Geometric thinking permeates mathematics and relates to most of the applications of mathematics. Geometric concepts provide a vehicle for exploring the rich connections between arithmetic concepts (such as numbers and calculations) and physical concepts (such as length, area, and shape). Even as children experience arithmetic notions in their daily lives, they encounter geometric notions. Common opinion recognizes that each child should begin developing a number sense early. There is no sound argument to postpone the development of geometric sense.

A benefit of early development of geometric thinking is that it provides alternative opportunities for "slow learners." Children develop number concepts at individual rates. Relative lateness of a child's numerical development often signifies a difficulty in remembering number facts. A common response to this problem is assignment of arithmetic drill, which often serves to confirm the diagnosis of slowness. However, numerical development may not be an accurate indication of other development, in particular, of geometric development. Often the labels "slow" and "fast" are incorrect when alternative activities are presented.

The lessons of this strand provide an opportunity to explore geometric notions informally. The activities are varied to meet other areas of the program and are diffused over a long period of time. As a natural consequence of their involvement in the activities, the students will develop knowledge and skills.

But the effects of such an informal program must be measured by their appearance in subsequent contexts. No goals for mastery can be posted here; none could be certified. The focus of this strand is experience.

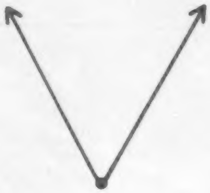
1

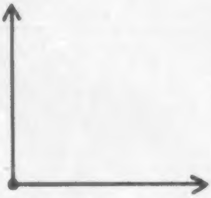
GEOMETRY OF ANGLES

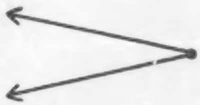
The geometry of CSMP Mathematics for the Intermediate Grades is based on a variety of constructions. Students use tools to explore geometric concepts, experiencing directly their properties and interrelationships. Angles and their measurement are the central ideas of the geometry strand for this semester. Our approach to angles is in keeping with an established pattern: provide tools with which to study a series of constructions.

Many treatments of angles simultaneously introduce the idea of angle and the measurement of an angle with a protractor. With this approach students often identify an angle with its measure. We take the position that the two should be realized separately, just as we learn about line segments long before measuring them with a ruler. Ten of these lessons are devoted to providing students an opportunity to explore angles and some of their properties through constructions before angle measurement is introduced in the last two lessons of the chapter.

The first lessons introduce the idea of an angle as two rays which have a common endpoint and which are not part of the same line.







In the discussion students observe angles implicit in physical objects and notice the prevalence of right angles, defined simply through a paper folding construction (see page 16). In the process, an informal comparison of angles with right angles leads naturally to the idea of acute and obtuse angles.

Using only a right angle template to determine acute and obtuse angles, the students attempt to construct polygons with various numbers of sides and of acute angles. Through this exercise, they focus on angles as controlling elements in the structure of a polygon, whereas their previous experience is likely to have emphasized the line segments which appear as its sides.

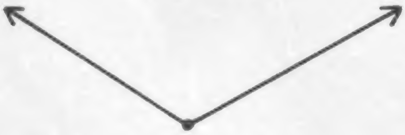
Given two supplementary pairs of angles, the students attempt to construct quadrilaterals under the condition that each angle of the quadrilateral is congruent to one of the four given angles and vice versa. They discover the several solutions and note that two pairs (the supplementary pairs, of course) together can be used to construct a quadrilateral, but no other pairing will work. This informal introduction to supplementary pairs of angles is carried through three more lessons. In the lessons students use eight angle templates, four supplementary pairs, to perform a series of constructions designed to emphasize supplementary pairs. Through the constructions, the students develop an appreciation of basic facts of Euclidean geometry. For example, three angles of a quadrilateral determine the fourth.

Triangle constructions are more tightly controlled since they admit fewer choices. Working with seven angle templates, students discover all triangles with each angle congruent to one of the seven. Performing the constructions, students are alert for patterns among the successes and the failures and discover a relation similar to the idea of supplementary pairs of angles. Thus they discover the classical result that, in terms of measurement, the sum of the measures of the angles of a triangle is 180°.

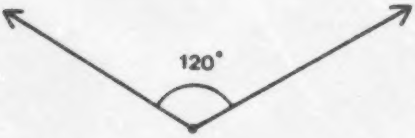
Using templates in this series of constructions provides an opportunity to develop an understanding of angles without the necessity to develop skill with a protractor. Experience in constructing polygons and a growing sense of angle relations, which are independent of angle measures, are preliminary to learning how to use a protractor. Its use is foreshadowed by the use of the angle templates, and the idea of angle measurement is a natural extension of the informal comparison of angles in the earlier lessons. From experience with the templates comes an appreciation of the ease, efficiency, and versatility of the protractor as a tool for constructing angles. A reprise of some of the earlier problems recouched in terms of measurement concludes the chapter.

NOTE ON ANGLE MEASURE

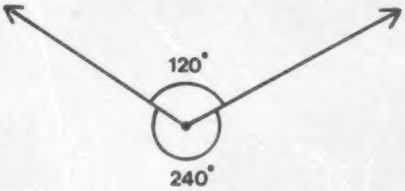
It is important to understand that only one measure between 0° and 180° is associated with an angle in these lessons. This may be a difficult viewpoint for those who are accustomed to always associating two measures with an angle. For example, given this angle . . .



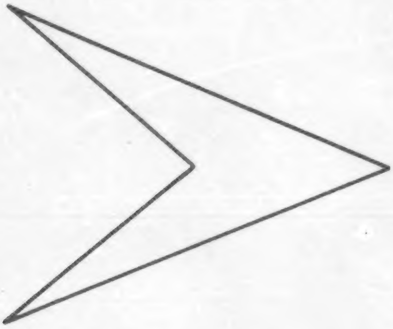
we assign one measure, namely 120°.



rather than

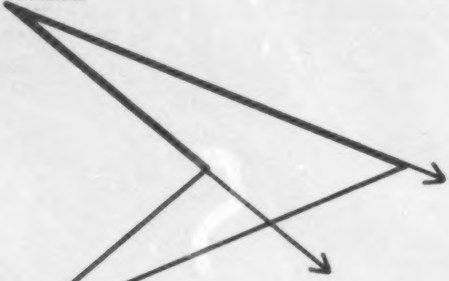


Now suppose we ask: What are the angles determined by this polygon?

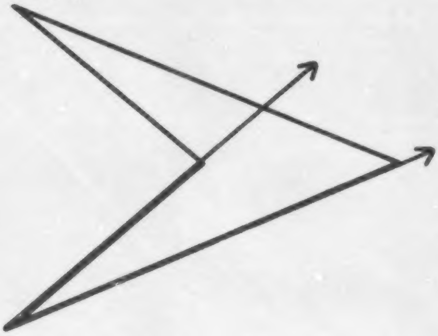


To answer, we recall that an angle consists of two rays with a common endpoint and are emphatic in the following identifications which result from taking the four corners in turn as vertices.

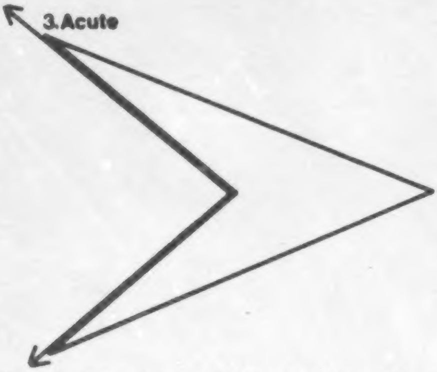
1. Acute



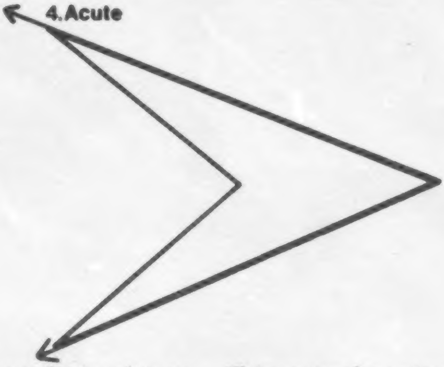
2.Acute



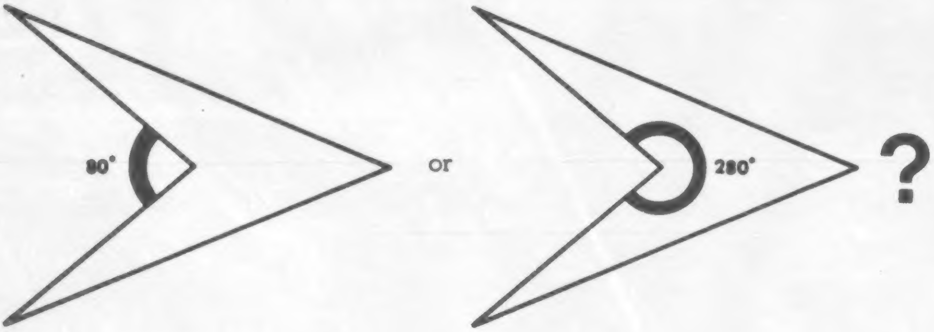
3.Acute



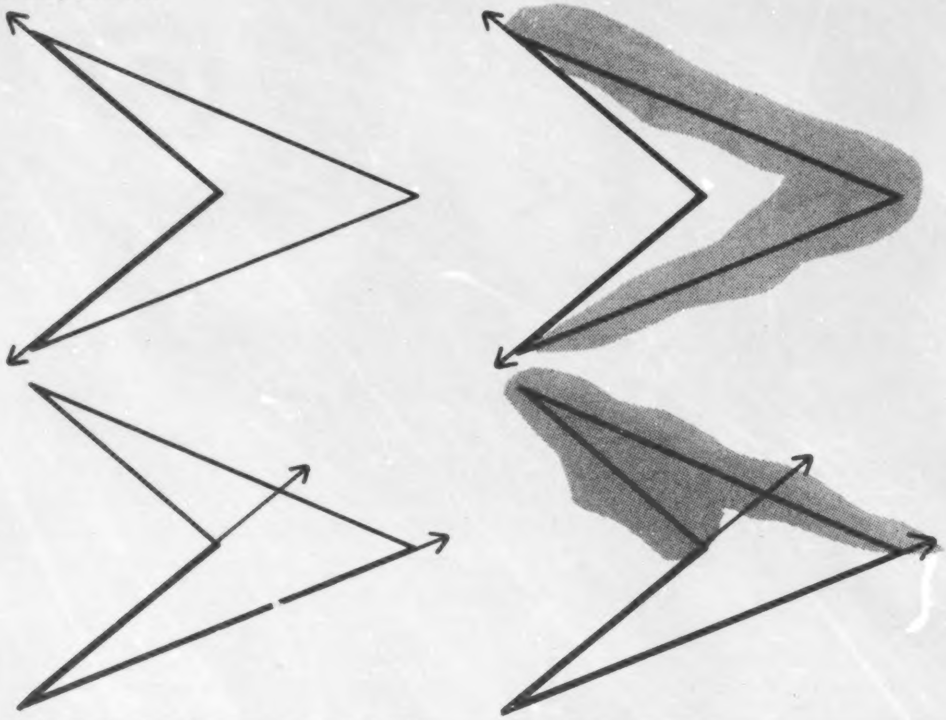
4.Acute



However, some might say that the third angle is obtuse. This apprehension results from the dual measurement system.



To assist in breaking through this prejudice, cover the extraneous sides of the polygon momentarily or trace the pertinent sides. Then examine the angle in isolation. The angles determined by a polygon can be viewed independently of the polygon. If necessary, extract them from the polygon for inspection.



CAPSULE LESSON SUMMARIES

Section 1.1:	Introduce angles and practice recognizing the angles implicit in various geometric objects. Define a right angle and compare it with other angles to classify them as acute or obtuse. (Worksheets G01, G02)
Section 1.2:	Construct polygons with a prescribed number of sides and acute angles.
Section 1.3:	Discuss the charts from Section 1.2. Discuss indentations in shapes and then look for solutions with no indentations to fill the cells of the chart.

Section 1.4: Construct quadrilaterals using four given angles. (Worksheet G03)

Section 1.5: Use two angle templates to construct polygons with three, four, five, and six sides.

Section 1.6: Use four angle templates to construct quadrilaterals. Recognize the special pairs which give parallelograms and trapezoids.

Section 1.7: Explore supplementary pairs of angles. Compare the template angles according to an intuitive idea of size. (Worksheet G04)

Section 1.8: Use seven angle templates to construct triangles.

Section 1.9: Complete a collection of triangles constructed with seven angle templates. Explore the relationship among the triples of angles which yield triangles.

Section 1.10: Explore the relationship among the three angles of a triangle. Extend the experimentation to the four angles of a quadrilateral.

Section 1.11: Introduce the protractor and its use in measuring angles and constructing angles of given measures. Discover that the sum of the measures of the angles of a triangle is 180°.
(Worksheets G05, G06)

Section 1.12: Discover relations among the measures of angles of polygons without indentations.

1.1 Introduction to Angles

CAPSULE LESSON SUMMARY

Introduce angles and practice recognizing the angles implicit in various geometric objects. Define a right angle and compare it with other angles to classify them as acute or obtuse.

MATERIALS

Teacher:	Straightedge; colored chalk; paper for a square corner
Student:	Pencil; straightedge; paper for a square corner; Worksheets G01 and G02

DESCRIPTION OF LESSON

Begin with a discussion to review the ideas of line and line segment and to introduce angles and related concepts. Maintaining the distinction between line and line segment is worthwhile for students in organizing their understanding of the geometry. The dialogue which follows occurred in one our development classes.

Draw a line segment on the board, emphasizing the endpoints with small dots.

T: What do we call this?

S: A line segment.

T: What is it related to?

S: A line.

T: Tell us about a line.

G1

S: It is straight.

S: It doesn't matter how long it is, but it doesn't curve.

S: It goes on and on.

S: It can be vertical or horizontal or diagonal.

S: A wiggly line is not straight.

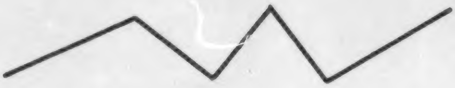
S: How about a zigzag?

S: A zigzag is a bunch of lines.

S: A zigzag is a bunch of line segments connected to each other.

S: But they must be connected at the endpoints.

Draw a zigzag on the board.



S: But that isn't a line or a line segment.

S: A curve isn't a line either.

S: A line is a straight segment that can be long as you like.

T: . . . a kilometer long?

S: Maybe.

S: No, a line just keeps going on and on.

S: A line segment has a beginning and an ending.

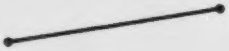
S: That's right, and a line has no beginning and no ending.

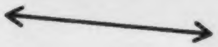
Your class should conclude that a line is straight and has no endpoints.

T (pointing to the line segment on the board): Can you indicate a line that is closely related to this line segment?

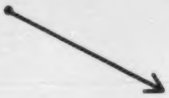
A volunteer should indicate with gestures the extension of the line segment in both directions without end.

T: Since we cannot draw all of a line on the board, we draw part of it and then use arrowheads to indicate its continuation.





T: How many endpoints does a line segment have? (Two)
And how many does a line have? (None)
What about something that has only one endpoint, like this?



T: Is it a line segment?

S: No, it has only one endpoint.

T: Is it a line?

G 1

S: No, it keeps going, but in only one direction.

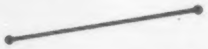
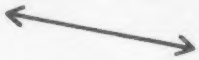
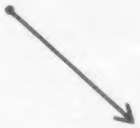
S: It has one endpoint and a line has none.

T: Think about a flashlight. When you turn it on, the light beams out in only one direction.

S: Light also comes from the sun.

S: Sunbeams — sunlight — sunrays.

T: We call this a ray. Now we have three ideas: line segment, line, and ray. And we have a way to show each in a drawing.

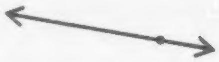




Draw another line on the board.

T: I am thinking of a ray which is part of this line. What do you need to know to find it?

S: Where is its endpoint?

Mark a point on the line.



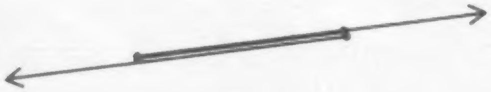
S: You could be thinking of either of two different rays.

Invite students to indicate the two rays which are determined by the chosen point. Then repeat the activity with one or two other points until the class understands that a line can be broken at any point into two different rays. For each pair, emphasize the fact that the two rays are different even though they have the same endpoint.

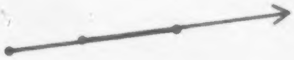
Draw a line segment on the board.

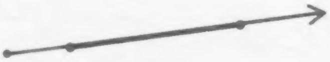
T: Who can use this line segment to draw a ray?

Invite students to extend the segment in either direction to produce one of the two rays which it most naturally determines. They are indicated here in red and in blue.



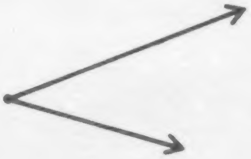
NOTE: There are several other ways in which a student might use the line segment to describe a ray, such as those shown below. However, the two shown above are those which are important for these lessons. If a student suggests another, affirm its correctness and continue the discussion until the principal two are discovered. If no alternatives are suggested, there is no need to interject them.





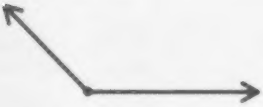
G1

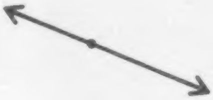
Now draw another pair of rays, this time with the same endpoint, different directions, but not on the same line.

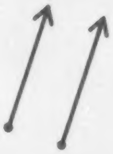


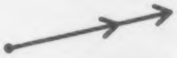
Again ask students to indicate the rays by tracing them. Emphasize, if necessary, that there are two rays explicit in the drawing.

Repeat the exercise with several other configurations.

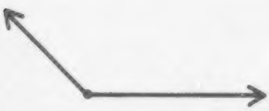




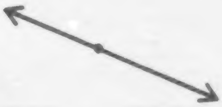




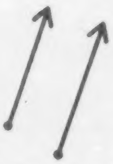
The only case likely to cause confusion is the last, since students may think that is shows two different rays. However, consistently tracing a ray from its endpoint in the direction of the arrow will aid the realization that although there might appear to be two rays because there are two arrowheads, they are in fact one and the same ray. The location of the arrowhead or the length of the line segment used in the drawing is not important. Only the endpoint and the direction of the arrow is important in defining the ray.



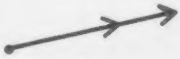
two rays



two rays

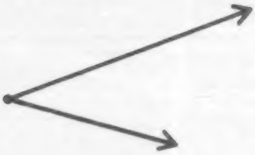


two rays



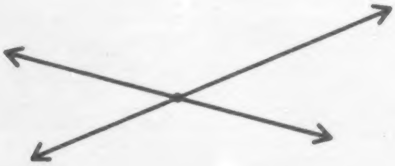
one ray

T: Let's look at these two rays again.



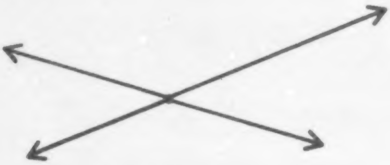
T: We call this an *angle*. An *angle* is two rays with the same endpoint and different directions, so long as they are not both part of the same line. The common endpoint of an angle's two rays is called the *vertex* of the angle.

Draw this picture of several rays.



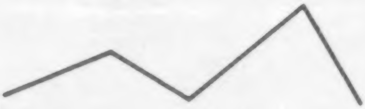
T: Who can show us an angle in this picture?

Let students indicate the four angles and their common vertex. You might use colored chalk, a different color for each ray. Then it will be easy to talk about the rays and the angles.



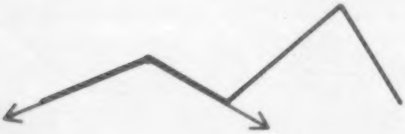
G1

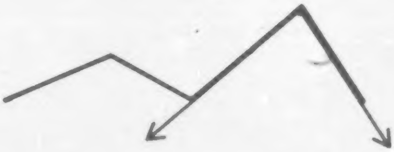
Draw a zigzag with several segments.

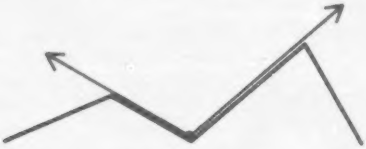


T: Who can show us an angle that uses segments of this zigzag?

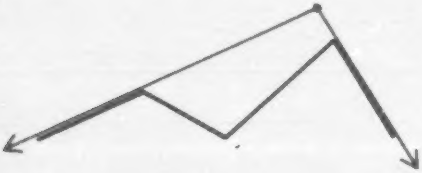
Let students indicate angles by putting dots at the corners of the zigzag and by extending the segments of the zigzag to define rays.

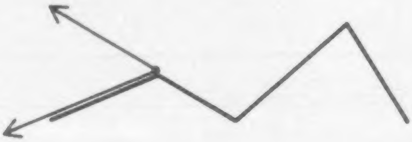




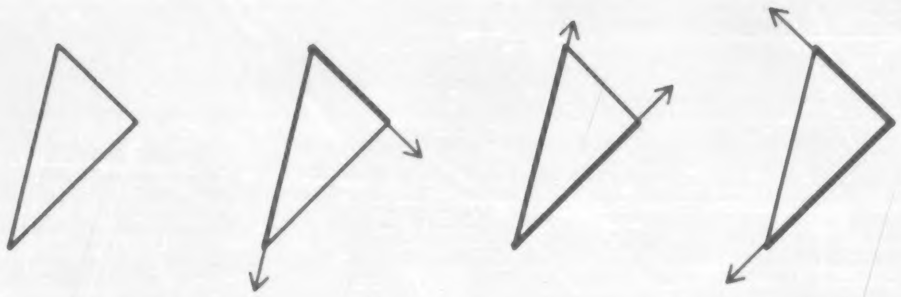


NOTE: The angles shown are the ones which arise most naturally from the zigzag. Several other angles could be defined, which should be accepted as correct, but not kept for further use. Two are shown below. Recognizing the angles implicit at the corners of a zigzag is basic to many of the constructions which follow in these lessons.

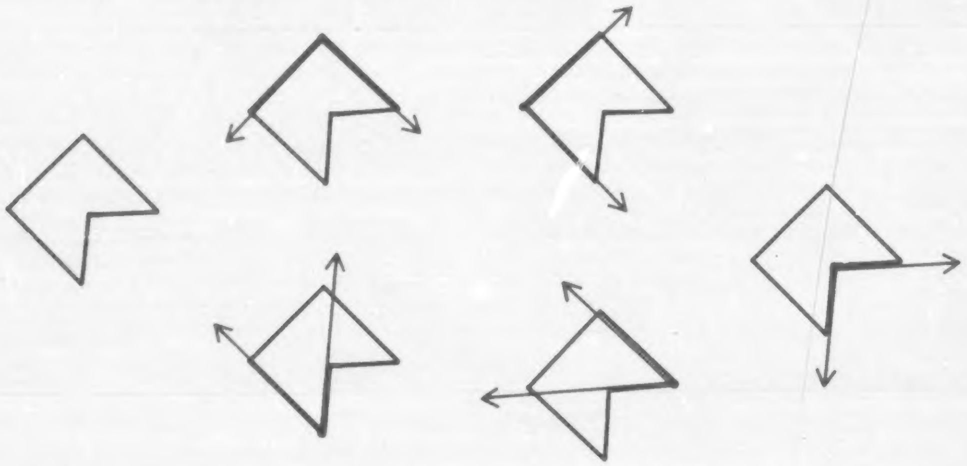




Repeat this activity with a triangle.



Repeat the activity again with a shape similar to the following.



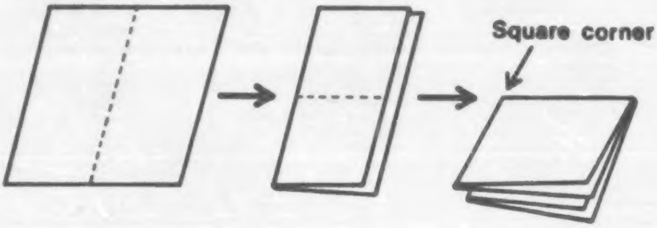
Distribute copies of Worksheet G01 for students to check their understanding of the various terms and geometric elements. Circulate among the class to correct any evident misconceptions. Allow only a few minutes for this work.

Recall the class's attention and initiate a discussion of the angles that are implicit in objects of the classroom (there are many, of course). In each case, let a student indicate the rays and the vertex of each angle under consideration.

T: Many of the angles found in the classroom are special. Watch me as I fold this piece of paper.

G 1

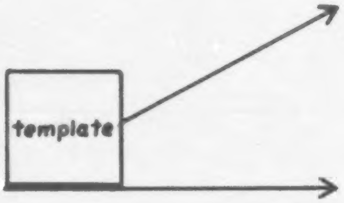
Fold a piece of paper as shown in the next illustration.

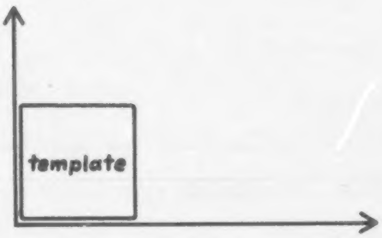


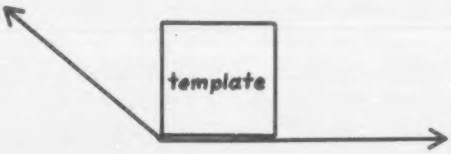
T: The corner of the paper makes an angle which I can compare to other angles. It is a special angle and has several names. We will call it a square corner or a right angle. Why do you think it is called a square corner?

S: The angle at each corner of a square is a right angle.

Draw some angles on the board using your paper square corner as a template, decide whether or not they are right angles. Proper technique consists of matching the vertex of the paper template with the vertex of the angle being compared and then aligning one edge of the paper square corner to one ray of the angle. If the second ray of each angle also matches, then the other angle is a right angle.



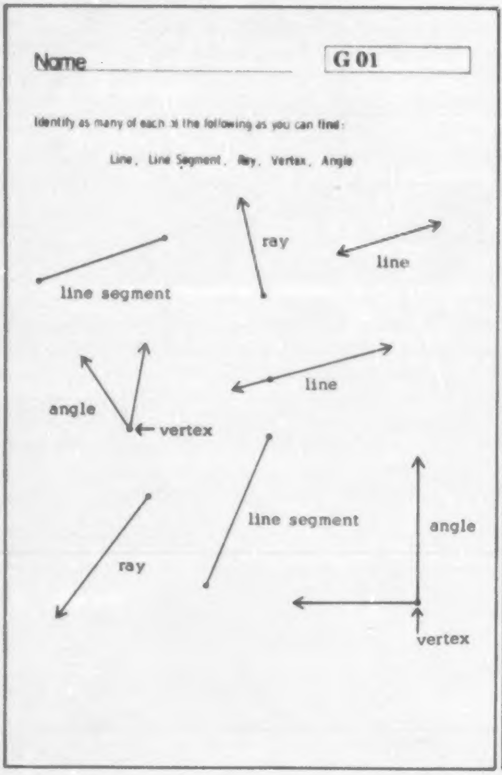




Provide a piece of paper to each student with which to make a paper square corner. When all are equipped, instruct them to check the angles on Worksheet G02. While the class is working, check student use of the paper template to compare angles. When most of the class have finished the worksheet, review the answers collectively, asking students to recheck their work, if necessary. In discussing the worksheet, encourage calling certain angles smaller or larger than a right angle. In a sense, angles B, D, and H are each smaller than a right angle and angles C and F are each larger. "Narrower" and "wider" are suggestive terms which naturally arise and correspond to "smaller" and "larger". Later, when the protractor is introduced, these usages will be justified; but for the moment students can work with them in a natural and intuitive manner.

T: An angle which is smaller than a right angle is called acute, and and angle which is larger than a right angle is called obtuse.

Ask students to label each angle on Worksheet G02 as "acute", "right", or "obtuse". Conclude the lesson by asking the students to draw three angles of each type on the reverse of the worksheet. A ruler or an edge of the paper square corner can be used as a straightedge for drawing. Prohibit freehand sketching of line segments.



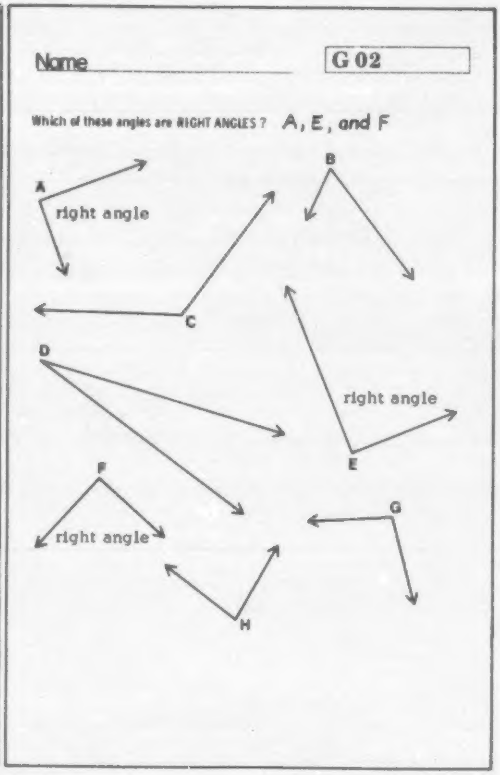
Name

G 01

Identify as many of each of the following as you can find:

Line, Line Segment, Ray, Vertex, Angle

line segment
ray
line
angle
vertex
line
ray
line segment
angle
vertex



Name

G 02

Which of these angles are RIGHT ANGLES? A, E, and F

A
right angle
B
C
D
E
right angle
F
right angle
G
H

1.2 Constructions with Acute Angles #1

CAPSULE LESSON SUMMARY

Construct polygons with a prescribed number of sides and acute angles.

MATERIALS

Teacher: Straightedge; colored chalk; paper square corner

Student : Pencil; straightedge; paper square corner; a large sheet of paper

DESCRIPTION OF LESSON

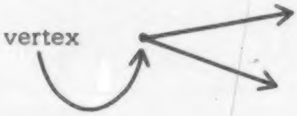
NOTE: Lessons G1.2 and G1.3 are closely related. Student work on the major exercise of this lesson may be carried over into the next.

Begin the lesson by reviewing the following concepts and terms, illustrating each in turn.

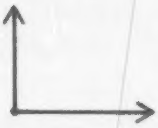
line segment

line

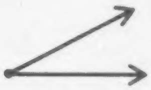
ray



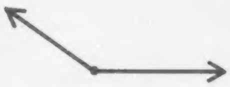
angle: two rays with a common endpoint which are not part of the same line.



right angle



acute angle

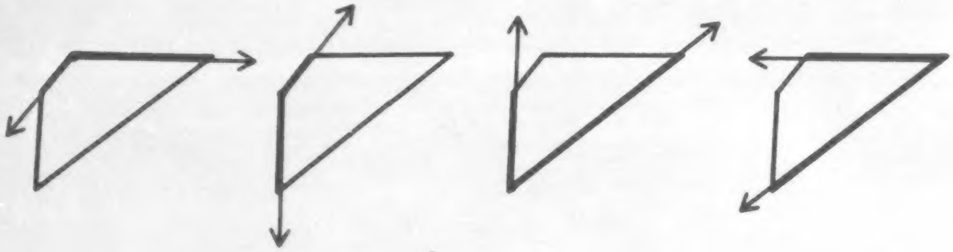


obtuse angle

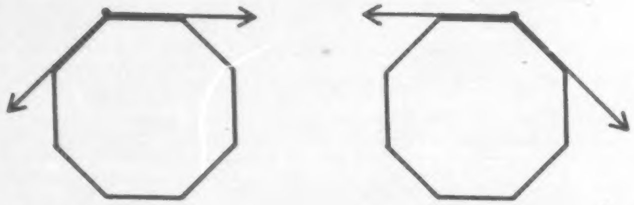
Ask each student to construct a paper square corner to use as a template. See Lesson G1.1 for a description of the construction.

G 2

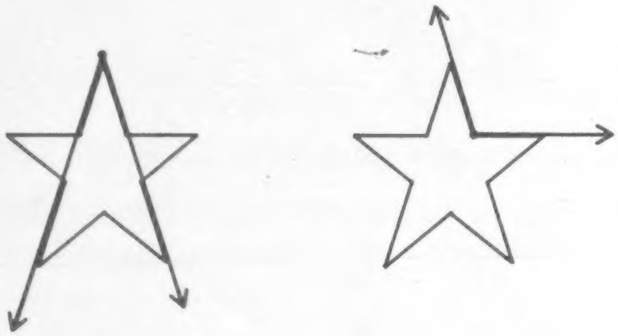
Draw a quadrilateral on the board and invite students to indicate the corner angles of the quadrilateral. For example,



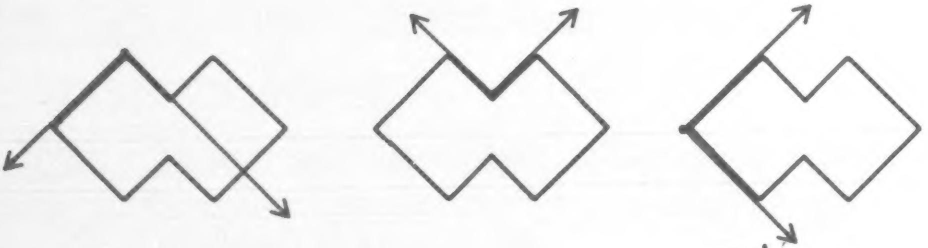
Repeat this activity with the following three shapes or similar ones.



and so on.

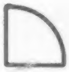
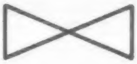
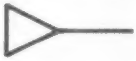
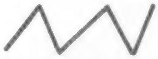
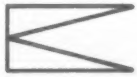
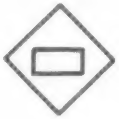


and so on.



and so on.

Ask students to sketch shapes with specified numbers of angles (e.g. six, twelve, seven) on a piece of paper. Students should use a straightedge or an edge of the template to draw straight line segments. Invite several students to copy their drawings on the board. Keep as many of the drawings on the board as possible. The following restrictions need only be introduced if they are violated in students' drawings.

- sides must be line segments — rules out 
- sides do not cross — rules out 
- no dangling edges — rules out 
- closed — rules out 
- three different sides do not meet — rules out 
- connected — rules out 

Following these rules, students will consistently construct simple (closed) polygons. Shapes of this type will be the central focus of the lessons and exercises to follow. In the discussion of each polygon, be sure that the class observes that the number of sides equals the number of angles.

G 2

NOTE: Students will begin to learn the standard names for polygons with specific numbers of sides through your regular usage.

quadrilateral	4 sides
pentagon	5 sides
hexagon	6 sides
septagon	7 sides
octagon	8 sides
nonagon	9 sides
decagon	10 sides

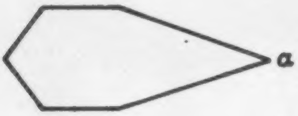
After several polygons have been drawn on the board, assign a student to each to use a paper square corner to label its angles appropriately as acute, square, or obtuse. Encourage class discussion and several checks of any disputed labeling. This exercise should provide an opportunity to emphasize the importance of careful, neat drawings, especially of straight line segments and of precisely defined vertices. NOTE: When appropriate, mention that "vertices" is the plural of "vertex".

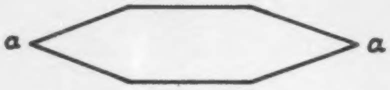
T: On a clean sheet of paper, draw a hexagon with exactly one acute angle.

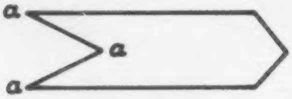
Monitor the class work. List a series of similar constructions for the students to perform in sequence.

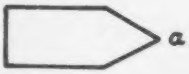
number of sides	number of acute angles
6	1
6	2
6	3
5	1
4	1
5	2

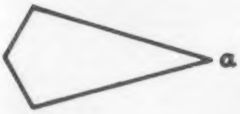
On the board collect a variety of solutions to the sequence of constructions for inspection and discussion. One complete set follows. Acute angles are marked "a".

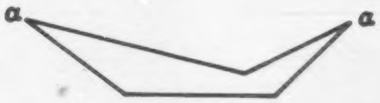












Draw a chart on the board as in the following illustration to collect the information and to suggest a related collection of constructions. The checkmarks indicate that these six constructions are possible, as already verified.

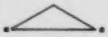
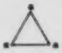
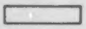
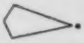
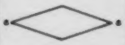
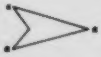
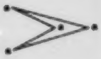
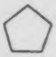
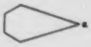
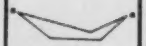
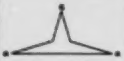
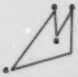
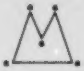
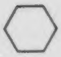
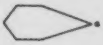
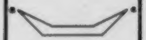
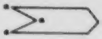
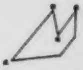
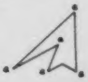
		number of acute angles					
		0	1	2	3	4	5
number of sides	3						
	4		✓				
	5		✓	✓			
	6		✓	✓	✓		

Distribute large sheets of paper to the students. Ask them to draw a chart similar to the one on the board, but with cells large enough to draw polygons within them.

For the remainder of the lesson, students should continue to construct polygons with the various combinations of numbers of sides and numbers of acute angles specified in the chart. Of course, the number of sides of a polygon equals its number of angles, so some of the constructions are automatically impossible. For example, a polygon with three sides cannot have four or more acute angles. In addition, a triangle must have two or three acute angles. But the experience of attempting these constructions will be valuable. The discussion of their impossibility is postponed until Lesson G1.3. Completion of the chart could be reserved for homework. A large chart might make an interesting bulletin board project.

For each construction which is possible, there are many solutions. One set of solutions is shown below.

number of acute angles

number of sides	0	1	2	3	4	5
3	impossible	impossible			impossible	impossible
4						impossible
5						
6						

1.3 Constructions with Acute Angles #2

CAPSULE LESSON SUMMARY

Discuss the charts from Lesson G1.2. Discuss indentations in shapes, and then look for solutions with no indentations to fill the cells of the chart.

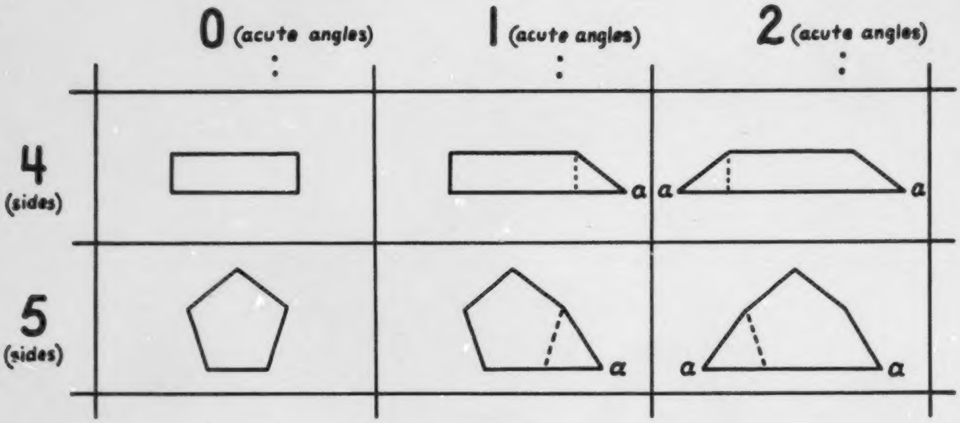
MATERIALS

Teacher: Straightedge; paper square corner

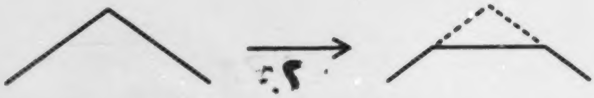
Student: Pencil; completed chart from Lesson G1.2; straightedge; paper square corner; large paper

DESCRIPTION OF LESSON

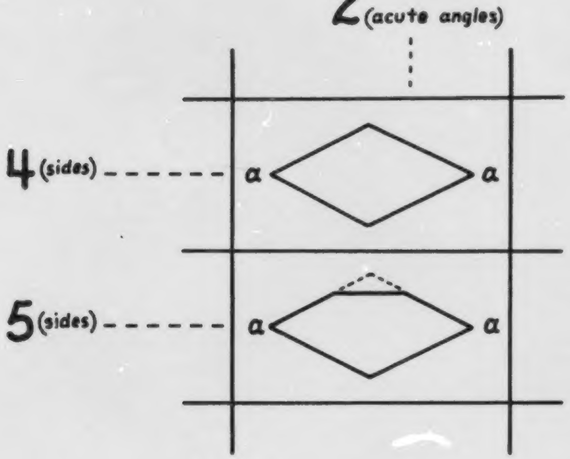
Begin with a discussion of the students' charts of constructions from Lesson G1.2. Center the discussion around the variety of techniques which students discovered for finding solutions. For example, these illustrations show two progressions of related shapes in which the number of acute angles increases from 0 to 1 to 2 with no change in the number of sides.



The next picture suggests a technique for replacing one obtuse angle with two obtuse angles.



If one has a construction to fit one cell of the chart, one can apply this technique to fill the cell below it — the number of sides increases by one, but the number of acute angles does not change. For example,



Ask the class which constructions they think are impossible. Students should refer to their charts displayed on their desks.

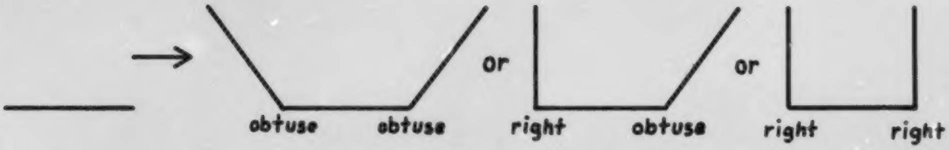
T: Which of the constructions were you not able to do? Do you think that they are impossible?

S: A triangle cannot have more than three angles nor can a quadrilateral have more than four angles; a polygon has the same number of angles as it has sides.

S: It is impossible to have a triangle with no acute angles or with just one acute angle.

T: Why?

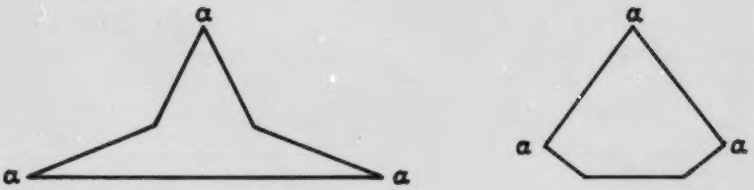
S: If I draw a side of a triangle that doesn't go with an acute angle, then the other sides can never meet to finish the triangle.



T: What can we say about these two other line segments when we draw a side with right angles at both of its endpoints?

S: They are parallel.

Students will often spontaneously construct polygons with "indentations." Copy on the board two solutions for the same cell of the chart from students' work, one with indentations and one without. For example,



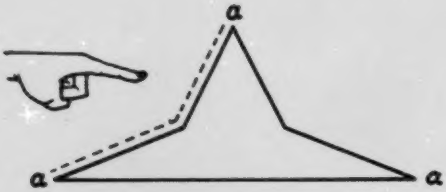
T: Each of these pentagons has exactly three acute angles. But how do they differ?

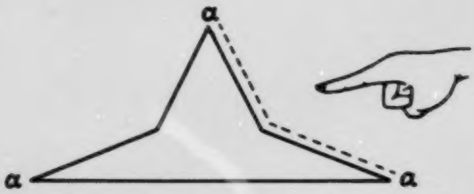
S: The sides have different lengths.

S: The three acute angles are next to one another in the pentagon on the right, but not in the pentagon on the left.

G 3

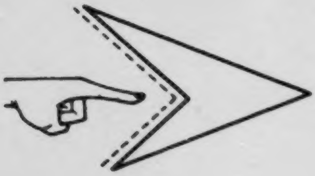
You may have to introduce the concept of indentation yourself. While this concept can be defined carefully, it is best left informal and intuitive for these lessons. Trace each of the obtuse corner angles of the shape on the left.

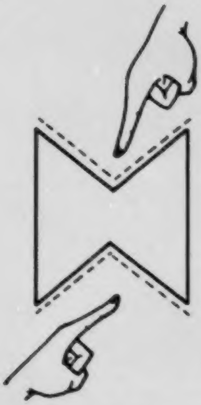


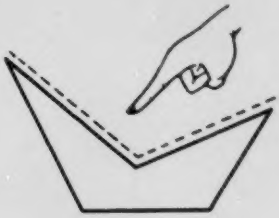


T: This polygon has two indentations, whereas the other polygon has none.

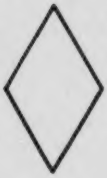
Invite several students to draw polygons from their charts on the board. Many of the students' constructions will have indentations. Then ask the class to identify the identations, if any, in each polygon. Thus,

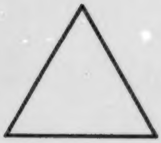






all have indentations, whereas . . .



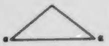
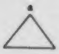
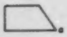
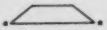
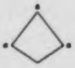
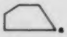
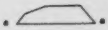
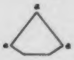
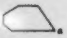
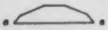
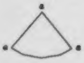


do not have indentations.

When the idea of indentation is clear, ask the students to reconsider their constructions and to produce solutions with no indentations, whenever possible. Furnish each student with a large sheet of paper on which to draw another chart.

Under this additional restriction, more of the constructions are impossible. However, if at least one solution is possible, then many solutions are. A set of solutions is shown below.

number of acute angles

number of sides	0	1	2	3	4	5
3	impossible	impossible			impossible	impossible
4					impossible	impossible
5					impossible	impossible
6					impossible	impossible

G 4

1.4 Polygon Constructions #1

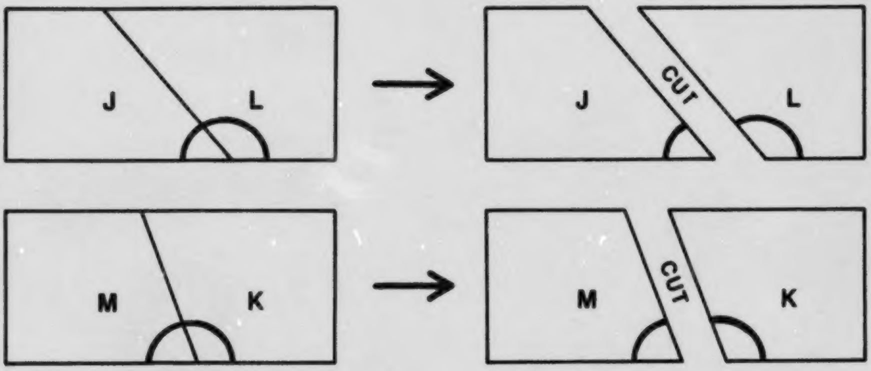
CAPSULE LESSON SUMMARY

Construct quadrilaterals using four given angles.

MATERIALS

Teacher : Straightedge ; four cardboard templates
Student : Pencil ; straightedge ; tracing paper ; Worksheet G03

NOTE : Cut four templates out of cardboard, one to match each of the angles on Worksheet G03. The angle measures are J = 50°, K = 110°, L = 130°, M = 70°. This is for your information only ; do not tell the class. Use a protractor and a straightedge to lay off each angle as shown here.



DESCRIPTION OF LESSON

Distribute copies of Worksheet G03 and tracing paper.

T: This worksheet shows four angles. What do we call a polygon with four angles?

S: Four angles — four sides — that is a quadrilateral.

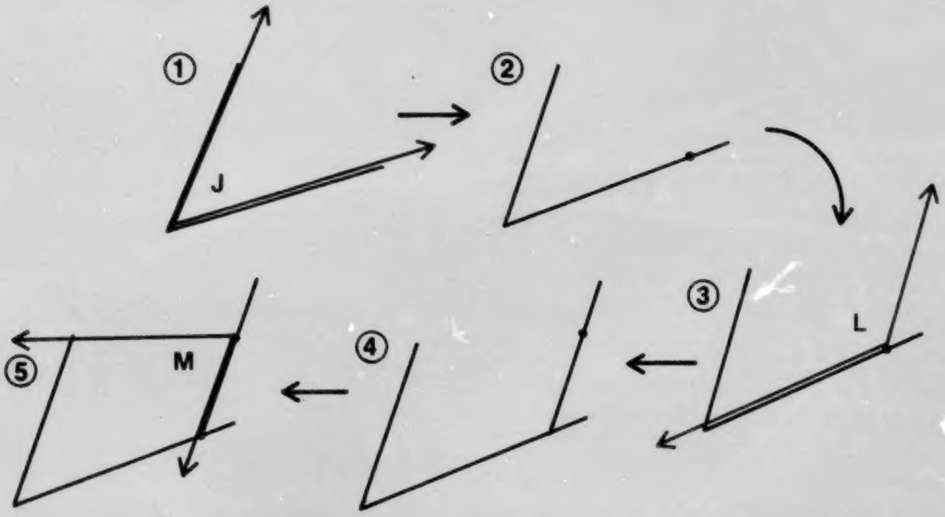
T: I have four templates, one to match each of the angles on the worksheet. Which of the angles are acute? Which are obtuse? Which are right angles?

S: J and M are acute; K and L are obtuse.

T: Can you construct a quadrilateral using all four of the angles shown on the worksheet? Use a piece of tracing paper to trace the angles as you build the quadrilateral.

Circulate to monitor students' use of tracing paper and, in particular, to check that they use a straightedge to make accurate, neat drawings. Some students may need special attention in developing a technique for using tracing paper to build quadrilaterals. One procedure is as follows:

- Trace one of the angles using a straightedge and make the line segments as long as desired.
- Pick a point on one of the line segments to be a second corner of the quadrilateral and hence the vertex of the second angle to be traced.
- Lay the tracing paper on the second angle so that the vertex is under the chosen point and so that one ray points at the first corner.
- Trace along the other ray as long as desired.
- Pick a point for the third corner, and so on.



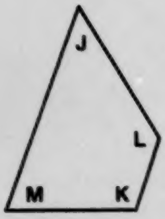
G4

As students succeed in constructing quadrilaterals, ask them to build others which are "different." Let students decide for themselves what might be considered different among quadrilaterals. As you observe students' work, invite some of them to draw solutions on the board using the templates. When most students have several solutions, initiate class discussion of solutions to develop criteria for deciding when two solutions differ.

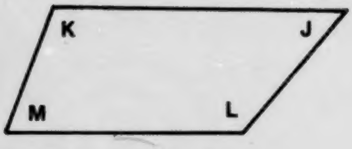
There are several ways in which solutions may differ:

- orientation on the page;
- size;
- sequence of the angles.

Orientation on the page is not important and should be readily discarded by the class. Size is more significant, but is still not very interesting. Lead the discussion to consider the relative position (or sequence) of the angles in each solution. Beneath each one of them, write the sequence of angles beginning at J and reading clockwise.

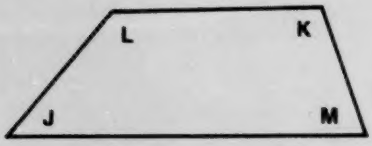


J – L – K – M

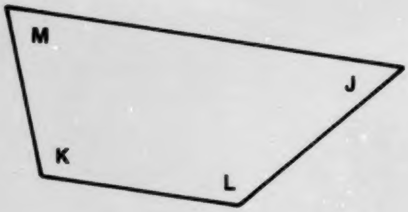


J – L – M – K

If two solutions have the same sequence, they must be the same except for size or orientation on the board.



J – L – K – M



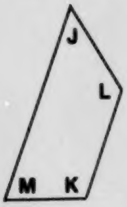
J – L – K – M

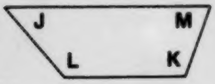
Ask students to label their individual solutions in this way and to decide how many different solutions they have found. Then invite students to draw solutions on the board which are not already there.

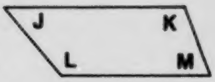
NOTE: A complete list can be found as follows. In any solution, either angles J and K will be opposite each other or they will be next to each other. And if they are next to each other, one will precede the other in a clockwise reading. The possibilities are:

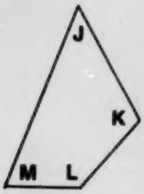
- J - ? - K - ?
- J - ? - ? - K
- J - K - ? - ?

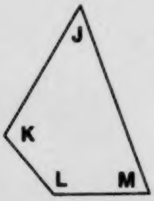
Now L and M can be inserted in the remaining positions to yield only six configurations of angles and hence six different quadrilaterals.

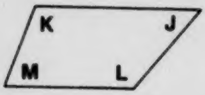






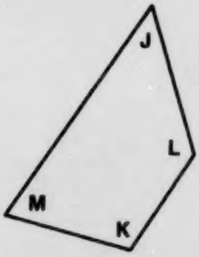






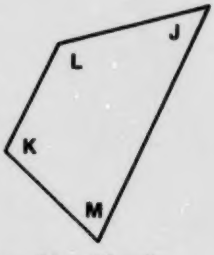
Some students may notice that in turning over the tracing paper, one solution yields another. This can lead to further discussion. For example,

turning over



J – L – K – M

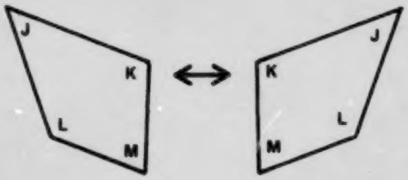
yields



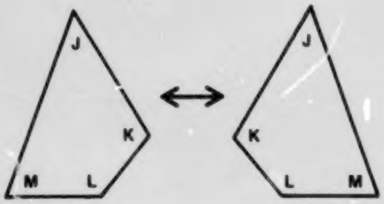
J – M – K – L

G 4

The other four solutions similarly fall into two pairs.



J – K – M – L   J – L – M – K



J – K – L – M   J – M – L – K

If there is time and interest in your lesson, you might introduce this situation yourself, if no student has done so, by simply suggesting that the class turn over their papers and compare the quadrilaterals with the originals.

1.5 Polygon Constructions #2

CAPSULE LESSON SUMMARY

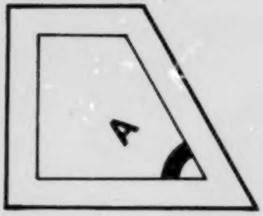
Use two angle templates to construct polygons with three, four, five, and six sides.

MATERIALS

Teacher: Two templates corresponding to A and B in the student set; straightedge

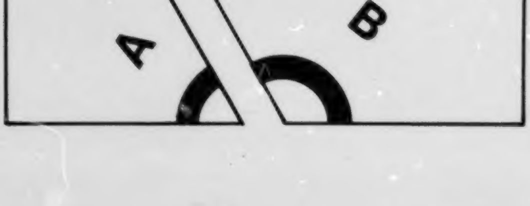
Student : Pencil; straightedge; templates A and B; large paper

NOTE: Cut two templates out of cardboard to correspond to A and B of the student set. They should be larger and thicker than those in the student set. Angle A measures 60° and angle B measures 120°. This is for your information only; do not tell the class. You can use templates A and B from a student set as a guide for making your own.



DESCRIPTION OF LESSON

Introduce the class to the angle templates A and B. Make clear that the angles marked with the black arcs are the ones to be used in the constructions they will be doing. In the lesson descriptions, we will use the same name for a template and the angle indicated on that template. Provide templates A and B to each student.

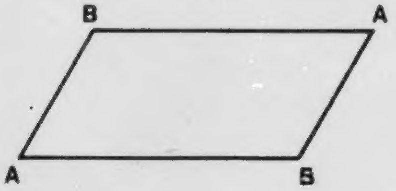


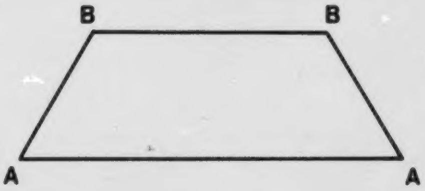
G 5

Ask the students to construct as many different quadrilaterals as possible, subject only to the condition that each angle be congruent to A or to B. For our purposes, we say that an angle traced using template A is congruent to the angle of A and an angle traced using template B is congruent to the angle of B, and so on.

Remind the class, if necessary, of their experience in Lesson G1.4, where quadrilaterals were distinguished by their sequences of angles, rather than by size or by orientation on the page.

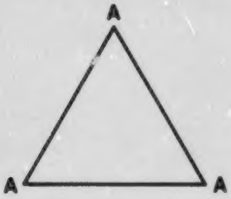
There are essentially two quadrilaterals to be found: a parallelogram and a trapezoid. Each has two occurrences of A and two of B. In the parallelogram, the angles congruent to A are diagonally opposite; in the trapezoid, they are adjacent. We will use this important characterization of the solutions in succeeding lessons.

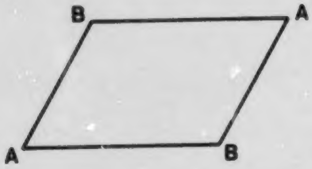


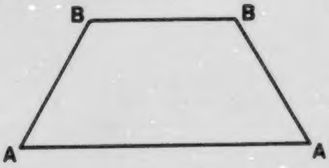


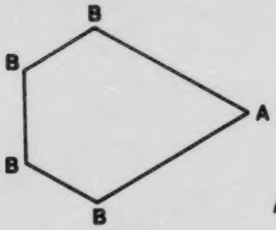
As students discover these solutions, ask them to construct polygons with three, five, and six sides using only A and B to construct the angles of their polygons. Invite students to copy solutions on the board using your templates. Label each angle appropriately as "A" or "B". A collection of solutions is shown on the next page.

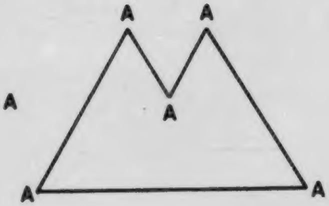
POLYGONS WITH EACH ANGLE CONGRUENT EITHER TO A OR TO B.

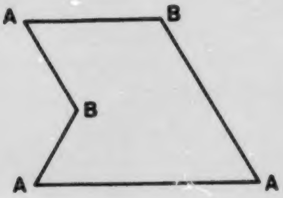


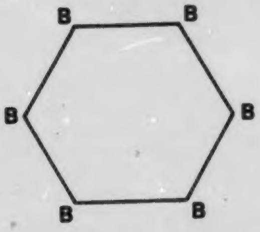


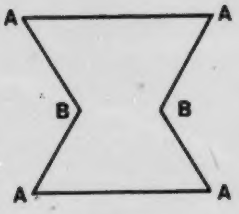


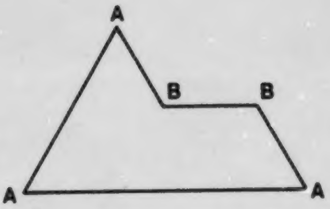


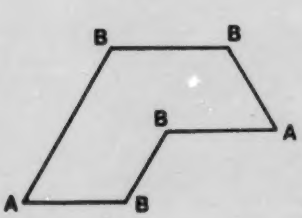


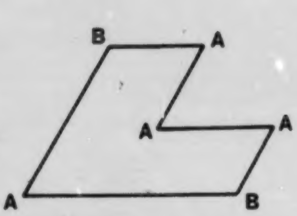


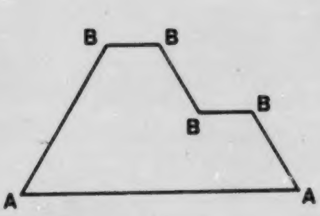










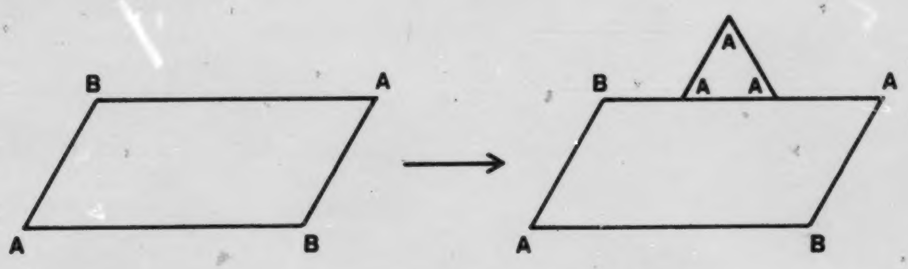


When many solutions are displayed on the board, lead a collective discussion of them. Your students should notice that there are only two solutions which are quadrilaterals, one of which is a parallelogram. The other quadrilateral you may introduce as a trapezoid. A trapezoid is a quadrilateral with exactly two parallel sides. In the discussion, call attention to whether or not solutions have indentations. All of the possible solutions without indentations are shown in the preceding illustration, although there are many more solutions with indentations. However, other than those shown, there is no other solution with three or four sides, with or without indentations.

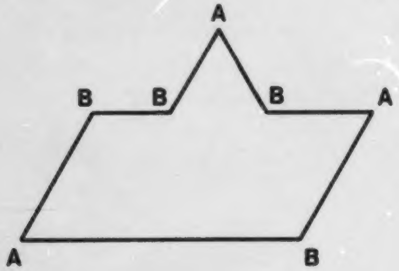
Preserve students' drawings for Lesson G1.6. Making a large poster of solutions might be an interesting class project, and the poster would serve well for reference in later lessons.

(Optional Activity)

An advanced problem is to find a way of building solutions with any given number of sides. Having constructed several solutions with three, four, five, and six sides, a student could look for solutions with seven, eight, nine, and ten sides, and so on. The idea is to find a general process for building solutions with more and more sides from solutions already found. One approach is to build a triangle with all angles congruent to A along one side of a solution. This procedure is successful since A and B are closely related: by constructing an angle congruent to A along a side, we automatically get a neighboring angle congruent to B. For example,



Now erase the common line segment.



The new polygon has seven sides, three more than the original. Using this technique, if we have a solution with twenty-six sides, we can build a solution with twenty-nine sides, and so on. In fact we can construct solutions with any number of sides (three or larger) by building the appropriate number of triangles on the edges of a three, four, or five-sided solution.

1.6 Polygon Constructions #3

CAPSULE LESSON SUMMARY

Use four angle templates to construct quadrilaterals. Recognize the special pairs which give parallelograms and trapezoids.

MATERIALS

Teacher: Eight templates corresponding to A, B, C, D, E, F, G, and H in the student set; straightedge

Student : Record of work from Lesson G1.5; pencil; straightedge; angle templates A, B, C, D, E, F, G, H; large paper

NOTE: As you did for templates A and B in preparation for Lesson G1.5, cut six templates out of cardboard to correspond to C, D, E, F, G, and H of the student set. The angles measure:

C: 75°, D: 105°, E: 45°, F: 135°, G: 30°, and H: 150°

This is for your information only; do not tell the class.

DESCRIPTION OF LESSON

In this lesson, students use four templates (C - D - E - F) to construct quadrilaterals. The object is their guided discovery that these four angles fall naturally into two pairs: C with D and E with F. The discovery of this pairing comes through recognizing that these pairs each yield two different quadrilaterals and further that they are related as mutually supplementary angles. The aim is for students to discover the idea of supplementary angles without its being defined in terms of the protractor or angle measure (two angles are mutual supplements if the sum of their measures is 180°). In this way students can develop a feeling for angles and their fundamental relationships without the distraction of measurement.

Display the poster of solutions from Lesson G1.5, if you have one. Otherwise, ask students to consult their drawings from that lesson.

T: Last week we used templates A and B to construct polygons with three, four, five, and six sides. You found many solutions, but there were only two solutions with four sides, two different quadrilaterals. Today we will use four other templates. Construct as many quadrilaterals as you can such that each angle is congruent to one of these four angles (hold up your templates C, D, E, and F).

Provide all eight templates to each student and instruct them to use only templates C, D, E, and F. Let the class begin the task with these four templates. There are twelve solutions. For the purpose of discussion, make a record of solutions on the board as students find them. By now the class will be sufficiently acquainted with the system of describing a quadrilateral by its sequence of angles so that drawing the solutions may not be necessary. Thus, you may record the solutions as in the next illustration; that is, simply name the angles of a solution in the order reported by a student.

C-D-E-F; D-C-D-C; D-D-C-C; *and so on.*

Part of the discussion should be directed toward recognizing and eliminating duplications in the list. If necessary, recall that size and orientation on the page are not to be considered in deciding whether solutions differ. And, of course, the corner from which the angle sequence is started is not important. Thus "C - D - E - F" is the same as "E - F - C - D".

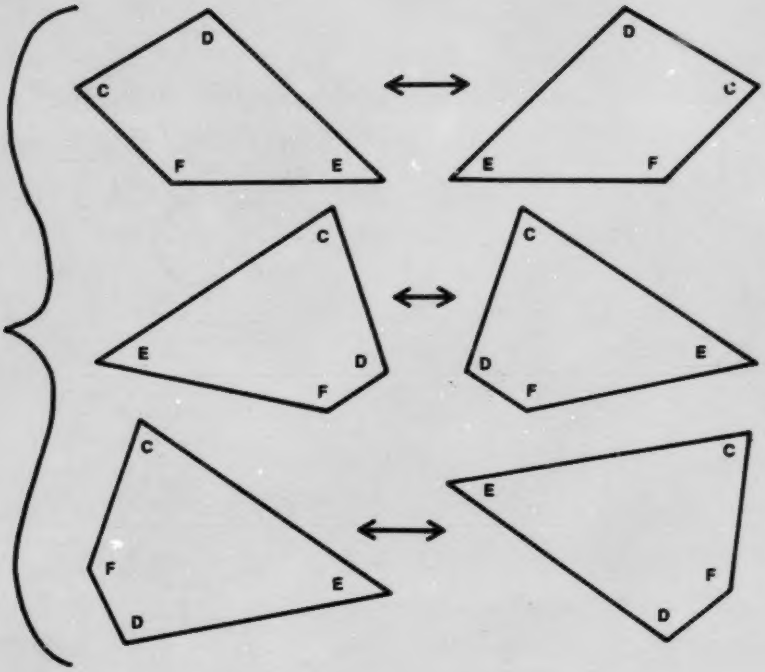
There are three types of solutions:

- four angles, none of which is congruent to any other;
- exactly three of the four angles are congruent;
- two pairs of congruent angles.

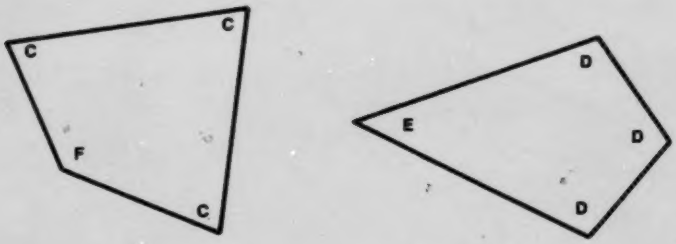
The solutions of the first type occur in pairs as did the solutions to the problem of Lesson G1.4. Each can be changed to its mate by turning it over. The illustration on the next page shows a complete set of solutions arranged by type.

G 6

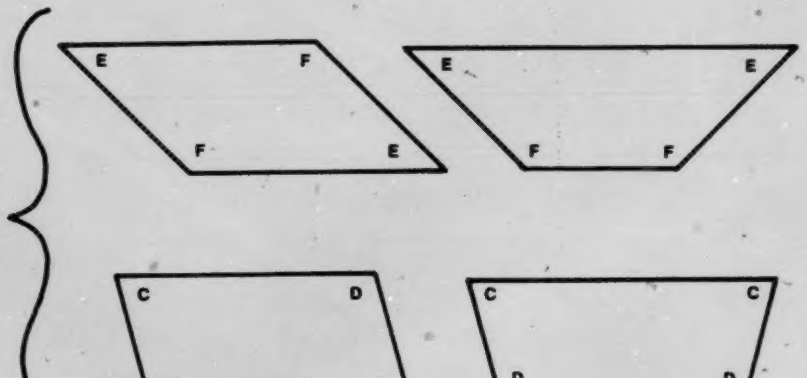
Type 1


D
C
F
E
D
C
E
F
C
E
D
F
C
D
E
F
C
F
E
D
E
C
F
D

Type 2



Type 3


E
F
F
E
E
E
F
F
C
D
D
C
C
C
D
D

T: How many quadrilaterals do we get with only two different templates, each used twice?

S: Four.

S: Two of them use C and D and two of them use E and F.

T: And what kinds of quadrilaterals appear?

S: Two are parallelograms and two are trapezoids.

S: One of each for each pair of angles.

Ask students to take out also templates A and B.

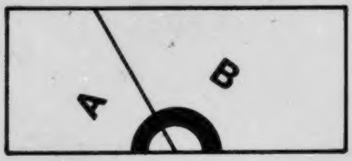
T: Look at your papers showing the polygons using templates A and B. How do the quadrilaterals there compare with these?

S: We found a parallelogram and a trapezoid with A and B, too.

T: So we have three special pairs: A with B, C with D, and E with F. Each pair gives us a parallelogram and a trapezoid. Is there anything else that you notice about these pairs?

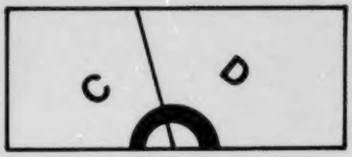
S: I can fit templates A and B together so that two sides are on a line.

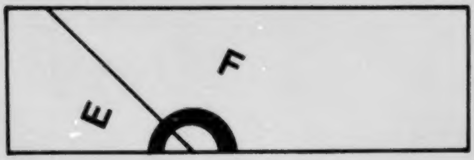
This is the important observation. Ask the student to demonstrate it with the templates.



G 6

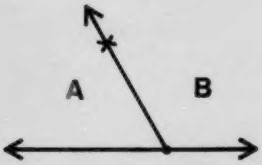
T: What about the other pairs? Can you fit them together so that two sides are on a line? Arrange them on your desks if you can.

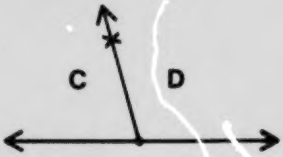


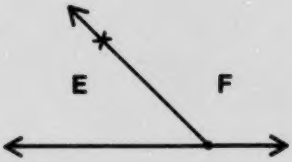


T: Let's draw these pairs of angles on the board.

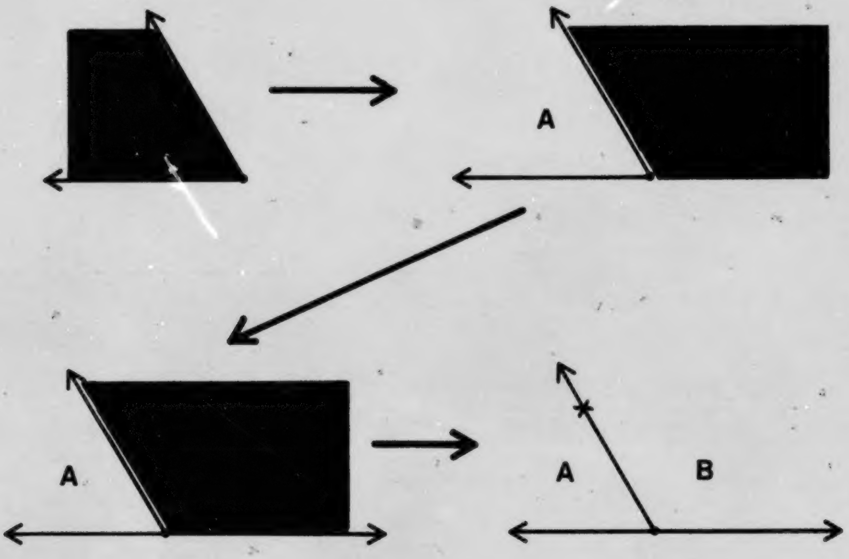
Use templates to construct the pairs of angles. In each, mark the common ray with an "**X**".







NOTE ON TECHNIQUE: The following sequence of drawings illustrates the process of juxtaposing angles, using templates A and B as an example.

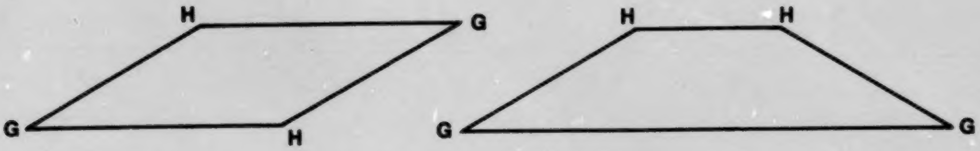


Now to reinforce the idea, ask students to consider templates G and H.

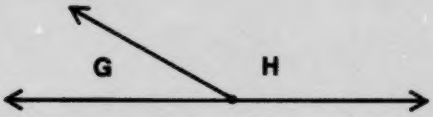
T: Do templates G and H fit together so that two sides are on a line?

S: Yes, they do.

T: Try to build a parallelogram and a trapezoid with G and H.



While the class performs the constructions with G and H, construct another angle pair on the board next to the three already there.



When the class has completed their constructions with G and H, continue the discussion.

T: A pair of angles which fit together[†] so that two sides are on line this way is called a supplementary pair. A and B are supplementary; C and D are supplementary; E and F are supplementary; and G and H are supplementary.

Making a poster showing the twelve constructions of quadrilaterals found in this lesson with templates C, D, E, and F might be an interesting bulletin board project. In any event, preserve student work for reference in Lesson G1.7.

[†] We rely on how templates fit together to convey what we mean by angles "fitting together." This is not precise terminology, but it is clear and natural.

1.7 Polygon Constructions #4

CAPSULE LESSON SUMMARY

Explore supplementary pairs of angles. Compare the template angles according to an intuitive idea of size.

MATERIALS

Teacher:	Straightedge; eight templates corresponding to A, B, C, D, E, F, G, and H in the student set; colored chalk
Student :	Record of work from Lesson G1.6; pencil; straightedge; angle templates A, B, C, D, E, F, G, and H; tracing paper; large paper; Worksheet G04

DESCRIPTION OF LESSON

Review the situation of Lesson G1.6, referring to a chart of the solutions, if you have one, or to the students' drawings from Lesson G1.6.

T: We found that among the eight angles A, B, C, D, E, F, G, and H there are four special pairs. What makes them special?

S: They were the pairs we could construct quadrilaterals with.

S: We made a parallelogram and a trapezoid with each pair.

S: Those pairs of angles could be fit together so that two of their sides are on a line.

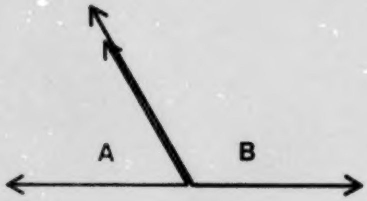
T: Which are the pairs?

S: A with B, C with D, E with F, and G with H.

T: What do we call these pairs?

S: Supplementary pairs of angles.

Invite a student to demonstrate that A and B form a supplementary pair. The student should match the vertex of A with the vertex of B and match one ray of each. See the note on technique in Lesson G1.6, page 44. The other two rays form a line.

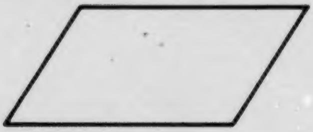


T: Test the other pairs of angles to see if they fit together with two sides on a line.

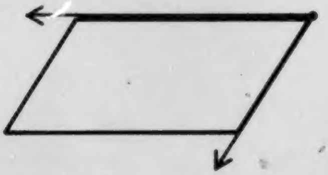
Allow some time for students to test other pairs and to build confidence that there are only four supplementary pairs. Distribute copies of Worksheet G04 and tracing paper.

T: There are some quadrilaterals on this worksheet (G04). See if you can find a supplementary pair in each one. Use the tracing paper to trace the angles and to fit them together.

Monitor the students' techniques in juxtaposing angles. One technique is illustrated here with a parallelogram.

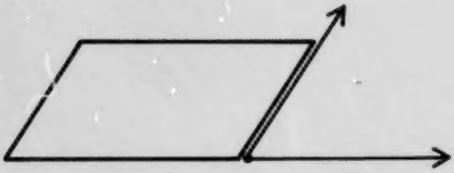


First trace one of the angles using a straightedge.

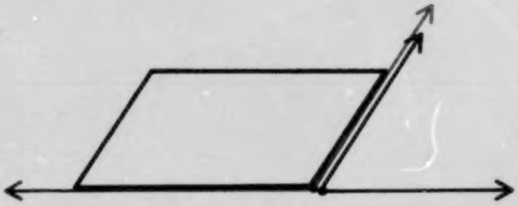


696

Then move the tracing paper to shift the vertex of the constructed angle to another corner of the parallelogram.



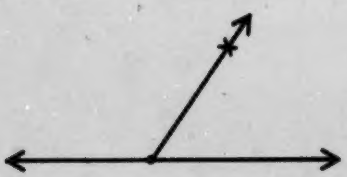
Then trace the angle at the second corner.



When most students have finished, discuss the results. See the answer key to Worksheet G04 on page 51. Notice the special case of the rectangle. All of its angles are right angles and hence congruent. One right angle is supplementary to another right angle. Perhaps someone will notice that there are four different angles forming two pairs of supplementary angles in one of the shapes and that it is a trapezoid.

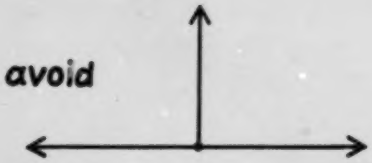
T: Who can show another supplementary pair? How can we build a supplementary pair on a line?

Invite a student to put the drawing defining a supplementary pair on the board.



T: Each of you draw a line on your paper, choose a point on the line, and draw a ray starting at that point. (Pause) Now each of you have a supplementary pair. Use the pair to construct a parallelogram and a trapezoid.

Again monitor the students' drawing. If anyone chooses a pair of right angles, suggest trying another pair to avoid this special case.

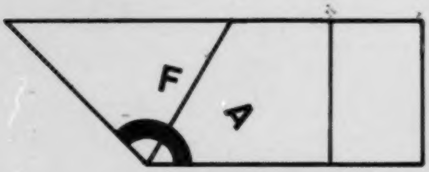


T: Each of you has chosen a different supplementary pair. Each of you has constructed the two quadrilaterals.

Conclude the lesson with a brief discussion of relative size of angles. Provide templates A, B, C, D, E, F, G, and H to each student.

T: Look at the eight angle templates. Which angles are acute? (A, C, E, and G)
Which are obtuse? (B, D, F, and H)
Which do you think is the smallest? (G)
Which is smaller, A or C? E or H? D or F? B or D?
Which is larger, C or E? F or H? G or B? E or G?

The idea of comparing angles was introduced in Lesson G1.1 in which angles were compared with right angles. We can extend that informal idea to a comparison of any two angles. With the templates in hand, it is easy to compare the angles directly by placing one on top of the other, matching the vertices, and aligning one edge of each template.



A on top of F

G7

T: Line up the templates on your desk in order of size.

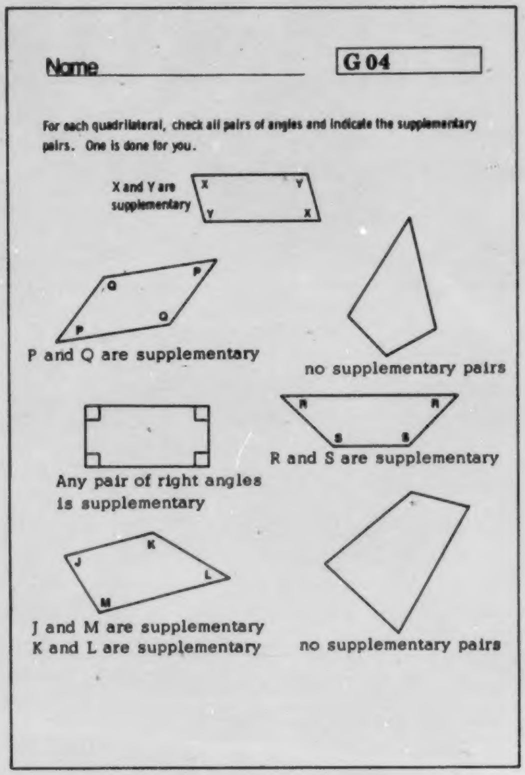
G E A C D B F H

T: Which is the smallest? (G)
And which is the largest? (H)
What do you notice about the supplementary pairs?

S: The largest and the smallest are supplementary; the second smallest and the second largest are also supplementary; and so on.

S: One angle in each pair is acute and the other is obtuse.

S: That's true for any supplementary pair, unless they are both right angles.


X
Y
Y
X
Q
P
P
Q
R
R
S
S
J
K
L
M

Name ______________________

G 04

For each quadrilateral, check all pairs of angles and indicate the supplementary pairs. One is done for you.

X and Y are supplementary

P and Q are supplementary

no supplementary pairs

Any pair of right angles is supplementary

R and S are supplementary

J and M are supplementary
K and L are supplementary

no supplementary pairs

1.8 Triangle Constructions #1

CAPSULE LESSON SUMMARY

Use seven templates to construct triangles.

MATERIALS

Teacher: Straightedge; seven templates corresponding to A, B, G, W, X, Y, and Z in the student set

Student : Pencil; straightedge; angle templates A, B, G, W, X, Y, and Z; large paper

NOTE: You have already made templates corresponding to A, B, and G, but you will need to make the other four. The angles measure:
A: 60°, B: 120°, G: 30°, W: 40°, X: 50°, Y: 70°, and Z: 90°.
This is for your information only; do not tell the class.

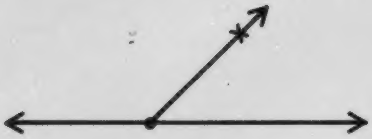
DESCRIPTION OF LESSON

Begin the lesson with a brief review of the work of Lesson G1.7.

T: Last week we used eight templates to construct quadrilaterals. How many supplementary pairs did we have among the templates?

S: We found four supplementary pairs.

Ask a student to illustrate a supplementary pair of angles.



" X " marks the ray common to the two angles.

T: How many supplementary pairs of angles are there?

S: More than we can count.

S: Infinitely many; we can point the ray in any direction.

T: Today we will use templates again, but not the same set of templates.

Provide templates A, B, G, W, X, Y, and Z to each student.

T: Which of these angles are acute? (A, G, W, X, and Y)
And which is obtuse? (Only B)
What about Z? (Z is a right angle.)
Which is the smallest of the seven angles? (G)
And which is the largest? (B)
Arrange them on your desk in order of size.

G W X A Y Z B

T: The problem today is to use these angles to construct triangles. There are many possibilities. How many?

S: We can start with any of the seven angles.

S: May we use an angle twice?

T: In constructing quadrilaterals, we often used an angle more than once; perhaps you can do the same with this problem.

S: Then we have seven choices for the second angle and seven choices for the third angle.

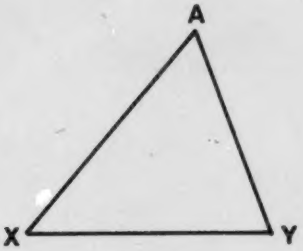
S: That's 7 x 7 x 7 choices in all; 343.

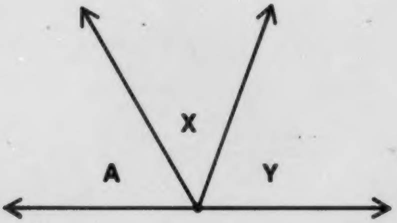
T: Do you think that you can construct 343 different triangles?

S: Probably some of the combinations won't work. Not all of the combinations worked when we were constructing quadrilaterals.

T: As you try different combinations, label the angles on your papers. Keep a record of the combinations that work and those that do not work. Also be looking for something special about the triples of angles which do work.

NOTE: The special relationship of successful triples is similar to that of supplementary pairs; namely, they can be juxtaposed to indicate a straight line. But leave this important fact for students to discover.

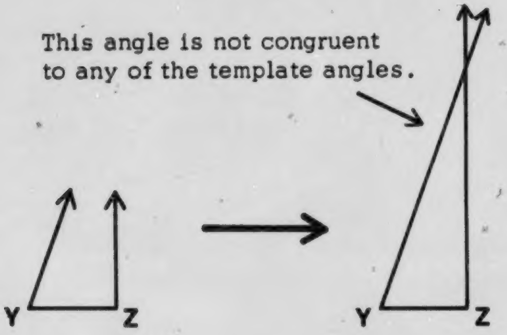




Allow the class to work for most of the remainder of the lesson. Many students will not immediately form a well-organized systematic program for testing triples of angles. Instead they are more likely to choose triples almost at random. However, encourage their keeping a chart of successes and failures.

B-G-G	yes
A-B-G	no
Z-Y-G	no

The students will discover through their experimentation that once two angles are chosen and drawn in the construction of a triangle, then the third angle is already determined. So if Y and Z are chosen first and drawn, extending their rays completes a triangle. Then the question is whether the third angle is congruent to any of the seven among the templates. In this case, the answer is "no".



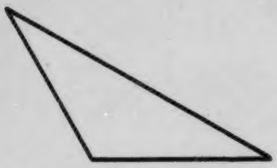
When students make this observation, either independently or at your instigation, pursue it. In particular, ask how this idea can be used to improve the class's estimate of the number of triangles.

With this information, there are seven choices for the first and seven for the second angle, but only one choice for the third angle. So the total number of combinations is at most 7 x 7 x 1 = 49, which is much smaller than 343. A better estimate appears naturally in the next lesson on this topic, Lesson G1.9.

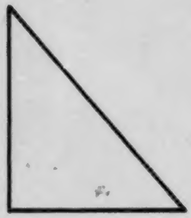
Reserve some time at the end of the period to begin a combined listing of successes and failures to be preserved for use in Lesson 1.9. Some suggestions for the list may not be correct and there may be some disagreement. However, list all suggestions for further discussion and checking, which will continue in Lesson G1.9. Also, save the students' drawings for that lesson so students can continue to experiment with triples of angles.

G 8

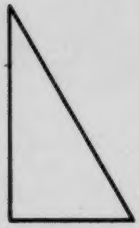
NOTE: Only six combinations yield triangles.



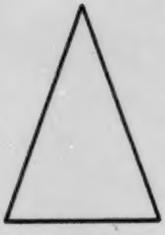
B-G-G



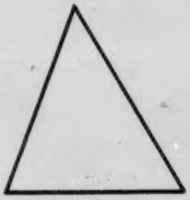
Z-W-X



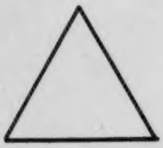
Z-G-A



Y-W-Y



Y-X-A



A-A-A

1.9 Triangle Constructions #2

CAPSULE LESSON SUMMARY

Complete a collection of triangles constructed with seven templates. Explore the relationship among the triples of angles which yield triangles.

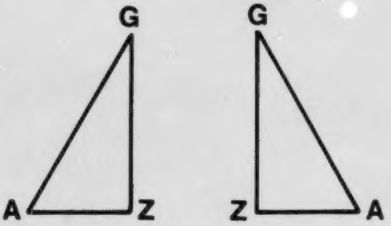
MATERIALS

Teacher:	Straightedge; seven templates corresponding to A, B, G, W, X, Y, Z in the student set
Student :	Results of constructions from Lesson G1.8; pencil; straightedge; angle templates A, B, G, W, X, Y, Z

DESCRIPTION OF LESSON

Return the students' worksheets from Lesson G1.8 and give each student templates A, B, G, W, X, Y, and Z. Copy the record from Lesson G1.8 of successes and failures on the board for further discussion. Instruct the students to check the listings for accuracy and to check for other triples not listed. Occasional discussion of the situation and the lists will direct the students toward completing the problem. Remind them to look for some special property of the successful triples.

Two observations help to make systematic lists. When two angles are drawn there is no choice for the third (as described at the end of Lesson G1.8), so a pair of angles can appear on the success list at most once. For example, A-G-Y and A-G-Z cannot both yield triangles since Y and Z are different angles. Another point is that the angles can be listed in any order. For example, A-G-Z and A-Z-G give the same triangle. Each can be turned over to yield the other.



With these observations, the class can produce a good systematic list of the possibilities.

T: Suppose we pick A first.

S: We have seven choices for the second angle.

List them.

A-A	A-B	A-G	A-W	A-X	A-Y	A-Z

T: Suppose we choose B first.

S: Then we have seven choices for the second angle again.

S: No, we already listed A and B, so we have only six choices.

Add them to the list.

A-A	A-B	A-G	A-W	A-X	A-Y	A-Z
	B-B	B-G	B-W	B-X	B-Y	B-Z

S: Then if we start with G, we have only five choices for the second angle since we already listed G with A and G with B.

Add to the list.

A-A	A-B	A-G	A-W	A-X	A-Y	A-Z
	B-B	B-G	B-W	B-X	B-Y	B-Z
		G-G	G-W	G-X	G-Y	G-Z

S: Starting with W there are four more; starting with X there are three more; starting with Y there are two more; and starting with Z just one more.

A-A	A-B	A-G	A-W	A-X	A-Y	A-Z
	B-B	B-G	B-W	B-X	B-Y	B-Z
		G-G	G-W	G-X	G-Y	G-Z
			W-W	W-X	W-Y	W-Z
				X-X	X-Y	X-Z
					Y-Y	Y-Z
						Z-Z

T: How many possibilities altogether?

S: 28.

T: Which of these pairs have we already found in the success list?

S: B-G-G gives a triangle, so we have B-G and G-G.

Circle "B-G" and "G-G" in the list of pairs. Do the same for each of the other successes on your list. The remaining pairs are all that remain to be checked for triangles. If the class has already found all six of the solutions, the remaining pairs will be struck from the list upon checking. If any of the six remains to be found, it will quickly appear in checking the pairs. The successful triples of angles are:

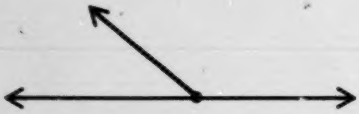
B-G-G	Z-X-W	Z-A-G
Y-Y-W	Y-A-X	A-A-A

G9

If no one has made the observation that a successful triple of angles can be juxtaposed to form a straight line, initiate a discussion as follows.

T: When we constructed quadrilaterals, we discovered the idea of supplementary pairs of angles.

Draw the defining picture of a supplementary pair.



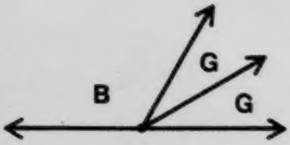
T: Pick one of your triangles and put its three angle templates together in a similar way. What do you find?

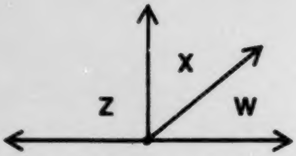
S: We get a line with them.

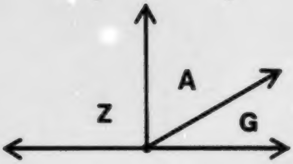
Demonstrate with a drawing on the board. The six combinations are shown below. Note that the order of the angles in each triple is insignificant.

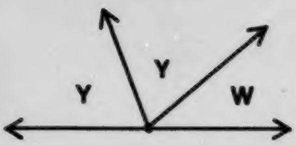
T: Check the other successful triples.

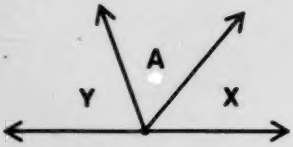
S: They all make straight lines.

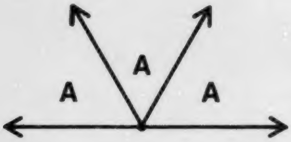






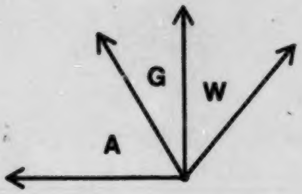


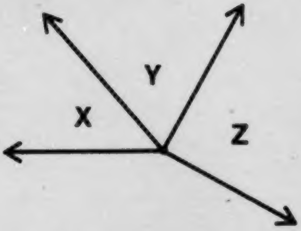




T: Now check a triple that does not yield a triangle.

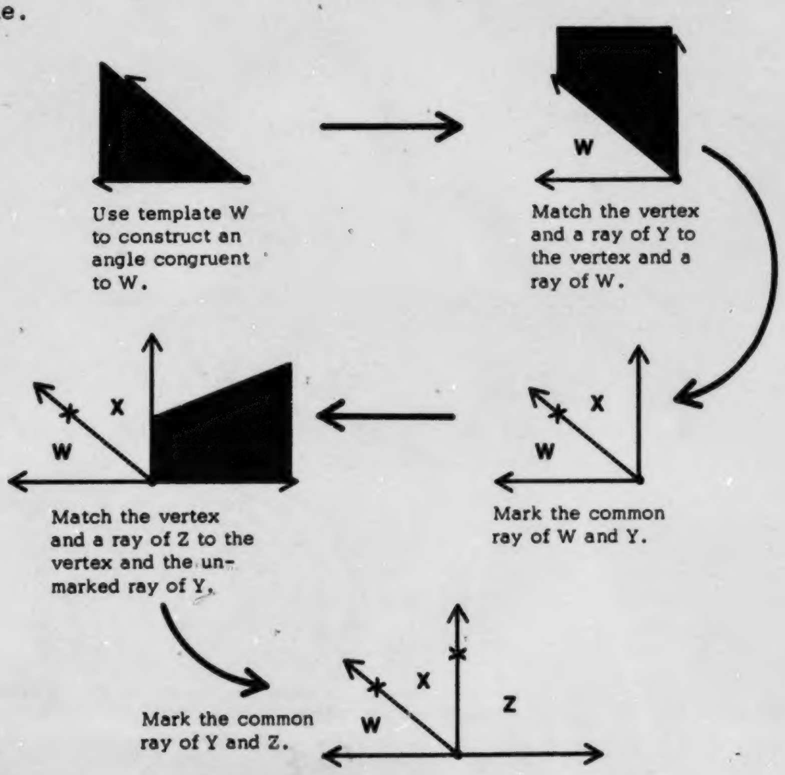
S: It doesn't make a straight line either.





We will return to this idea in Lesson G1.10. Preserve students' worksheets for reference in that lesson.

NOTE ON TECHNIQUE: The following sequence of drawings illustrates the process of juxtaposing three angles, using templates W, X, and Z as an example.



Use template W to construct an angle congruent to W.

Match the vertex and a ray of Y to the vertex and a ray of W.

Mark the common ray of W and Y.

Match the vertex and a ray of Z to the vertex and the unmarked ray of Y.

Mark the common ray of Y and Z.

G1.10 Triangle Constructions #3

CAPSULE LESSON SUMMARY

Explore the relationship among the three angles of a triangle. Extend the experimentation to the four angles of a quadrilateral.

MATERIALS

Teacher: Straightedge; templates for a supplementary pair of angles, e.g. A and B, which were constructed for Lesson G1.5.

Student : Results of constructions in Lessons G1.8 and G1.9; pencil; straightedge; angle templates A, B, G, W, X, Y, Z; large paper; tracing paper

DESCRIPTION OF LESSON

Begin with a review discussion of the work of Lessons G1.8 and G1.9.

T: We have been using these templates to construct triangles. What did we discover about the angles?

S: We have seven templates to use, but not all combinations of the templates gave us a triangle.

S: We could only build six different triangles with the seven templates.

T: Which combinations were successful? Let's list them again as you locate them on your papers.

B-G-G Z-X-W Z-A-G

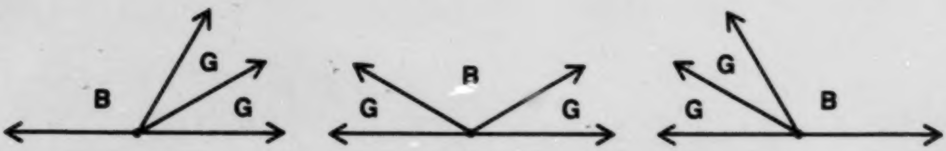
Y-Y-W A-X-Y A-A-A

T: There is something special about each of these triples of angles.

S: Yes, we can construct a line with each triple.

T: Let's try that with one of the triangles.

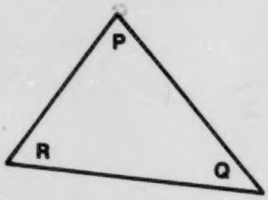
Ask a student to use the appropriate templates to construct the drawing. See the note on technique in Lesson G1.9, page 61. For example, if B - G - G is chosen, then depending on the order of the angles the student will produce one of these three drawings:



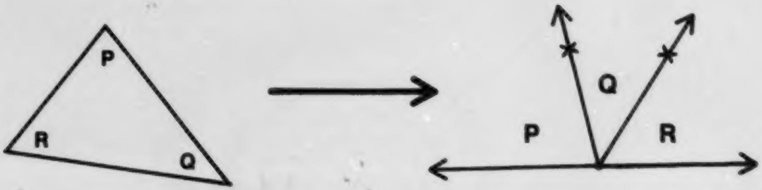
Regardless of the order, a straight line is determined by the juxtaposition of the angles.

T: So the successful triples give us a line and the other triples do not. Let's try an experiment. Each of you draw a triangle, any triangle you wish. Use your straightedge and draw carefully.

Draw a triangle on the board.



T: Label the angles of your triangle "P", "Q", and "R" as I have done. Now take a piece of tracing paper. Trace angles P, Q, and R with a common vertex just as we did with the templates. Each time you add another angle to the drawing, mark the common ray with a small "**X**", so you won't confuse the rays.

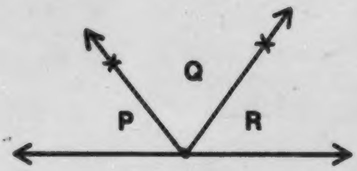


G 10

T: If you finish one triangle, try another.

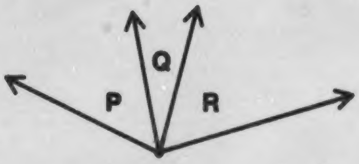
S: I get a straight line each time.

T (sketch a drawing as you describe it): Now draw a line, choose a point, and then construct any two rays from that point on the same side of the line. Label the three angles as I do here. See if you can draw a triangle with angles congruent to P, Q, and R.



S: Mine works.

T: This time make a drawing in which no two of the rays lie on a line, like this one. Try to construct a triangle with the three angles.



S: It doesn't work for me.

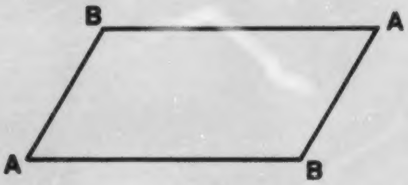
S: The angles of a triangle give us a straight line.

S: And if we start with a line, we have a way to construct three angles which give us a triangle.

T: Now let's look at a parallelogram.

G 10

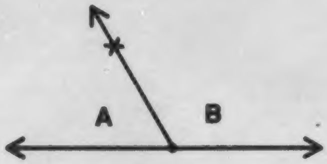
Draw a parallelogram on the board using templates A and B or any other supplementary pair of angles.



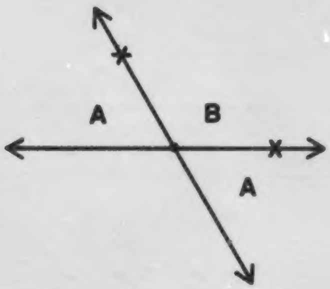
T: What do we know about angles A and B?

S: They are supplementary.

Illustrate the defining relation for A and B.

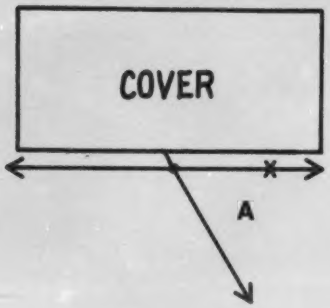


T: Suppose we add another angle congruent to A in this drawing. Match the vertex and one ray of the template A to the second ray of B and mark the common ray.

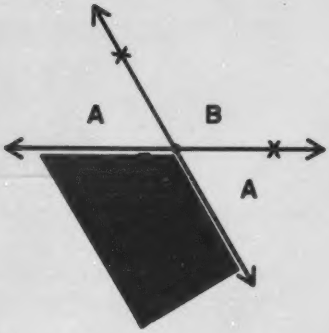


T: What can you say about the fourth angle, the angle formed by the rays without X's?

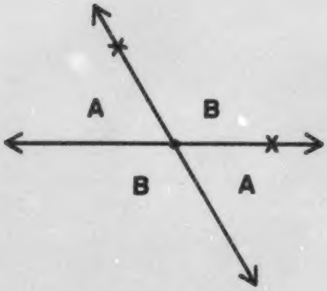
S: It is congruent to B. If you cover the top part of the drawing, we can see A and a supplementary angle, which must be B.



S: We could test it with template B.



Label the angle.

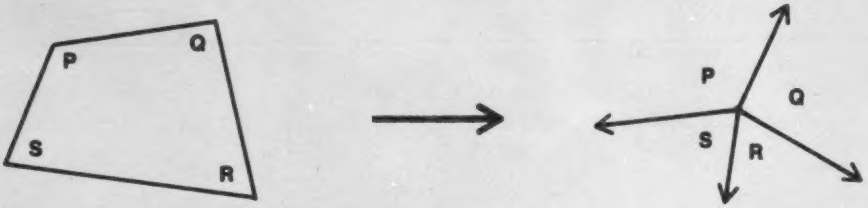


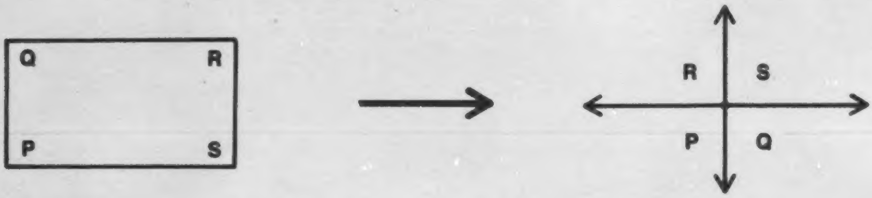
S: The four angles of that parallelogram bring us around in a circle.

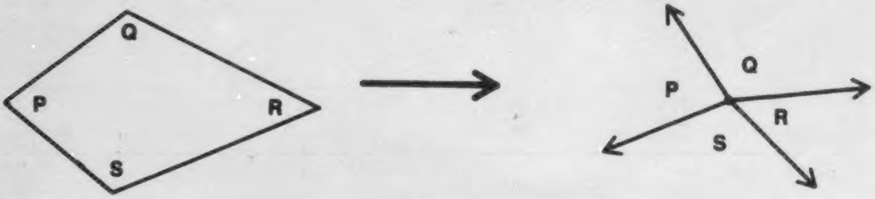
T: Now you try the same construction. Draw a quadrilateral. It doesn't have to be a parallelogram. Let's see whether your four angles "bring you around in a circle."

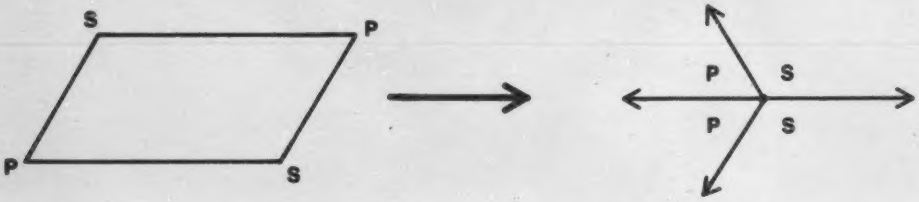
Circulate to assist students with the tracing paper, if necessary, as they trace the angles of their quadrilaterals. By adding angles consistently in a clockwise direction around the vertex and marking the common rays after each addition, they will be able to work systematically. As always, encourage careful, accurate drawings. Even a slight error will cause the experiment to fail.

Ask students who work quickly and accurately to try several other quadrilaterals. The collective experience of the class will support the inference that all quadrilaterals behave in this construction as does the parallelogram.









G 11

1.11 Introduction to the Protractor #1

CAPSULE LESSON SUMMARY

Introduce the protractor and its use in measuring angles and constructing angles of given measures. Discover that the sum of the measures of the angles of a triangle is 180°.

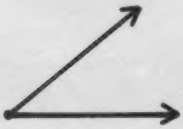
MATERIALS

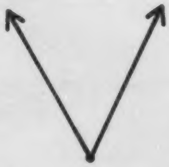
Teacher: Demonstration protractor; straightedge

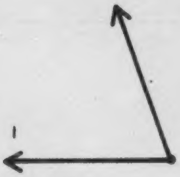
Student: Pencil; protractor; Worksheets G05 and G06; paper

DESCRIPTION OF LESSON

Before the lesson begins, use the demonstration protractor to draw three angles on the board. They should measure 40°, 55°, and 70°.







T: I have drawn three angles. How can we decide which is the largest and which is the smallest?

S: We could trace them on tracing paper and then compare them.

S: We could make a template to fit one so that we can compare it to the others.

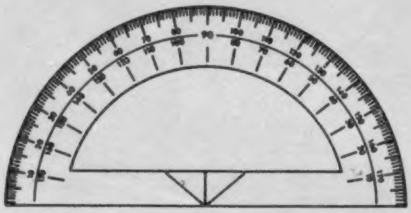
T: Those are both good ideas, but they are a little complicated. In one case, we would need tracing paper; in the other case we would need cardboard to make templates. Let's consider two line segments. (Draw two line segments on the board.) How do we decide which is longer?

S: We use a ruler to measure their lengths.

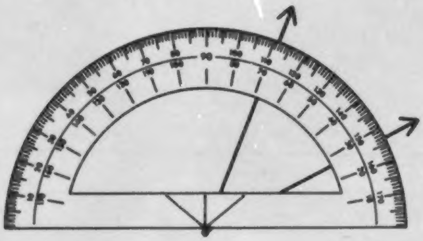
T: Of course. A ruler is marked with units and divisions of the units so that we can measure the length of one line segment, say in centimeters, and then measure the length of the second line segment, also in centimeters. Then we only need to compare the measures.

Here is a tool with which to measure angles. It is called a protractor.

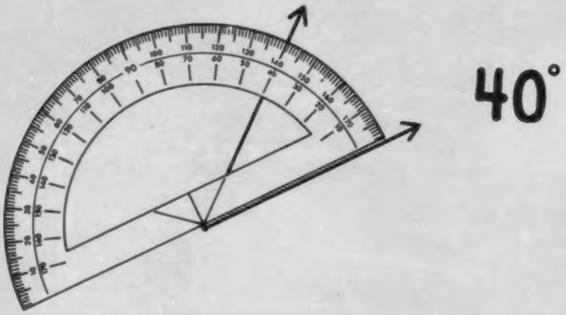
Display your demonstration protractor.



T: The protractor has a scale from 0 to 180. Each division is one degree. The degree is the unit for measuring angles. Now I will demonstrate how to measure one of these angles using the protractor. There is a mark in the center of the straight side of the protractor. I place it at the vertex of the angle I want to measure.

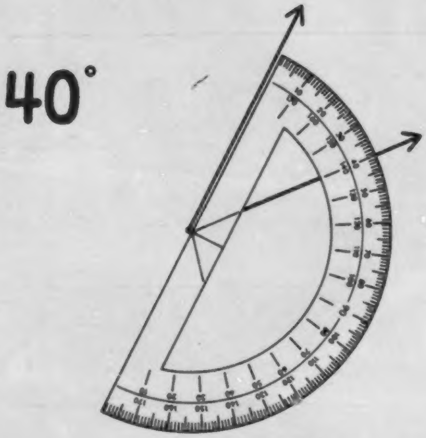


T: Then I align the edge of the protractor with one of the angle's rays (first ray) so that the other ray (second ray) cuts across the scale of the protractor. I count the number of a degrees starting with 0 (zero) at the first ray. The second ray crosses the scale at 40. So the angle measures 40 degrees.



T: There are numbers on the scale to help to count the degrees. In fact there are two rows of numbers. Watch out! We must be careful to use the correct row. Only one of them has 0 at the first ray. There are two scales because we can align the protractor with either of the two rays of the angle.

Demonstrate the alternative alignment and measurement.



Distribute protractors and Worksheet G05.

T: Use your protractor to measure each of the angles on the worksheet.

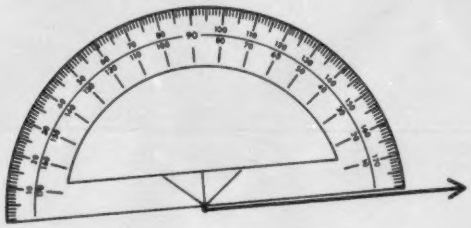
Monitor use of the protractor and correct faulty technique. When most of the class finish this task, check the angle measurements collectively. Then reopen the discussion.

T: In addition to measuring angles, we can use the protractor to construct angles. Suppose we want to construct an angle measuring 60°. How should we start?

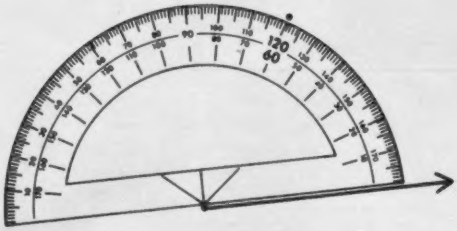
S: Pick a point for the vertex and draw a ray with that point as its endpoint.

Draw a ray on the board.

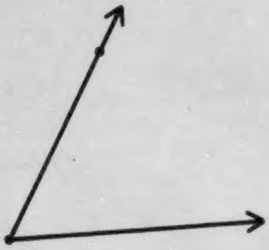
S: Now put the center mark of the protractor on the endpoint of the ray and align the protractor with the ray.



S: Now count from 0 degrees to 60 degrees on the scale and make a mark on the chalkboard at 60.



S: Draw a ray from the vertex through the mark.



Distribute Worksheet G06.

T: Here is a worksheet asking you to construct several angles with your protractor.

Again monitor the angle construction techniques. The most common mistake is to use the wrong scale. Simply remind students as necessary to begin counting degrees at 0. When most finish the constructions, resume the discussion.

T: Each of you draw a triangle. Measure all three angles of the triangle and add the measures.

Record the total for each student on the board. A typical record will be similar to the following.

181°	183°	180°	183°	186°
181°	185°	181°	180°	182°
178°	187°	187°	180°	180°

T: What do you think of these totals?

S: They are all 180° or close to 180°.

T: Can you give any reason for the differences?

S: We all drew different triangles.

S: That's true, but I think that it's hard to measure accurately. I couldn't decide whether to read 42° or 43° for one of my angles.

T: Different readings in measuring the angles would account for differences in the totals.

S: All of the totals are 180° or close to 180°, and the protractor scale is divided into 180 degrees.

S: Moving along the protractor is like going halfway around a circle.

T: When we were using templates, what did we discover when we fit the three angles of a triangle together?

S: They gave us a straight line, just as the straight edge of the protractor does.

S: So the three angles of a triangle together measure 180°.

T: Can you construct a triangle with angles measuring 40°, 60°, and 110°?

S: No. The total is 210° and that's too much.

T: If two angles of a triangle measure 40° and 60°, what does the third angle measure?

S: 80°.

T: A few weeks ago we used seven templates to construct triangles. How would you use a protractor to find all of the possible triangles?

G 11

S: I would first measure all seven angles. Then I would find which triples of angles give a total of 180°.

T: Here are the measures of the seven angles from the templates.

Write them on the board.

G	W	X	A	Y	Z	B
30°	40°	50°	60°	70°	90°	120°

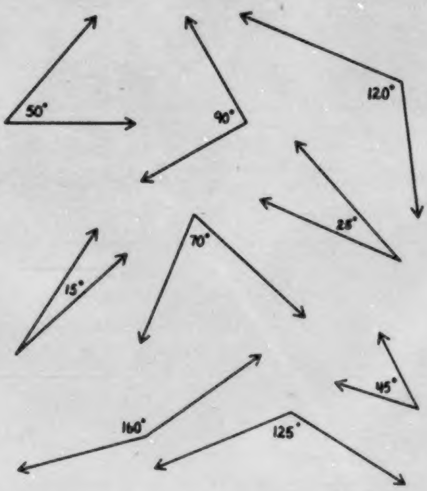
T: Which triples give triangles?

End the discussion with a list of the six triples which add to 180°.

B-G-G Z-X-W Z-A-G
Y-Y-W Y-A-X A-A-A

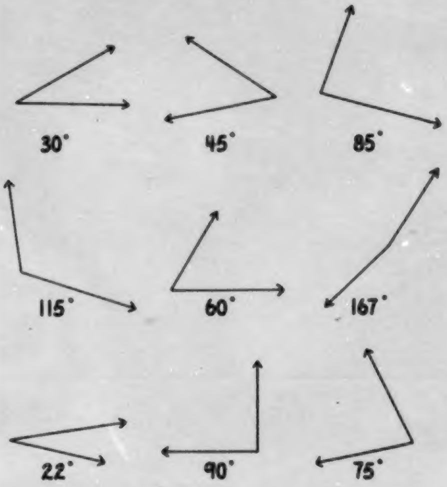
Name ______________________ **G 05**

Use a protractor to measure these angles.


50°
90°
120°
25°
15°
70°
45°
160°
125°

Name ______________________ **G 06**

Use a straightedge and a protractor to construct angles with the given measures.


30°
45°
85°
115°
60°
167°
22°
90°
75°

1.12 Introduction to the Protractor #2

CAPSULE LESSON SUMMARY

Discover relations among the measures of angles of polygons without indentations.

MATERIALS

Teacher: Demonstration protractor; straightedge
Student : Pencil; protractor; paper

DESCRIPTION OF LESSON

Begin the lesson with a brief review of the results of lesson G1.11.

T: Last week we used a protractor to measure angles. What did we learn about the three angles of a triangle?

S: If we measure the three angles and add the measures, the sum is 180°.

T: Did all of you get a sum of 180° when you worked with the triangle that you drew?

S: No, but that was because our measurements are not necessarily accurate. We were all close to 180°.

T: Try the same exercise again, but this time draw a quadrilateral instead of a triangle.

Allow time for individual work. Then record students' totals on the board. A typical set of results is shown here.

360°	358°	357°	356°	359°
355°	363°	360°	358°	361°
365°	362°	360°	364°	360°

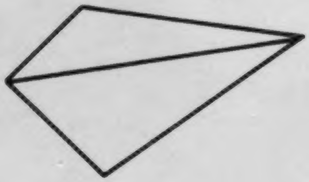
S: Most of the results are 360° or close to 360°.

T: Why do you think that 360° appears to be important for quadrilaterals?

S: We get 180° with a triangle and 360° with a quadrilateral.

S: I divided my quadrilateral into two pieces by drawing a segment between opposite corners. That made two triangles.

Sketch a similar drawing on the board.



S: There are six angles given by the two triangles. Together they measure $180° + 180° = 360°$. But if we think of the angles as fitting together to form the angles of the quadrilateral, all the measure of its angles is counted. So the angles of the quadrilateral together measure 360°.

T: Each of you construct a diagonal in your quadrilateral to make two triangles. Measure all of the angles of the two triangles and add the measures together.

Allow time for individual work.

T: Do you get the same total as before? (Yes)
What sum do you think we'll get if we add the measures of the angles of a pentagon with no indentations?

S: 540°, just add another 180°.

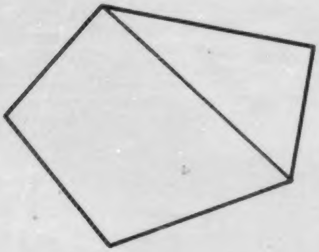
T: Each of you draw a pentagon without indentations and find the total of the measures of its angles.

G 12

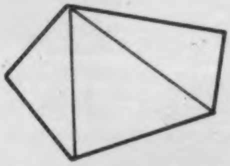
Allow time for individual work. Then let students share their results with the class.

T: Why do you think you are all getting sums of 540° or close to 540°?

S: I connected two corners of my pentagon to make a triangle and a quadrilateral. The angles of the triangle contribute 180° and the angles of the quadrilateral contribute 360°. So the sum of the measures of the angles of my pentagon should be 540°.



NOTE: Some students might indicate a division of the pentagon into three triangles and compute "3 × 180°" to justify the total of 540°. For example,



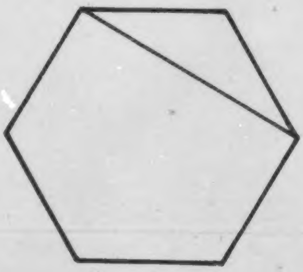
T: Let's make a table to show what information we have found.

number of sides	sum of angle measures
3	180°
4	360°
5	540°

T: What sum do you expect for a hexagon with no indentations?

S: Add another 180° in the chart; we can connect two vertices of a hexagon to make a pentagon and a triangle.

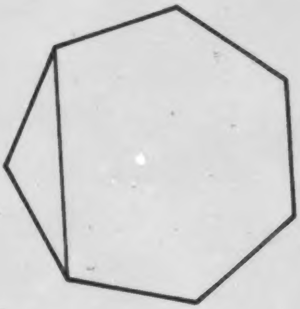
number of sides	sum of angle measures
3	180°
4	360°
5	540°
6	720°



T: And what about a septagon with no indentations?

S: Add another 180°.

number of sides	sum of angle measures
3	180°
4	360°
5	540°
6	720°
7	900°



T: What about a "kilogon" (1,000 sides)? Do you see a pattern?

S: For the sum of the measures, subtract 2 from the number of sides and then multiply times 180°.

S: For a kilogon — 998 x 180° = 179,640°.

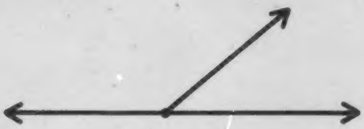
S: I see another pattern. Just take the number of sides times 180° and then subtract 360°.

Add this information to the table. Save the table for later reference.

number of sides	sum of angle measures
3	180°
4	360°
5	540°
6	720
7	900°
1,000	179,640°

T: Very good. Now who can construct a supplementary pair of angles?

Invite a student to draw the defining relation of a supplementary pair of angles on the board.



T: What does this picture suggest about the measures of a supplementary pair?

S: The sum of their measures is 180°.

T: What is the measure of a supplement to a 50° angle?

S: 130°.

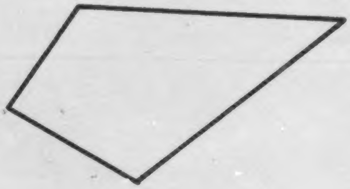
729

T: What is the measure of a supplement to a 135° angle?

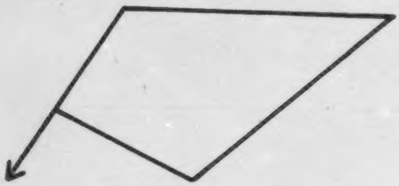
S: 45°.

Continue with a few more similar questions.

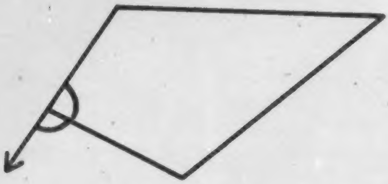
Draw a quadrilateral on the board. Then alter the drawing as in the next five pictures. Discuss the alterations as you perform them.



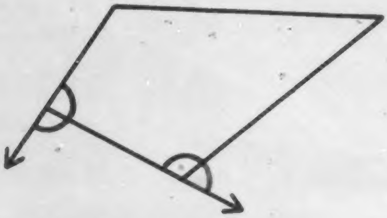
T: I extend one side of this quadrilateral to draw a ray.



T: Now I mark the angle of the quadrilateral in red and its supplement in blue.

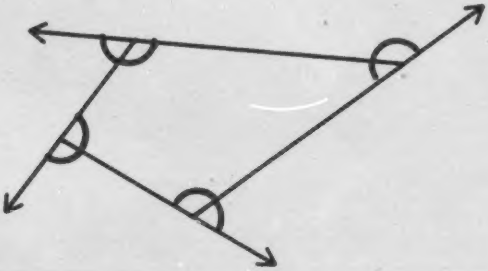


T: Then I do the same at the next corner.



G 12

T: I do the same again at the other two corners.



S: The rays make the drawing look like a pinwheel.

T: How many supplementary pairs do we have here?

S: Four.

T: And what is the sum of their measures?

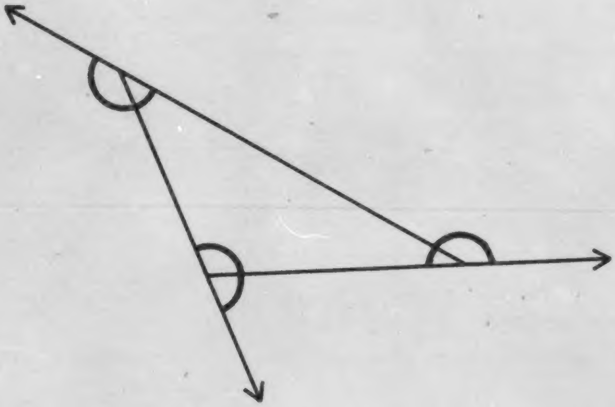
S: 720°; 4 × 180° = 720°.

Add another column to the table.

number of sides	sum of angle measures	
3	180°	
4	360°	720°
5	540°	
6	720°	
7	900°	
1,000	179,640°	

T: Let's do the same thing with a triangle.

Draw a triangle and repeat the process as pictured here.



T: How many supplementary pairs are there?

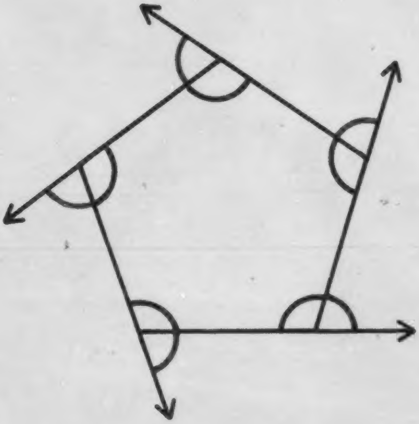
S: Three, and the sum of their measures is 540°; 3 × 180° = 540°.

Add the information to the table.

number of sides	sum of angle measures	
3	180°	540°
4	360°	720°
5	540°	
6	720°	
7	900°	
1,000	179,640°	

T: Let's do the same thing with a pentagon having no indentations.

number of sides	sum of angle measures	
3	180°	540°
4	360°	720°
5	540°	900°
6	720°	
7	900°	
1,000	179,640°	



T: And what about a hexagon (without indentations)? What do you predict?

S: The sum is always equal to the number of sides times 180°.

S: Of course, because there are as many corners as sides and we build a supplementary pair at each corner.

Complete the third column in the table.

T: Now let's go back to the quadrilateral. What is the sum of the measures of the blue angles? Use the table to compute.

S: The red angles are the angles of the quadrilateral and their measures add to 360°. The measures of the red and blue angles together add to 720°. So the measures of the blue angles must add to 360°.

Add another column to the table.

number of sides	sum of angle measures		sum of blue angle measures
3	180°	540°	
4	360°	720°	360°
5	540°	900°	
6	720°	1,080°	
7	900°	1,260°	
1,000	179,640°	180,000°	

T: What is the sum of the measures of the triangle's blue angles?

S: Red and blue together measure 540°. Red angles alone measure 180°. So the blue angles measure 360°.

S: The same is true for the pentagon and the hexagon.

number of sides	sum of angle measures		sum of blue angle measures
3	180°	540°	360°
4	360°	720°	360°
5	540°	900°	360°
6	720°	1,080°	360°
7	900°	1,260°	
1,000	179,640°	180,000°	

S: The same is always true. The number in the first column gives the total number of degrees for the red angles and the number in the second column gives the number of degrees for red and blue together.

G 12

We only have to subtract. So we always get 360° as the total number of degrees for the blue angles.

S: Even for the kilogon!

Complete the table.

PROBABILITY AND STATISTICS

Comprehensive School Mathematics Program

CSMP Mathematics for the Intermediate Grades, Part VI

Probability and Statistics

CEMREL, Inc.

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SUGGESTED CHRONOLOGICAL ORDER OF P-LESSONS

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PROBABILITY AND STATISTICS

INTRODUCTION

There are several reasons why people study mathematics. It is an interesting and challenging intellectual exercise when studied for its own sake. But it is also a supremely useful topic of study in that it has applications outside itself to other disciplines, as well as more generally to understanding the world around us. Some areas of mathematics are more easily applied than others, and the areas which particularly concern us in this strand are towards the easy end of the spectrum, both because of the small amount of background mathematical knowledge required before meaningful application can be made, and because it is usually fairly easy to see the link between the mathematical model and the "real" situation being modeled. For these reasons, it would be difficult to overstate the importance in today's world of probability and statistics and their applications to problems and situations within and outside the discipline of mathematics.

The lessons of Chapter 1 in this strand deal with systematic methods of analyzing games of chance and of finding numerical probabilities for outcomes or events. Situations which involve us in finding probabilities of certain events give rise to combinational problems and give support to our work with fractions in the World of Numbers strand. Furthermore, the situations employ logical and strategic thinking skills, so there is a further integration with the N-lessons and with the L-lessons. One lesson makes use of a geometric (measure) analysis for solving a probability problem.

Investigating topics of social interest, the lessons of Chapter 2 introduce new methods of graphing and analyzing statistical data. The lessons support real application of much of the work in the World of Numbers strand; for example, they make use of averages, ratios, percents, and a rounding operation.

1

PROBABILITY STORIES

Through their CSMP experiences, students have developed two methods of analyzing probability situations: a combinatorics approach which involves counting the number of favorable outcomes and the number of possible outcomes, and a geometric approach which involves dividing a square to reflect the choices and probabilities of a situation. The lessons in this chapter provide an opportunity to extend each of these techniques. In one lesson, the students' geometric analysis of a two-person game yields a paradoxical result. The second lesson employs Pascal's Triangle, originally generated in the story-workbook THE HIDDEN TREASURE, to reveal a relationship among several distinct situations. This brief chapter reflects a broad perspective of probability, both in terms of applications and solution techniques.

CAPSULE LESSON SUMMARIES

Section 1.1:	Analyze a two-person game played with four cubes to determine which player is favored and to find the best strategies. Discover some surprising results about the game which may be contraintuitive. (Worksheets P01, P02)
Section 1.2:	Review a method for counting the number of paths from a specified corner of a rectangular grid to each of its intersections. Use the results to solve a probability problem concerning Nabu's trips to the river to go fishing. Apply the same grid of numbers to determine the expected boy-girl distribution of the four oldest children in families with at least four children. Compare this expected distribution to the boy-girl distribution in your students' families. (Worksheets P03, P04, P05, P06)

P1

1.1 Four Number Cubes[†]

CAPSULE LESSON SUMMARY

. Analyze a two-person game played with four cubes to determine which player is favored and to find the best strategies. Discover some surprising results about the game which may be contra-intuitive.

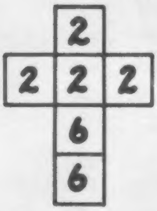
MATERIALS

Teacher: Colored chalk; four number cubes

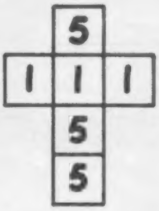
Student : Paper; ruler; Worksheets P01 and P02

NOTE: Before the lesson begins, put labels on the four color cubes as shown below.

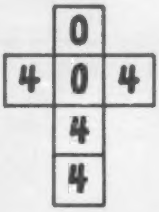
Red



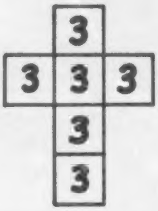
Blue



Yellow



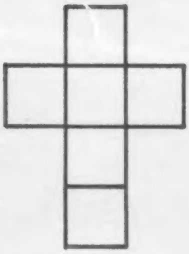
Green



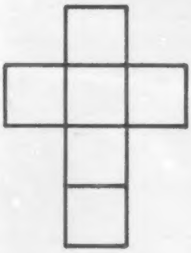
DESCRIPTION OF LESSON

Draw four maps of a cube on the board.

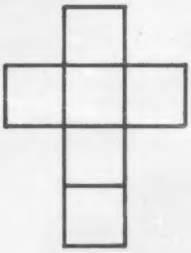
Red



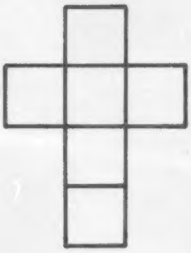
Blue



Yellow



Green



NOTE: Control the length of discussions and pace of the lesson so that the students begin their worksheets by the middle of the class period.

[†] These cubes were designed by Bradley Efron, a statistician at Stanford University. See Martin Gardner's "Mathematical Games", Scientific American, December 1970, page 110.

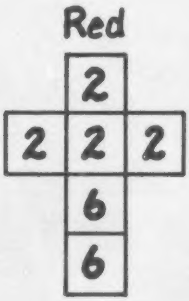
T: I have a game to show you. We play it with these four cubes.

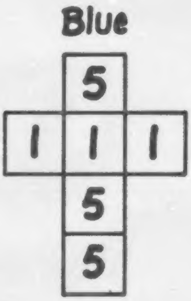
Hold up each of the four cubes for the class to observe. Select a student, say Kathy, to play the first game.

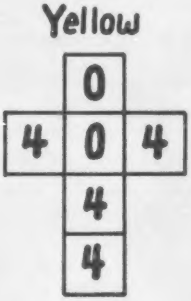
T: I will choose one of the four cubes and then you, Kathy, can select any one of the other three cubes. We'll roll the cubes we have chosen and the person who gets the higher number wins.

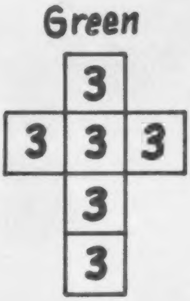
Play the game several times. Emphasize that each time you, the teacher, are first to select a cube. Encourage students to notice the different distributions of numbers on the cubes.

Give a cube to each of four students. Ask them to tell you how to label each map of the cube on the board. (The face positions of the numerals on each map is not important for this lesson.)









T: Do you think it is a fair game?

Encourage students to explain why they think that the game is fair or that the game favors either the teacher or the student. Students may suggest ways to make the game fair. Accept those suggestions, but remind the students that your question concerns whether this game is fair. After the discussion you may wish to take a class tally to determine their opinions.

P 1

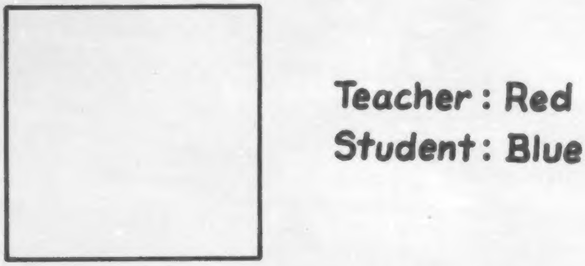
T: If you feel that the teacher is favored, what is the teacher's best strategy?

If you feel that the student is favored, what is the student's best strategy?

Discuss strategies the students feel each player should use.

T: Now we will use probability methods to determine who is favored in this game and to learn each player's best strategy.

Draw the following picture on the board.



T: Let's use this square to analyze the game supposing that the teacher selects the red cube and the student chooses the blue cube. Does it matter who rolls first?

S: No.

T: Let's say the teacher rolls first. What are the possible outcomes?

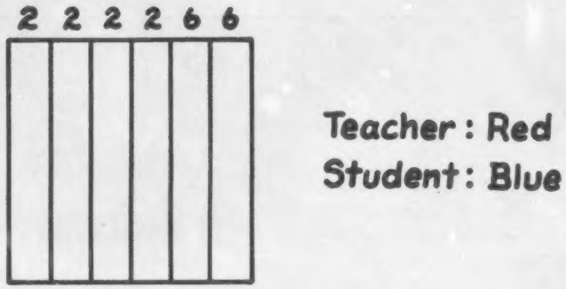
S: The teacher could roll a "2" or a "6".

T: How could I show this on the square?

S: Divide the square into six parts of the same size for the cube's six faces. Label four of the parts "2" and label the other two parts "6".

S: You could divide the square into three parts of the same size and label two parts "2" and one part "6".

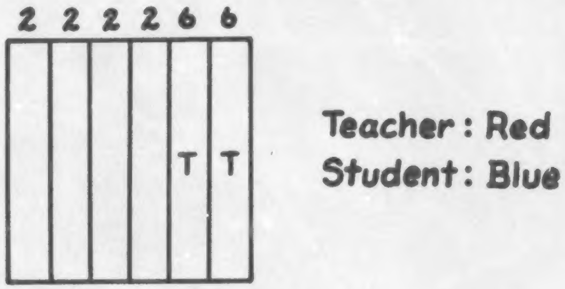
T: Either way would be okay, but we'll use six parts to remind us of the six faces of the cube.


2 2 2 2 6 6
Teacher : Red
Student : Blue

T: What happens if the teacher rolls a "6"?

S: The teacher wins for sure.

Label the two regions for "6" with a "T".


2 2 2 2 6 6
T T
Teacher : Red
Student : Blue

Point to one of the regions labeled "2".

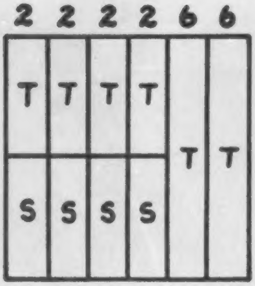
T: What happens if the teacher rolls a "2"?

S: The student wins about half the time since the blue cube has three "1"'s and three "5"'s.

T: How can I show that on the square?

P 1

S: Divide each region for "2" in half. Label one-half "T" and one-half "S".



Teacher : Red
Student : Blue

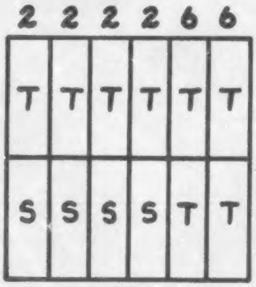
T: Who is favored if the teacher selects the red cube and the student chooses the blue cube?

S: The teacher, since the teacher has 8 chances out of 12 to win and the student has only 4 chances out of 12.

T: What is each player's probability of winning?

S: $\frac{8}{12}$ or $\frac{2}{3}$ for the teacher; $\frac{4}{12}$ or $\frac{1}{3}$ for the student.

Dividing the regions for "6" in half yields all pieces the same size. Eight pieces are labeled "T" and four pieces are labeled "S". Add the following information to the picture.



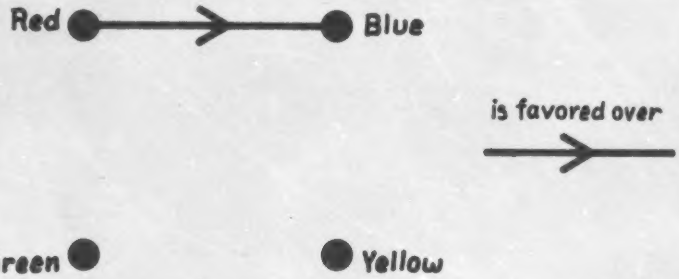
Teacher : Red $\frac{8}{12} = \frac{2}{3}$

Student : Blue $\frac{4}{12} = \frac{1}{3}$

$p(R,B) = \frac{2}{3}$ $p(B,R) = \frac{1}{3}$

T: I read this (point to "p (R,B) = $\frac{2}{3}$") as "the probability of red beating blue is $\frac{2}{3}$." I read this (point to "p (B,R) = $\frac{1}{3}$") as "the probability of blue beating red is $\frac{1}{3}$."

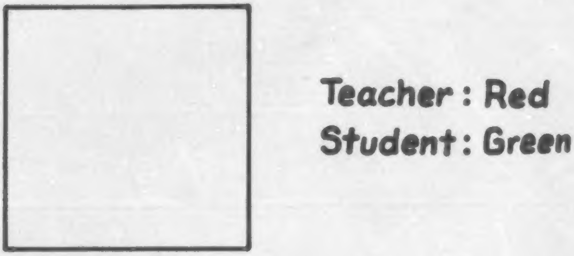
Draw the following picture on the board.



T: I'll use an arrow picture to record the fact that the red cube is favored over the blue cube in this game. We now know that I am favored if I select the red cube and a student picks the blue cube. Do you think that the green cube or the yellow cube would be a better choice for the student?

S: I think that the green cube is favored over the red cube.

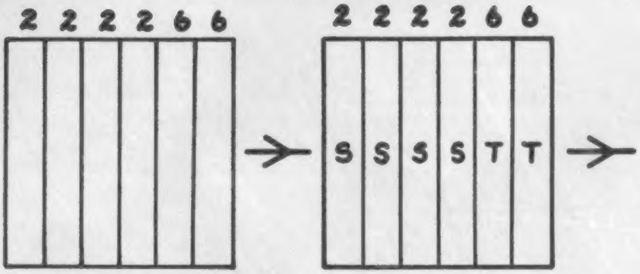
T: Let's analyze the game supposing that the teacher selects the red cube and the student chooses the green cube.



Analyze this game in a manner similar to the previous game.

P 1

The following sequence of squares indicates the steps your class might suggest.


2 2 2 2 6 6
2 2 2 2 6 6
S S S S T T

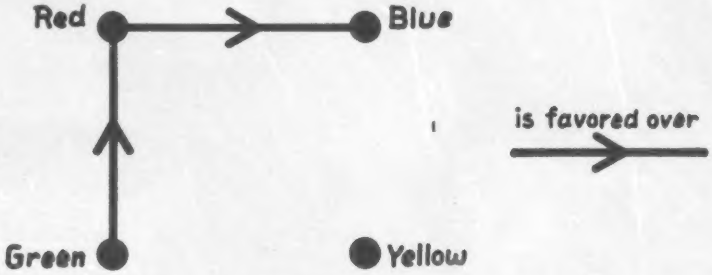
Teacher : Red $\frac{2}{6} = \frac{1}{3}$

Student : Green $\frac{4}{6} = \frac{2}{3}$

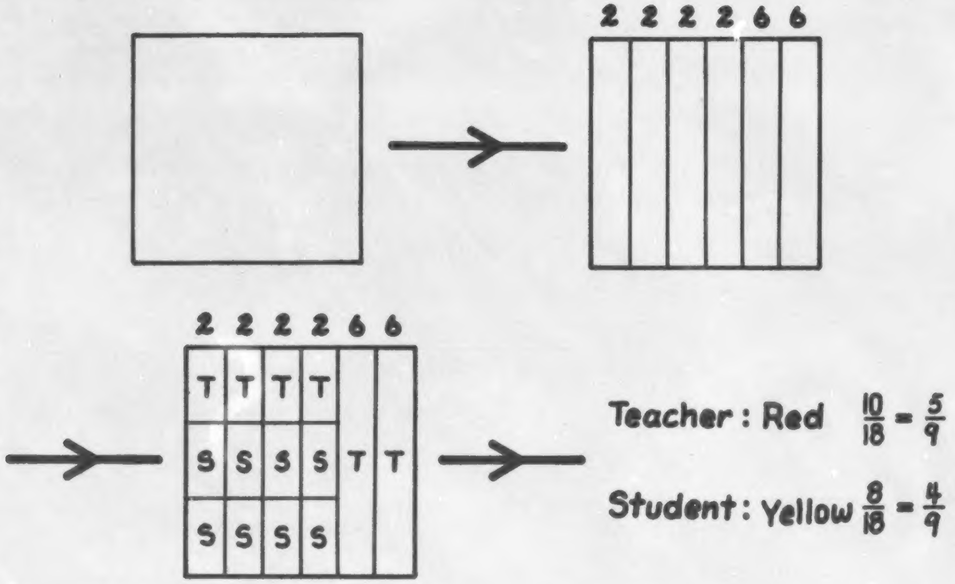
$p(G, R) = \frac{2}{3}$ $p(R, G) = \frac{1}{3}$

T: How can I indicate this result in the arrow picture?

S: Draw a red arrow from green to red.


Red
Blue
is favored over
Green
Yellow

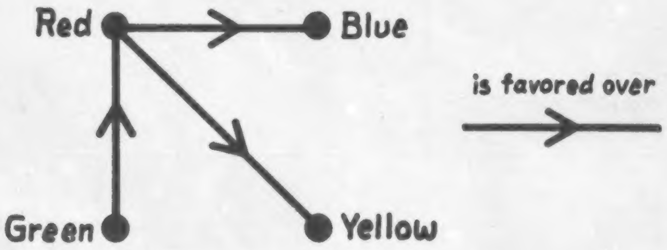
In a similar manner analyze the game in the case in which the teacher selects the red cube and the student chooses the yellow cube.


2 2 2 2 6 6
2 2 2 2 6 6
T T T T
S S S S T T
S S S S

Teacher : Red $\frac{10}{18} = \frac{5}{9}$

Student : Yellow $\frac{8}{18} = \frac{4}{9}$

$p(R, Y) = \frac{5}{9}$ $p(Y, R) = \frac{4}{9}$


Red
Blue
Green
Yellow
is favored over

Call the class' attention to the arrow picture.

T: In this game, is it a good strategy for the teacher to select the red cube?

S: No, because the student could choose the green cube and would be favored to win.

T: Which cube might be better for the teacher?

S: The green cube, because the green cube is favored over the red cube.

P 1

T: Do you think there are any other red arrows that could be drawn in the picture?

Among their comments students may suggest that you could draw a red arrow from green to blue since green is favored over red and red is favored over blue. Similarly, they may suggest drawing a red arrow from green to yellow. Indicate that these are interesting ideas and they might be right, but that these suggestions need to be checked by further analysis of the game.

T: You'll complete the analysis of this game on worksheets. I'll divide the class into three groups to reduce the amount of work each of you has to do.

Distribute Worksheets P01 and P02. Divide the class into three parts and designate them "Group #1", "Group #2", and "Group #3". Write the following information on the board.

Group #1	Group #2	Group #3
Teacher selects the blue cube.	Teacher selects the yellow cube.	Teacher selects the green cube.
$p(B,G) =$ ___	$p(Y,G) =$ ___	$p(G,B) =$ ___
$p(B,Y) =$ ___	$p(Y,B) =$ ___	$p(G,Y) =$ ___

T: Each of you has two problems to solve, one problem for each worksheet. Those in Group #1 should assume that the teacher selects the blue cube. You must calculate the probability of the blue cube winning over the yellow cube and the probability of the blue cube winning over the green cube.

Referring to the board, explain in a similar manner the responsibilities of the students in Group #2 and in Group #3.

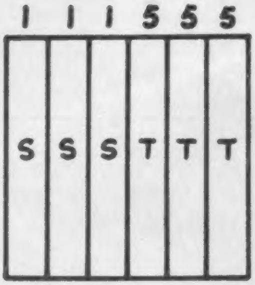
T: On your worksheets, write your group number and use the squares to solve the two problems assigned to your group.

Observe your students' individual work and assist students needing help. If some students finish both of their problems quickly suggest that they solve the problems from another group. The following pictures indicate possible solution techniques for the six problems.

Group #1

Teacher selects the blue cube (1,1,1,5,5,5).

Student selects the green cube (3,3,3,3,3,3).



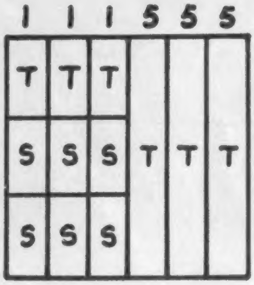
Teacher: $\frac{3}{6} = \frac{1}{2}$

Student: $\frac{3}{6} = \frac{1}{2}$

$p(B, G) = \frac{1}{2}$

$p(G, B) = \frac{1}{2}$

Student selects the yellow cube (0,0,4,4,4,4).



Teacher: $\frac{12}{18} = \frac{2}{3}$

Student: $\frac{6}{18} = \frac{1}{3}$

$p(B, Y) = \frac{2}{3}$

$p(Y, B) = \frac{1}{3}$

P 1

Group #2

Teacher selects the yellow cube (0,0,4,4,4,4).

Student selects the green cube (3,3,3,3,3,3).

0	0	4	4	4	4
S	S	T	T	T	T

Teacher : $\frac{4}{6} = \frac{2}{3}$

Student : $\frac{2}{6} = \frac{1}{3}$

$p(Y, G) = \frac{2}{3}$

$p(G, Y) = \frac{1}{3}$

Student selects the blue cube (1,1,1,5,5,5).

0	0	4	4	4	4
S	S	T	T	T	T
		S	S	S	S

Teacher : $\frac{4}{12} = \frac{1}{3}$

Student : $\frac{8}{12} = \frac{2}{3}$

$p(Y, B) = \frac{1}{3}$

$p(B, Y) = \frac{2}{3}$

Group #3

Teacher selects the green cube (3,3,3,3,3,3).

Student selects the blue cube (1,1,1,5,5,5).

3	3	3	3	3	3
T	T	T	T	T	T
S	S	S	S	S	S

Teacher : $\frac{6}{12} = \frac{1}{2}$

Student : $\frac{6}{12} = \frac{1}{2}$

$p(G, B) = \frac{1}{2}$

$p(B, G) = \frac{1}{2}$

Student selects the yellow cube (0,0,4,4,4,4).

3	3	3	3	3	3
T	T	T	T	T	T
S	S	S	S	S	S
S	S	S	S	S	S

Teacher : $\frac{6}{18} = \frac{1}{3}$

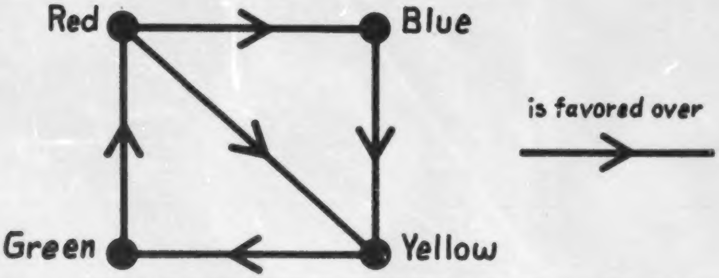
Student : $\frac{12}{18} = \frac{2}{3}$

$p(G, Y) = \frac{1}{3}$

$p(Y, G) = \frac{2}{3}$

When about ten minutes of the class period remain, stop the students' individual work in order to collect and discuss their results. Ask each group for the solutions to their two problems. If there is not a consensus on the correct solution to a problem, quickly solve the problem at the board. Record the results on the board and invite students to add arrows to the arrow picture whenever possible.

Group #1	Group #2	Group #3
$p(B,G) = \frac{1}{2}$	$p(Y,G) = \frac{2}{3}$	$p(G,B) = \frac{1}{2}$
$p(B,Y) = \frac{2}{3}$	$p(Y,B) = \frac{1}{3}$	$p(G,Y) = \frac{1}{3}$



NOTE: Since $p(B,G) = \frac{1}{2}$, a red arrow cannot be drawn between the dots for the blue cube and the green cube.

As the results are recorded, students may make several observations:

- $p(B,Y) = \frac{2}{3}$ and $p(Y,B) = \frac{1}{3}$ both imply that a red arrow can be drawn from blue to yellow. Similarly, $p(Y,G) = \frac{2}{3}$ and $p(G,Y) = \frac{1}{3}$ both imply that a red arrow can be drawn from yellow to green.
- $p(B,Y) + p(Y,B) = \frac{2}{3} + \frac{1}{3} = 1$; $p(Y,G) + p(G,Y) = \frac{2}{3} + \frac{1}{3} = 1$; $p(B,G) + p(G,B) = \frac{1}{2} + \frac{1}{2} = 1$.
- Even though green is favored over red and red is favored over blue, green is not favored over blue. Similarly, green is favored over red and red is favored over yellow, but green is not favored over yellow.

P 1

T: We have analyzed all the possible games with these four cubes. Who is favored, the teacher or the student?

S: The student, because no matter which cube the teacher selects the student can choose a cube which is favored over the teacher's cube.

T: Using the arrow picture, tell me what the student should do for each cube I could select. For example, what if I select the red cube . . . ?

S: If you select the red cube, the student should choose the green cube.

S: If you select green, the student should choose yellow.

S: If you select yellow, the student should choose red or blue.

S: If you select blue, the student should choose red.

1.2 Probability Problems

CAPSULE LESSON SUMMARY

Review a method for counting the number of paths from a specified corner of a rectangular grid to each of its intersections. Use the results to solve a probability problem concerning Nabu's trips to the river to go fishing. Apply the same grid of numbers to determine the expected boy-girl distribution of the four oldest children in families with at least four children. Compare this expected distribution to the boy-girl distribution in your students' families.

MATERIALS

Teacher: Probability Posters #1 and #2 for IG-VI; felt-tip markers
Student: Worksheets P03, P04, P05, and P06

NOTE: Before teaching this lesson, take the following survey in some class other than your own. Instruct all the students with at least three brothers and/or sisters to stand up. Ask each of the standing students the following question: "Among the four oldest children, possibly including yourself, in your family, how many girls?" Record the answers for use in Exercise 2 of this lesson. For example,

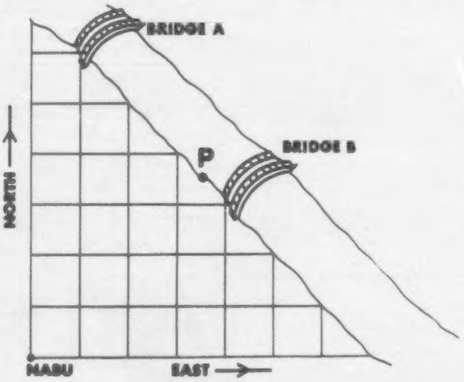
Mr. Lionel's class:

number of girls	0	1	2	3	4
number of families	1	3	3	0	1

DESCRIPTION OF LESSON

Exercise 1

Display Probability Poster #1 for IG-VI.



Do not write the letter "P" on the poster. It is here just to facilitate the lesson description.

T: This is a map showing the streets in part of Nabu's home town. He lives at the intersection labeled "Nabu". Nabu enjoys fishing in the river. How far does Nabu live from the river?

S: Seven blocks if he walks along the streets.

S: It's shorter if he walks straight to the river (from "Nabu" to "P"), how far does he walk?

A student might suggest $3\frac{1}{2}$ blocks by counting the number of diagonals from Nabu to P. If this occurs, point out that each diagonal is longer than one block. Therefore, the straight line walk from "Nabu" to "P" is longer than $3\frac{1}{2}$ blocks.

T: Can anyone figure out a way to measure the length in blocks of Nabu's walk from his home to here? (Trace a line segment from "Nabu" to "P".)

Many methods are possible. One solution is to measure the distance from "Nabu" to "P" with a ruler or board compass and then determine by comparison the approximate number of blocks that this length represents. The length of a line segment from "Nabu" to "P" is about five blocks.

T: When Nabu walks to the river he does walk along the streets. Which route would he choose, if he wants a shortest walk?

S: Straight east, seven blocks.

S: Straight north, seven blocks.

S: It doesn't matter. As long as Nabu only walks north or east, all of the routes are seven blocks; that is, he always walks toward the river.

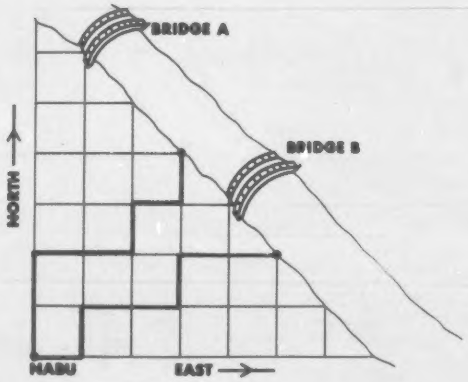
T: Usually, Nabu doesn't care where he fishes along the river. As he walks to the river he flips a coin at each street intersection. If he gets a "head", he walks east one block. If he gets a "tail", he walks north one block. How many times will he flip the coin on a walk to the river?

S: Seven.

Write **TTHHTHT** on the board.

T: If Nabu flips heads and tails in this order, which route would he take to the river?

Invite a student to trace the appropriate path, shown in red in the next illustration. Similarly, invite a student to trace a path for **HTHHTHH** , shown in blue.



T: What series of coin flips would allow Nabu to end at Bridge A?

Any series with six **T**'s and one **H** could land Nabu at Bridge A; for example, **TTTHTTT**.

T: How could Nabu land at Bridge B?

P 4

Any series with three **T**'s and four **H**'s would land Nabu at Bridge B; for example, **HHTHHTT**

T: Sometimes the fishing is better under the bridges. If Nabu uses his coin flipping method, what do you think Nabu's probability is of arriving at one of the bridges?

Let students express their opinions and give estimates. Write the following information on the board.

$$p(\text{Bridge A}) = \underline{\qquad\qquad\qquad}$$

T: After walking seven blocks, Nabu arrives at the river. What two quantities must we calculate to determine the probability that he arrives at Bridge A?

S: The number of paths to Bridge A.

S: The total number of paths to the river.

Complete the equation on the board.

$$p(\text{Bridge A}) = \frac{\text{number of paths to A}}{\text{total number of paths to the river}}$$

T: How many different seven-block paths are there from Nabu's house to Bridge A?

S: Seven.

Invite a student to trace the seven paths from Nabu's house to Bridge A.

T: How many seven-block paths are there to the river?

Invite students to suggest methods of determining this number.

Encourage suggestions that they have solved similar problems in earlier lessons by constructing a grid of numbers. Once the students remember this, distribute Worksheet P03 and display Probability Poster #2.

T: This is the grid of numbers we've used before. What do the numbers on the grid represent?

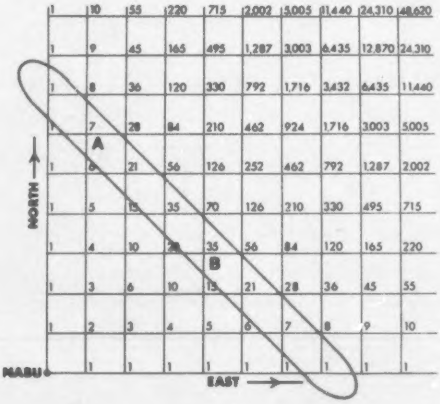
S: They indicate the number of paths from Nabu's house to each intersection.

T: Which intersections on the grid represent places on the river in Nabu's town?

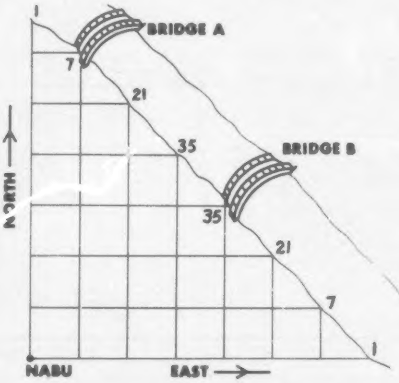
S: The intersections which are exactly seven blocks from Nabu's home.

S: The intersections along the diagonal labeled "1, 7, 21, 35, 35, 21, 7, 1".

Highlight this diagonal on the grid. Ask someone to locate the intersections for Bridge A and Bridge B and label them. On the map, invite a student to mark the number of paths to each intersection on the river.



1	10	55	220	715	2,002	5,005	11,440	24,310	48,620
1	9	45	165	495	1,287	3,003	6,435	12,870	24,310
1	8	36	120	330	792	1,716	3,432	6,435	11,440
1	7	28	84	210	462	924	1,716	3,003	5,005
1	6	21	56	126	252	462	792	1,287	2,002
1	5	15	35	70	126	210	330	495	715
1	4	10	20	35	56	84	120	165	220
1	3	6	10	15	21	28	36	45	55
1	2	3	4	5	6	7	8	9	10
	1	1	1	1	1	1	1	1	1



P 4

T: How many paths are there from Nabu's house to the river?

S: 128, $1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128$.

S: $2^7 = 128$.

T: Of those 128 paths to the river, how many end at Bridge A?

S: 7.

T: What is Nabu's probability of arriving at Bridge A?

S: 7 chances out of 128 or $\frac{7}{128}$.

Write the following information on the board.

$$p(\text{Bridge A}) = \frac{\text{number of paths to A}}{\begin{array}{c}\text{total number of paths}\\ \text{to the river}\end{array}} = \frac{7}{128}$$

$$p(\text{Bridge B}) =$$

T: What is Nabu's probability of arriving at Bridge B?

S: $\frac{35}{128}$. The grid indicates that there are 35 paths from Nabu's house to the river.

Complete: $p(\text{Bridge B}) = \frac{35}{128}$.

T: What is the probability that Nabu arrives at either Bridge A or Bridge B?

S: $\frac{42}{128}$, since $7 + 35 = 42$.

T: What simple fraction is close to $\frac{42}{128}$?

S: $\frac{42}{128}$ is about $\frac{1}{3}$ since $3 \times 42 = 126$.

T: Therefore, Nabu's probability of arriving at one of the bridges is about $\frac{1}{3}$.

Compare your students' estimates to $\frac{1}{3}$.

Leave Poster #2 with the grid of numbers on the board. Poster #1 is no longer needed.

Exercise 2

Have the survey data from another class available for this exercise. The dialogue uses the data from Mr. Lionel's class, given at the beginning of this lesson.

Draw the following table on the board.

number of of girls	number of families: our class			
0				
1				
2				
3				
4				
total				

T: How many of you have at least three brothers and sisters? Please stand up if you do.

P 4

Ask each of the standing students the following question and record the data in the table.

T: Consider the four oldest children in your family. Count yourself if you are one of the four oldest. How many girls are there among those four oldest children?

The following data was collected in one of our development classes.

number of of girls	number of families: our class			
0	1			
1	0			
2	3			
3	2			
4	1			
total	7			

Invite comments on any characteristics of your data.

T: Do you think that these results are quite normal or quite unusual? I asked the same question in Mr. Lionel's class. Do you expect the data from his class to be exactly the same, nearly the same, or quite different?

Let students respond to these questions. Then enter the data from Mr. Lionel's class into the table. Invite comments on similarities or differences between the two classes. Then total the results from the two classes.

number of of girls	number of families: our class	number of families: other class	total	
0	1	1	2	
1	0	3	3	
2	3	3	6	
3	2	0	2	
4	1	1	2	
total	7	8	15	

T: It would be interesting to know whether the survey results in our two classes are normal or unusual.

Let's consider some different orderings of boys and girls among the four oldest children in a family. For example, a family could have two boys, a girl, and then another boy.

Write **BBGB** on the board.

T: What are some other orderings of the first four children?

Write a few of the student's suggestions on the board.

BBGB **GBBB** **GGGG**

P 4

T: How many orderings would represent a family with three girls and one boy?

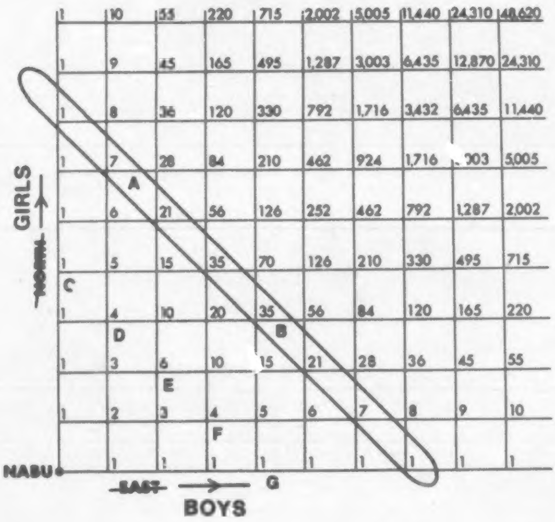
S: Four: **GGGB** , **GGBG**, **GBGG** , and **BGGG** .

T: How many would represent a family with four girls?

S: Only one: **GGGG** .

T: So there are more ways for a family to have three girls and one boy than to have four girls. We could count the types of families with zero, one, or two girls in a similar way, but there is also a way to do this using the grid of numbers on the poster.

On the poster, cross out "EAST" and write "BOYS". Also cross out "NORTH" and write "GIRLS".



1	10	55	220	715	2,002	5,005	11,440	24,310	48,620
1	9	45	165	495	1,287	3,003	6,435	12,870	24,310
1	8	36	120	330	792	1,716	3,432	6,435	11,440
1	7	28	84	210	462	924	1,716	[illegible]003	5,005
1	6	21	56	126	252	462	792	1,287	2,002
1	5	15	35	70	126	210	330	495	715
1	4	10	20	35	56	84	120	165	220
1	3	6	10	15	21	28	36	45	55
1	2	3	4	5	6	7	8	9	10
NABU	1	1	1	1	1	1	1	1	1

T: Which dot on this grid represents a family with three girls and one boy?

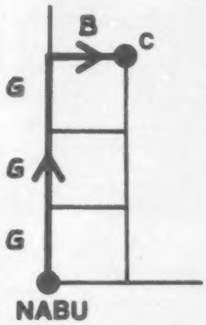
S: Here (D) because from the dot labeled "Nabu" you go up three and over one.

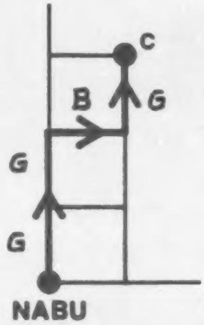
T: What does the "4" written near this dot (D) represent?

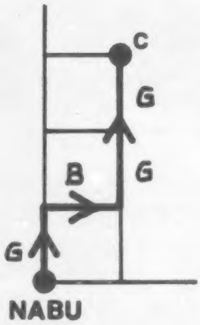
S: There are four different paths from Nabu's house to that dot (D).

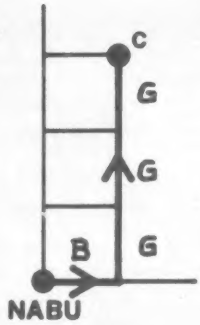
S: There are four different orderings by which a family could have three girls and one boy.

Let a student trace the four paths from "Nabu" to "D" reading them as: **GGGB**, **GGBG**, **GBGG**, and **BGGG** to indicate the four orderings with three girls and one boy.









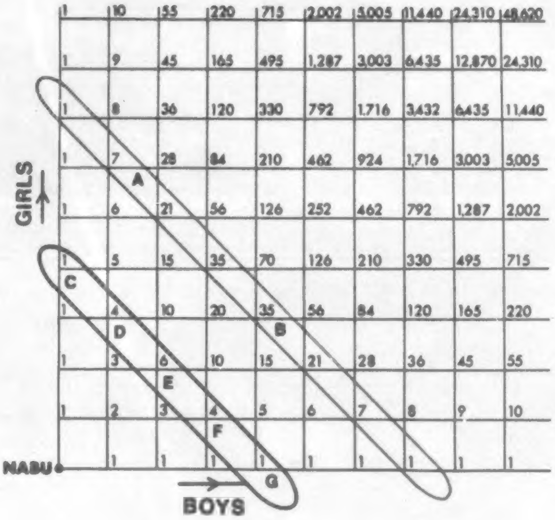
T: Which other dots on this grid represent families with four children?

Students should indicate dots C, E, F, and G and reach the following conclusions about these dots.

- C represents families with four girls and no boy. This can occur in only one way.
- E represents families with two girls and two boys. This can occur in six ways.
- F represents families with one girl and three boys. This can occur in four ways.
- G represents families with no girl and four boys. This can occur in one way.

P 4

Circle the dots C, D, E, F, and G on the poster.



1	10	55	220	715	2,002	5,005	11,440	24,310	48,620
1	9	45	165	495	1,287	3,003	6,435	12,870	24,310
1	8	36	120	330	792	1,716	3,432	6,435	11,440
1	7	28	84	210	462	924	1,716	3,003	5,005
1	6	21	56	126	252	462	792	1,287	2,002
1	5	15	35	70	126	210	330	495	715
1	4	10	20	35	56	84	120	165	220
1	3	6	10	15	21	28	36	45	55
1	2	3	4	5	6	7	8	9	10
	1	1	1	1	1	1	1	1	1

T: Let's record the expected number of families of each type.[†]

number of of girls	number of families: our class	number of families: other class	total	expected number of families
0	1	1	2	1
1	0	3	3	4
2	3	3	6	6
3	2	0	2	4
4	1	1	2	1
total	7	8	15	16

[†] In using the grid of numbers to determine the expected number of families of each type, an assumption is made that the probability of having a boy is the same as the probability of having a girl. In fact, this is not completely accurate based on empirical data, but the probabilities are close enough to 1/2 to make that assumption here.

Invite students to compare the numbers in the last two columns: the actual distribution of families in the two classes versus the mathematically expected distribution. For example, the preceding results from one of our development classes were very similar to the expected distribution.

Distribute Worksheet P04. Answer Question #1 collectively.

T: Which intersection on our graph represents families with exactly six girls and two boys?

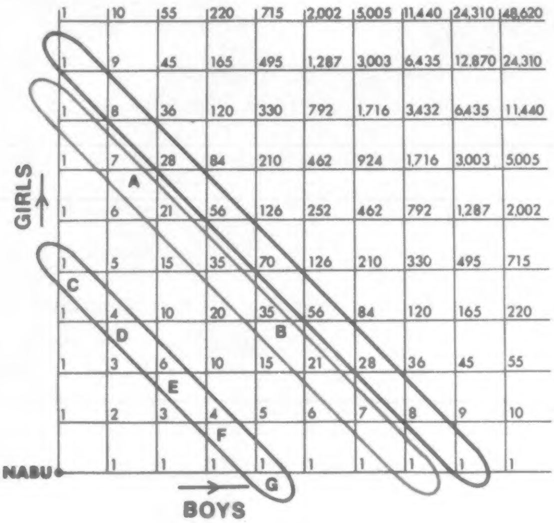
S: The intersection labeled "28" which is six up and two across from the original labeled "Nabu".

T: How many ways are there for a family to have six girls and two boys?

S: 28.

T: Which other intersections represent families with exactly eight children?

Circle the intersections as the students indicate them.



1	10	55	220	715	2,002	5,005	11,440	24,310	48,620
1	9	45	165	495	1,287	3,003	6,435	12,870	24,310
1	8	36	120	330	792	1,716	3,432	6435	11,440
1	7	28	84	210	462	924	1,716	3,003	5,005
1	6	21	56	126	252	462	792	1,287	2,002
1	5	15	35	70	126	210	330	495	715
1	4	10	20	35	56	84	120	165	220
1	3	6	10	15	21	28	36	45	55
1	2	3	4	5	6	7	8	9	10
NABU	1	1	1	1	1	1	1	1	1

P 4

T: How many different ways can the boys and girls be ordered in a family with eight children?

S: 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 = 256.

S: $2^8 = 256$.

T: What is the probability that a family with eight children has six girls and two boys?

S: 28 chances out of 256.

Let students complete Worksheet P04 individually. Worksheets P05 and P06 are also available. You may wish to discuss some of the solutions collectively.

Name ____________________ **P04** *

1. Find the probability that a family with eight children has exactly six girls and two boys. $\frac{7}{64}$

$$\frac{28}{2^8} = \frac{28}{256} = \frac{7}{64}$$

2. Find the probability that a family with eight children has exactly six boys and two girls. $\frac{7}{64}$

3. Find the probability that a family with eight children has at least seven girls. $\frac{9}{256}$

4. Find the probability that a family with eight children has more boys than girls. $\frac{93}{256}$

Name ____________________ **P05** **

Use the available space to show your work.

1. Bernie forgot to read a chapter in his history book and his history teacher gave a surprise test the next day. The test had ten true - false questions. Bernie simply guessed at each answer. Find the probability that Bernie scored at least 75% on the test. $\frac{7}{128}$

$$\frac{56}{2^{10}} = \frac{56}{1{,}024} = \frac{7}{128}$$

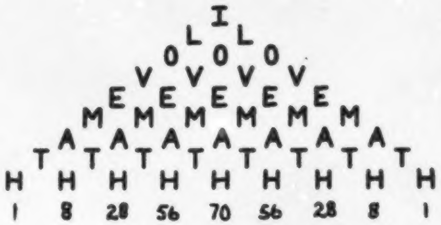
2. Sarah took the same history test with ten true - false questions. She knew the right answer for five of the questions, but had to guess on the other five questions. Find the probability that Sarah scored at least 75% on the test. $\frac{1}{4}$

$$\frac{16}{2^5} = \frac{16}{64} = \frac{1}{4}$$

Name ____________________ **P06** ***

After complaining loudly about a math assignment, Fred was told by his math teacher to write "I LOVE MATH" 250 times.

Fred, being clever, wrote the following:



Did Fred complete his assignment? Yes Explain. The number of ways to spell "I LOVE MATH" using this picture is 2^8 or 256.

(Hint: How many different ways can "I LOVE MATH" be spelled using the picture?)

2

POPULATION GROWTH

Internationally, population growth causes a major impact on the policies of many governments. Domestically, the changing age distribution of the United States population affects educational systems and senior citizens' programs. A prerequisite to comprehending the implications of these phenomena is an understanding of the population statistics which reveal changes.

The lessons in this chapter introduce methods of graphing and analyzing United States population data for 1910-1975. Since the various graphs are for the same era, the students discover relationships among the graphs and at the same time learn that each graph provides a distinct perspective on recent United States history. The influences of two World Wars and the Great Depression are apparent. A comparison of current population distribution graphs of the United States, Mexico, and Sweden provide an effective conclusion to this series of lessons. The drastic differences among the graphs suggest many social implications as well as reveal the strong visual impact of graphs in conveying comparative information.

The problems addressed in these lessons require the application of many mathematical concepts. Reporting growth rates as annual growth rate per 1,000 people employs averages and ratios. The age distribution data is given in percents to allow comparison between countries. Appropriate approximations are made with a rounding operation. And hand-calculators provide the support which allows students to handle real population data.

CAPSULE LESSON SUMMARIES

Section 2.1: Review the rounding operation. Define "population growth rate". Using population data, determine the net population gain or net growth rate of several cities. (Worksheets P07, P08)

Section 2.2:	Review calculations dealing with population growth data. Organize population data into a table and a graph. Ask quantitative and social questions about the data and the graph. (Worksheets P09, P10)
Section 2.3:	Compare graphs of the population growth rate and the total population for the United States in 1910-1975. Discuss and graph the 1975 age-group distribution data for the United States. (Worksheets P11, P12, P13)
Section 2.4:	Discuss historical and social questions related to a graph of the United States population by age groups. Find the median age of the 1976 U.S. population. Compare age-group population graphs of the United States, Sweden, and Mexico. (Worksheets P14, P15, P16, P17, P18)

P2.1 Population Growth Rate #1

CAPSULE LESSON SUMMARY

Review the rounding operation. Define "population growth rate". Using population data, determine the net population gain or net growth rate of several cities.

MATERIALS

Teacher: None
Student : Hand-calculator; Worksheets P07 and P08; pencil

NOTE: The inclusion of up-to-date local data (school, city, or state statistics) is highly recommended for this lesson. Recent almanacs are a reliable source of population data.

DESCRIPTION OF LESSON

Exercise 1

Write these two problems on the board.

$$387 \circledR 10 = \boxed{} \qquad 387 \circledR 100 = \boxed{}$$

T: Who can explain the rounding operation? You may use these problems to help you explain it.

S: In each of those problems, the answer is the multiple of the second number which is closest to the first number. For example, $387 \circledR 10 = 390$ since 390 is the closest multiple of 10 to 387.

S: $387 \circledR 100 = 400$ because 400 is the closest multiple of 100 to 400.

P 2

Write the following problems on the board and solve them in a similar manner. Answers are in the boxes.

$73 \circledR 10 = \boxed{70}$

$129 \circledR 10 = \boxed{130}$

$58{,}711 \circledR 1{,}000 = \boxed{59{,}000}$

$16.84 \circledR 0.1 = \boxed{16.8}$

$0.372 \circledR 0.1 = \boxed{0.4}$

$671 \circledR 100 = \boxed{700}$

$8{,}438 \circledR 100 = \boxed{8{,}400}$

$5{,}555{,}444 \circledR 1{,}000 = \boxed{5{,}555{,}000}$

$9.36 \circledR 0.1 = \boxed{9.4}$

$0.372 \circledR 0.01 = \boxed{0.37}$

For problems involving decimals, remind students of the hand-calculator relation $\boxed{+}$ $\boxed{=}$ . . . . For example, to solve 16.84 ® 0.1, consider putting 0.1 on the display and pressing $\boxed{+}$ $\boxed{=}$ . . . .

Exercise 2

Distribute hand-calculators; every student should have one, if possible.

T: In 1790, the United States' population was only about 4 million. That's about half the present population of New York City. Today we'll learn about the United States' population growth.

Births can increase a country's population. What are other ways for a country's population to increase or decrease?

S: Deaths.

S: Immigration and emigration.

You may need to explain that emigrants from a country are people who leave that country to live elsewhere.

T: When we put all four of these factors together, we get a country's net growth.

Write the following statement on the board.

In 1977, the United States showed a net gain in population. Its rate was an additional 7.5 people for every 1,000 people.

Invite students to discuss what this statement means. They may suggest some of the following ideas:

- Consider the United States' population being divided into groups of 1,000. For each group of 1,000 people at the beginning of the year, there were 1,007 or 1,008 people at the end of the year.
- You can say "7.5 people" because 7.5 is an average calculated from many groups of 1,000 people.
- 7.5 people for every 1,000 means the same as 15 people for every 2,000 or 75 people for every 10,000.

Begin a table with this line of information on the board.

Population	Annual Growth Rate per 1,000 people	Annual Net Gain
5,000	6.4	

P 2

T: A city has a population of 5,000 and its annual growth rate is 6.4 per 1,000. What is that city's net population gain in one year? You may use your hand-calculator.

S: 32, since 5 × 6.4 = 32.

T: Why did you calculate 5 × 6.4?

S: You can imagine that the city's population is divided into five groups of 1,000 people. Each group of 1,000 gains, on the average, 6.4 people. Therefore we multiply 5 × 6.4 to find the net gain.

Record this answer and extend the table.

Population	Annual Growth Rate per 1,000 people	Annual Net Gain
5,000	6.4	32
17,000	10.7	

T: If a city with 17,000 people has a growth rate of 10.7 per 1,000, what is its net gain?

S: 181.9, since 17 × 10.7 = 181.9.

Write "181.9 Ⓡ 1" on the board.

T: We'll use the rounding relation to round each net gain to the nearest person. What does 181.9 Ⓡ 1 equal?

S: 182.

Record 182 in the table.

Proceed similarly for the following lines of data. Answers are provided in the boxes. When considering the population of 1,086,000, you may need to discuss that 1,000,000 people could gather in 1,000 groups of 1,000 people. Thus there are 1,086 thousands in 1,086,000. Students may notice that this could be determined by "covering" the last three zeroes in 1,086,000.

Population	Annual Growth Rate per 1,000 people	Annual Net Gain
62,000	10.7	663
965,000	2.3	2,220
1,086,000	2.3	2,498

Add to the table the following line of data for Chicago.

Population	Annual Growth Rate per 1,000 people	Annual Net Gain
2,828,000 (Chicago)	¯14	

T: What do you think a negative growth rate means?

S: The city lost population in that year.

T: What was Chicago's net gain that year?

S: 2,828 x ¯14 = ¯39,592.

Record this result in your table.

P 2

Add the following line of data to the table.

Population	Annual Growth Rate per 1,000 people	Annual Net Gain
7,800	15.3	

T: How should we calculate net gain when a city's population is 7,800?

Accept suggestions from the students. Make sure that both of the following methods are discussed.

- Use 7,800 Ⓡ 1,000 = 8,000 to round 7,800 to the nearest 1,000. Then calculate "8 × 15.3 = 122.4" and "122.4 Ⓡ 1 = 122".
- Calculate "7,800 ÷ 1,000 = 7.8" (i.e. there are 7.8 thousands in 7,800). Then calculate "7.8 x 15.3 = 119.34" and "119.34 Ⓡ 1 = 119".

Discuss the effect of rounding off by comparing the two answers, 122 and 119. Note that the two answers are quite close and that either answer would be acceptable for most uses of this data.

Proceed similarly for the following lines of data. Let the students choose whether or not to round off the population to the nearest thousand before applying the annual growth rate. Both answers are provided in the boxes.

Population	Annual Growth Rate per 1,000 people	Annual Net Gain
12,360	3.8	46 or 47
711,518 (Washington, D.C.)	$^{-}12.1$	$^{-}8,615$ or $^{-}8,609$

Ask the students to turn off their hand-calculators. Erase the previous data from the table and put in these lines of information.

Population	Annual Growth Rate per 1,000 people	Annual Net Gain	Students' Predictions
63,000	16.4	1,033	✕
63,000	8.6		
93,720	16.4		
93,720	10.4		

T: I have calculated that a city of 63,000 people with a growth rate of 16.4 per 1,000 has an annual net gain of 1,033 people. Is the net gain for a city of 63,000 people and growth rate of 8.6 per 1,000 larger or smaller than 1,033?

S: Smaller, since that city's population is the same as the first city but its growth rate is lower.

S: Smaller, since we would calculate "63 x 8.6" instead of "63 x 16.4".

Write "smaller than 1,033" in the column for students' predictions. Consider the next two lines of data similarly, asking students to predict whether the net gain will be larger or smaller than 1,033. Students are likely to predict that the net gain in a city with 93,720 people and a growth rate of 16.4 will be larger than 1,033. The class may be undecided about the last line of data since 93,720 > 63,000 but 10.4 < 16.4.

P 2

If you wish, ask students to estimate the net gain for the three lines of data. Then invite students to use their hand-calculators to calculate the three unknown net gains. Note whether the students' predictions were accurate.

Population	Annual Growth Rate per 1,000 people	Annual Net Gain	Students' Predictions
63,000	16.4	1,033	
63,000	8.6	542	smaller than 1,033
93,720	16.4	1,542 or 1,537	larger than 1,033
93,720	10.4	978 or 975	undecided

Add this line of data to the table.

Population	Annual Growth Rate per 1,000 people	Annual Net Gain
5,000		42

T: If a city of 5,000 gains 42 people in one year, how can we calculate its growth rate?

Two methods of solution are likely to be suggested:

- Consider the equation $5 \times \boxed{} = 42$. Fill in the box by trial and error. For example, students may try $5 \times 8 = 42$, $5 \times 9 = 45$, $5 \times 8.5 = 42.5$, and so on until they determine that $5 \times 8.4 = 42$.
- Calculate "$42 \div 5 = 8.4$".

Proceed similarly with the following problems, allowing students to use either method of solution. Answers are provided in the boxes, using Ⓡ 0.1 to round each answer to the nearest tenth.

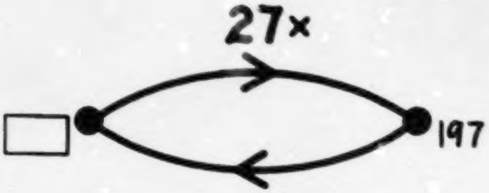
Population	Annual Growth Rate per 1,000 people	Annual Net Gain
23,000	$\boxed{1.8}$	42
164,812	$\boxed{2.2}$	366
27,000	$\boxed{7.3}$	197

If your students used the division method to solve the above problems, distribute Worksheets P07 and P08. Otherwise, lead to that method of solution in the following way.

Refer to the last line of data and write "27 x ☐ = 197" on the board.

T: Can anyone suggest another number story using these numbers?

If no one suggests 197 ÷ 27 = ☐, draw this arrow picture.



T: What can the blue arrow be for?

S: ÷27.

P 2

T: How can I find the number in the box?

S: Calculate "197 ÷ 27".

Write "197 ÷ 27 = ☐" beneath "27 x ☐ = 197".

T: What is the growth rate for that city?

S: About 7.3, since 197 ÷ 27 is about 7.296.

Invite students to solve the following two problems using the division method. Answers are in the boxes.

Population	Annual Growth Rate per 1,000 people	Annual Net Gain
82,281	$\boxed{^{-}13.6}$	$^{-}$1,118
1,216,000	$\boxed{6.8}$	8,269

Worksheets P07 and P08 are available for individual work.

Name ______________________ **P07 ***

Complete.

33 ⓡ 5 = 35

33 ⓡ 8 = 32

82 ⓡ 10 = 80

637 ⓡ 10 = 640

73.27 ⓡ 1 = 73

58.18 ⓡ 0.1 = 58.2

16,837 ⓡ 10 = 16,840

16,837 ⓡ 100 = 16,800

16,837 ⓡ 1,000 = 17,000

5.862 ⓡ 1 = 6

5.862 ⓡ 0.1 = 5.9

5.862 ⓡ 0.01 = 5.86

Name ______________________ **P08 ****

Fill in the boxes.

Country or City	Population	Annual Population Growth Rate per 1,000 People	Net Population Gain - One Year
Hondo, Texas	5,000	9.8	$\boxed{49}$
Harrisburg, Pennsylvania	79,697	-14.7	$\boxed{-1,172}$
Honolulu, Hawaii	705,381	23.7	$\boxed{16,718}$
New York, New York	7,895,563	-10.5	$\boxed{-82,903}$
Sunnyside, Oregon	6,208	$\boxed{8.7}$	54
Bogulusa, Louisiana	21,823	$\boxed{-13.8}$	-301

A city with a population of 32,000 on January 1, 1978 has a population growth rate per 1,000 people of 13.6 in 1978 and 12.9 in 1979. What is that city's population on December 31, 1978? 32,435

. . . on December 31, 1979? 32,853

Answer may vary slightly due to rounding.

P2.2 Population Growth Rate #2

CAPSULE LESSON SUMMARY

Review calculations dealing with population growth data. Organize population data into a table and a graph. Ask quantitative and social questions about the data and the graph.

MATERIALS

Teacher: Probability Poster #3 for IG-VI; felt-tip marker
Student: Hand-calculator; Worksheets P09 and P10; pencil

DESCRIPTION OF LESSON

Distribute hand-calculators; every student should have one, if possible.

Exercise 1

Begin this table on the board.

Year	Population Jan.1	Annual Growth Rate per 1,000	Annual Net Gain
1975	32,000	12.4	
1976			

T: This population data is for the city of Portland. We'll study Portland's population changes over a five-year period. What was Portland's net population gain in 1975?

S: 397, since 32 x 12.4 = 396.8 and 396.8 ® 1 = 397.

T: Therefore, what was Portland's population on January 1, 1976?

S: 32,000 + 397 = 32,397.

Record these answers in the table. Then adding one line of data at a time, continue in a similar manner until the following table is complete. Let students choose whether or not to round off the "January 1" population to the nearest thousand before applying the annual growth rate, but ask that they be consistent throughout the activity. Your students should arrive at one of the answers provided in each box.

Year	Population Jan.1	Annual Growth Rate per 1,000	Annual Net Gain
1975	32,000	12.4	397
1976	32,397	4.7	150 or 152
1977	32,547 or 32,549	-4.7	-155 or -153
1978	32,392 or 32,396	7.5 or 7.4	240
1979	32,632 or 32,636	-0.6	-19

Write this data on the board.

	Population	Growth Rate per 1,000
U.S. 1979:	218,000,000	7.2 per 1,000

T: In 1979, the United States population was about 218,000,000 and our growth rate was 7.2 per 1,000. What was the net population gain for the United States in 1979?

S: 218,000 x 7.2 = 1,569,600.

You may need to remind students that one million people make up 1,000 groups of 1,000 people each. Therefore, there are 218,000 thousands in 218,000,000.

T: The annual net gain was 1,569,600. What was the daily net gain?

S: About 4,300 per day, since there are 365 days in a year and 1,569,600 ÷ 365 = 4300.2739.

T: Yes, the United States was gaining about 4,300 people every day. How many is that per hour?

S: About 179. There are 24 hours in a day and 4,300 ÷ 24 = 179.16666.

T: What was the net gain per minute?

S: About 3 new people every minute, since there are 60 minutes in an hour and 179 ÷ 60 = 2.9833333.

Exercise 2

Distribute Worksheet P09.

T: Pretend that this is a newspaper article. I'll ask you some questions about it. When was the population growth rate 15.2? 9.8?

S: It was 15.2 in 1910 and 9.8 in 1940.

T: What was the growth rate in 1930?

S: 7.3.

T: When was the growth rate the highest?

S: In 1920, when the rate was 18.8.

T: In which five-year periods did the growth rate decline?

Accept any of the following answers: 1910-1915, 1920-1925, 1925-1930, 1930-1935, 1955-1960, 1960-1965, 1965-1970, and 1970-1975.

T: What do you think of the way this data is presented? Could you suggest better ways to organize the data?

S: Put the information into a table.

Following students' suggestions, draw a table on the board. It is not necessary to complete the table. For example, you might draw:

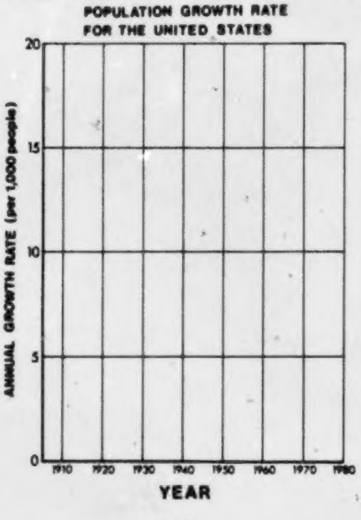
Year	Growth Rate per 1,000
1910	15.2
1915	14.9

T: Are there other ways to organize the data besides in a table?

S: We could graph the data.

T: Tell me how you would draw the graph.

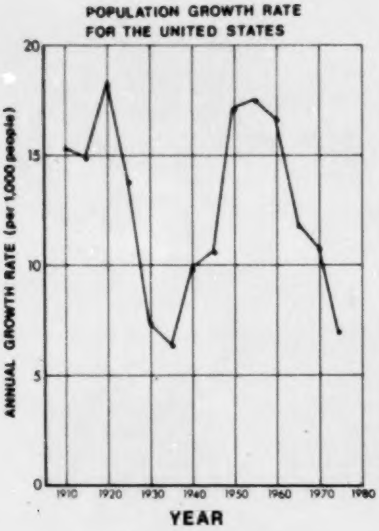
Following students' directions, draw a rough sketch on the board of the graph they describe. They may suggest this format for the graph.



P 3

Once the students describe how to label and mark the axes, distribute Worksheet P10 and display Probability Poster #3 for IG-VI. Tell the students to graph the growth rate data on their worksheets. As they are working individually, invite students to the board to draw the graph on the poster. Encourage accurate placement of dots.

T: I'll connect the dots on the poster so it is easier to see when the growth rate increased or decreased.


POPULATION GROWTH RATE FOR THE UNITED STATES

T: Now I have a few more questions about this data. When was the growth rate the lowest?

S: 1935.

T: When was the next lowest growth rate?

S: 1975.

T: In which five year periods did the growth rate increase?

S: 1915-1920, 1935-1940, 1940-1945, 1945-1950, and 1950-1955.

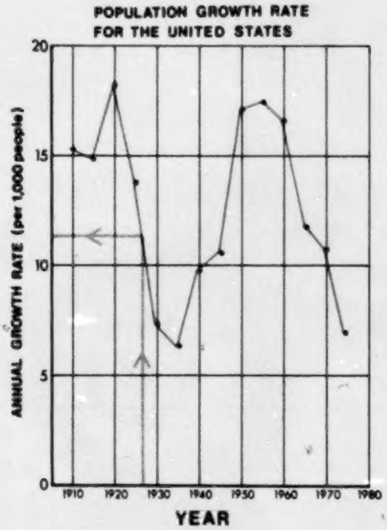
Discuss how the graph made these questions easier to answer as compared to using the original, disorganized data.

Point to the dots on the graph for 1925 and 1930.

T: Since we know the growth rates in 1925 and 1930, we can draw these two dots. Then I joined the dots with a red line segment. Do we know the growth rate for 1927? Could we estimate it?

S: We don't know it exactly, but could use the graph to estimate it.

Your students are likely to use the following method (see the blue arrows in the next illustration) to estimate the growth rate for 1927 to be about 11 or 12 per 1,000.


POPULATION GROWTH RATE FOR THE UNITED STATES

T: This is probably a good estimate for 1927, but we cannot be sure. If 1927 has a very large or very small growth rate, it would not be shown on this graph since our graph includes data from 1925 and 1930, not 1927.

The growth rates were quite low from 1930 to 1945. What historical events might have caused those low growth rates in the United States?

S: The Great Depression in the early 1930's.

S: World War II in the 1940's.

T: The growth rate in 1975 was also quite low. Do you think it was low for the same reasons as in 1930 to 1945?

The students should realize that there are other reasons besides poverty and war to cause a low growth rate in 1975. For example, due to social reasons people may be marrying later on or may be choosing to have fewer children.

T: From 1910 to 1975, did the total United States population ever decrease?

S: No, the growth rate was always above zero, so the population was always increasing.

Some students are likely to observe that the growth rate decreased often (1920 to 1935 and 1955 to 1975) and conclude that the total population also decreased in those years. For these students, point out that the graph is of growth rate, not of total population. Even in 1935 and 1975, the growth rate was positive. Therefore the population was always increasing, although slowly at times. You might use this analogy: a child's rate of growth can be slow or fast, but the height of a child increases even when the growth rate is very slow.

T: If the United States' population does decrease in 1995, how would that be shown on the growth rate graph?

S: The growth rate would be negative. You would have to extend the graph to allow negative numbers.

The following question is optional and need not be considered if too little time remains in the class period.

T: When was the net gain of the United States' population larger, in 1940 or in 1975?

S: It's hard to tell. The growth rate was larger in 1940, but the population was larger in 1975.

Write this information on the board.

	Population	Growth Rate per 1,000
1940:	132,000,000	9.8
1975:	214,000,000	7.0

T: These are the growth rates and populations for 1940 and for 1975. Let's calculate the net gain of population in each year.

S: The net gain in 1940 was 9.8 x 132,000 = 1,293,600 people.

S: The net gain in 1975 was 7.0 x 214,000 = 1,498,000 people.

S: So the net gain of population was larger in 1975 than in 1940 even though the growth rate was lower.

Note: Save Poster #3 with the completed graph for use in Lesson P2.3. Or before Lesson P2.3, you may prefer asking a student to draw a more accurate graph of the same data on the extra, unused Probability Poster #3 for IG-VI.

P2.3 Population Pyramid

CAPSULE LESSON SUMMARY

Compare graphs of the population growth rate and the total population for the United States in 1910-1975. Discuss and graph the 1975 age-group distribution data for the United States.

MATERIALS

Teacher:	Probability Posters #3 and #4 for IG-VI; colored chalk
Student:	Hand-calculator; ruler; Worksheets P11, P12 and P13; colored pencils

Note: Use the copy of Probability Poster #3 on which a graph of the U.S. Population Growth Rate was drawn in Lesson P2.2.

DESCRIPTION OF LESSON

Exercise 1

Display Probability Posters #3 and #4 for IG-VI.

T (pointing to Poster #3): Who recognizes this graph?

S: It is the graph that we drew of the United States Population Growth Rate from 1910 to 1975.

T (pointing to Poster #4): This is a graph of the total United States Population from 1910 to 1975. What was the United States population in 1950?

S: About 150 million.

T: . . . in 1925?

S: About 117 million.

T: When did the United States population reach 100 million?

S: In 1914 or 1915.

T: . . . 200 million?

S: About 1968.

T: Last time we agreed that the United States population has always increased. How does each of these graphs show that fact?

S: The graph of total population is always rising.

S: The growth rate graph shows that the growth rate was always positive.

T: Both of these graphs concern the United States population from 1910 to 1975. Do you see any similarities in the shapes of these two graphs?

S: When the growth rate was very low from 1930 to 1945, the total population graph became almost level; there was little increase of population.

S: When the growth was high in the 1950's, the total population grew very quickly.

S: The total population graph seems to be leveling off around 1975 due to low growth rates from 1965 to 1975.

Exercise 2

Distribute hand-calculators; every student should have one, if possible. Write this information on the board.

United States : K-8

1970 : 34,300,000 students
1975 : 32,000,000 students

T: In 1970, there were about 34,300,000 students in kindergarten through the eighth grade in the United States. In 1975, there were only 32,000,000 students in the same grades, even though the total population increased from 1970 to 1975. Can you explain this?

Invite students to express their opinions.

T: What could be some effects of this large decrease in the number of elementary school students? Who might be very concerned?

Students may suggest effects such as closing schools and laying off teachers in some school districts.

Distribute Worksheet P11.

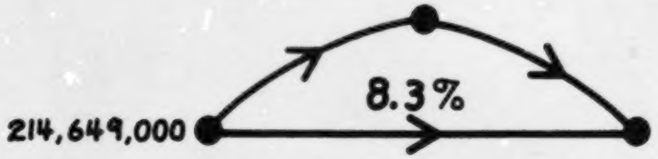
T: This table shows the percent of Americans in various age groups in 1976. What percent of Americans were in the 25-29 age group?

S: 8.3%

T: How can we calculate the number of Americans in that age group?

S: Calculate 8.3% of the total U.S. population, 214,649,000.

Draw this arrow picture on the board.

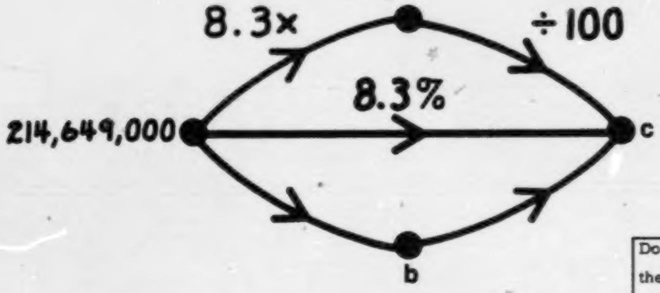


T: How could I calculate 8.3% of 214,649,000?

S: Multiply 214,649,000 by 8.3 and divide the answer by 100.

S: Divide 214,649,000 by 100 and multiply the answer by 8.3.

Encourage both orders for doing the calculation. Label the red arrow and blue arrow and add arrows to indicate that the reverse order may be used in doing the calculations.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Can you do these calculations on your calculator?

S: No, 214,649,000 does not fit on the hand-calculator display.[†]

[†] We assume that the calculator displays only eight digits.

P 5

T (pointing to b): Can you calculate 214,649,000 ÷ 100 without a hand-calculator?

S: Yes. 214,649,000 ÷ 100 = 2,146,490. When you divide by 100 you can "drop" the last two zeros.

Label b "2,146,490".

T (pointing to c): Use your calculator to complete the calculation of 8.3% of 214,649,000.

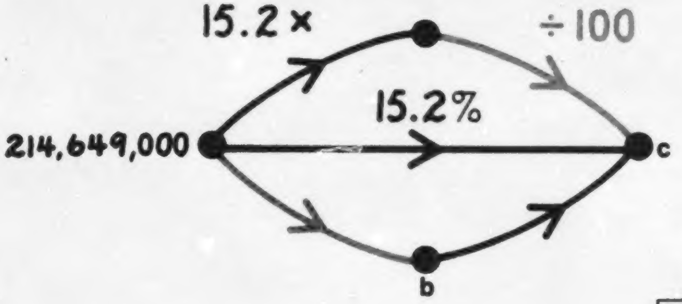
S: 8.3 x 2,146,490 = 17,815,867.

Label c "17,815,867".

T: What percent of Americans in 1976 were less than 10 years old?

S: 15.2%, because 7.1% are in the 0-4 age group and 8.1% are in the 5-9 age group and 7.1 + 8.1 = 15.2.

Relabel the arrow picture.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: How many Americans in 1976 were less than 10 years old?

S: About 32,626,648 since 214,649,000 ÷ 100 = 2,146,490 and 2,146,490 x 15.2 = 32,626,648.

Label b "2,146,490" and c "32,626,648" as you use the arrow picture to explain the method of calculation.

T: What do you expect is the sum of all of the numbers in the "Percent" column of your worksheet?

Some students may be able to explain why the total should be 100.

T: Use your hand-calculator to add those numbers.

What sum do you get?

S: 100.

T: Why should the answer be 100?

S: 100% of the total population means all Americans. Every American must be in one of the age groups.

T: We calculated that about 32,600,000 Americans are less than 10 years old. Of these, 51% are boys. Is that more or less than one-half of the Americans under 10 years old?

S: More, since 50% is $\frac{1}{2}$x .

T: Why do you think that there are more boys than girls under 10 years old?

P 5

After a brief discussion, explain that slightly more than one-half of all births are boys.

T: If 51% of Americans under 10 years old are boys, what percent are girls?

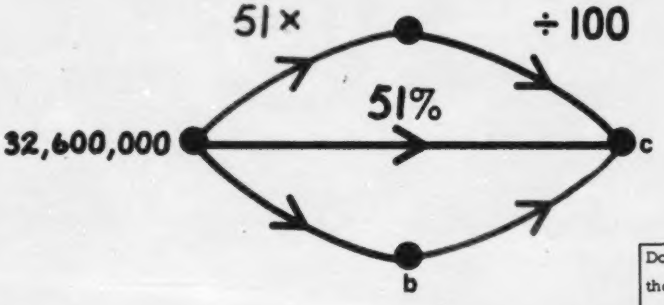
S: 49%, since 51 + 49 = 100.

S: 49%, since 100 - 51 = 49.

T: We know that in 1976 there were about 32,600,000 Americans under 10 years old. If 51% of them are boys, how do I calculate the number of boys under 10 years old?

S: Calculate 51% of 32,600,000; divide 32,600,000 by 100 and then multiply that number by 51.

Ask students to relabel the arrow picture to indicate these calculations.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (pointing to b): What is 32,600,000 ÷ 100? Do not use your calculator.

S: 326,000.

Label b "326,000".

T (pointing to c and tracing the green arrow): So what is 51% of 32,600,000?

S: 51 × 326,000 = 16,626,000.

T: How could we calculate the number of girls less than 10 years old?

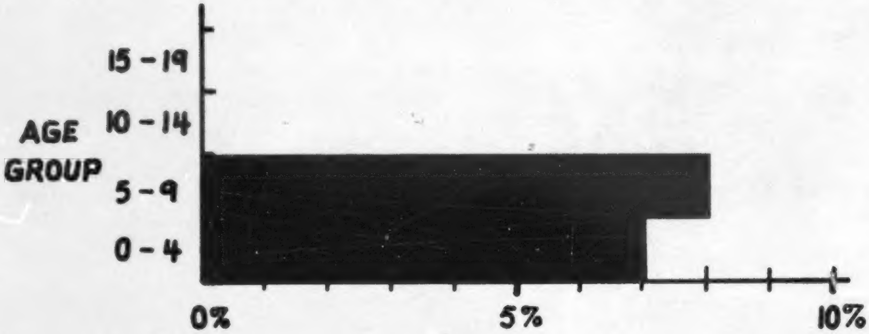
S: Subtract the number of boys under 10 from the total number of children under 10: 32,600,000 - 16,626,000.

S: Calculate 49% of 32,600,000.

Using either method determine that there were 15,974,000 girls less than 10 years old in 1976.

T: Now let's graph the data in the table on your worksheet.

Explain what you are doing as you draw this part of a graph on the board.



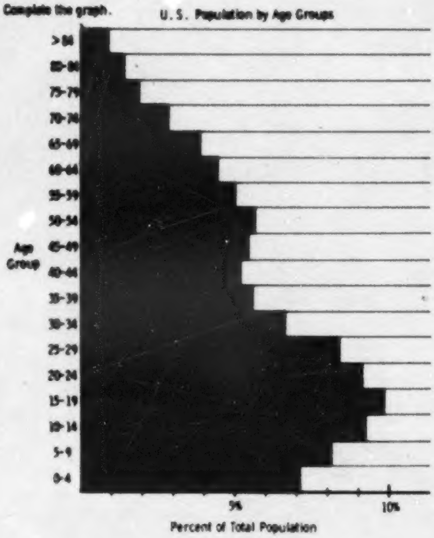
T: I have drawn bars to indicate that 7.1% of the population is in the 0-4 age group and 8.1% is in the 5-9 age group. Who can continue this graph?

Invite students to graph the data for the 10-14 and 15-19 age groups. Distribute Worksheets P12 and P13. Tell the students to complete the graph on Worksheet P12, and then go on to answer the questions on Worksheet P13. You may wish to discuss some of the questions on P13 collectively.

Name ____________________

P12 *

Complete the graph.



Name ____________________

P13 **

1. What percent of Americans in 1976 were 10-14 years old? 9.2 %
   How many Americans in 1976 were 10-14 years old? 19,747,708

2. What percent of Americans in 1976 were 65 years old or older? 10.8%
   How many Americans in 1976 were 65 years old or older? 23,182,092
   What percent of Americans were less than 65 years old in 1976? 89.2%

3. Many Americans still remember World War II. What percent of the Americans in 1976 were born before 1941 (near the beginning of World War II)? 41.8% (35 years old or older)
   (Hint: Think about their age.)

4. What percent of Americans were 5-19 years old in 1976? 27.1%
   How many Americans were 5-19 years old in 1976? 58,169,879
   In 1976, 82% of Americans 5-19 years old were in elementary or secondary schools.
   How many students is that? 47,699,300

Answers may vary due to rounding.

P2.4 Comparative Population Graphs

CAPSULE LESSON SUMMARY

Discuss historical and social questions related to a graph of the United States population by age groups. Find the median age of the 1976 U.S. population. Compare age-group population graphs of the United States, Sweden, and Mexico.

MATERIALS

Teacher:	Probability Poster #5 for IG-VI; felt-tip marker
Student:	Hand-calculator; ruler; Worksheets P14, P15, P16, P17, and P18; pencil

DESCRIPTION OF LESSON

Distribute hand-calculators; every student should have one, if possible.

Exercise 1

Display Probability Poster #5 for IG-VI.

T: From the graph, we see that in 1976 there were fewer Americans in the 40-44 age group than in 36-40 or 45-49 age groups. What could have caused this?

S: The Depression.

S: World War II.

T: Let's determine which is more reasonable. When did the Depression occur?

S: It started in 1929 and continued through the 1930's.

T: When was the United States involved in World War II?

S: From 1940 to 1945.

If your students do not know these facts, provide them yourself.

T:	What are the possible years of birth of people who are in the 40-44 age group in 1976?
S:	1932 to 1936; 1976 - 44 = 1932 and 1976 - 40 = 1936.
S:	Therefore, the Depression is the likely reason for fewer people in that age group.

On the graph, write "1932 - 1936" near the label "40 - 44".

T:	Another unusual feature of the graph is that a smaller percent of Americans are in each of the age groups 0-4, 5-9, and 10-14, than in the age group 15-19. What could cause this decrease?
S:	People might be marrying later or they are deciding to have fewer children.
S:	The fewer number of Americans 40 to 44 years old meant that there were fewer people to have children for awhile. Their children might be 10 to 14 years old.
T:	In the last lesson on statistics, I told you that there were about 2 million fewer students in elementary school in 1975 than in 1970. Does the graph help explain why this is possible, even though the total population is increasing?
S:	The number of children in the lower age groups is decreasing.

Exercise 2

Note: Conduct this activity using either the students' mass in kilograms or weight in pounds. This activity could be related to a project in science or physical education in which the students weigh themselves.

Write "Median Weight" on the board.

T: I want to find the "median weight" of students in this class. The median weight is a weight such that the number of students heavier than that weight is equal to the number of students lighter than that weight.[†]

What might be the median weight of this class?

Accept any reasonable estimate, for example, 35 kg or 90 lbs.

T: How many weigh less than 35 kg? More?

If this estimate is not the median weight, invite a student to suggest another weight for the median and ask the same questions. Continue until the median weight, say 37 kg, is found.

T: The number of you who weight less than 37 kg is the same as the number who weigh more than 37 kg. Therefore, the median weight of this class is 37 kg.

What do you think is meant by the "median age" of the United States' population?

S: The age that one-half of the people are younger than.

T: That's right! What do you think the median age is in the United States?

Let students make estimates of the median age.

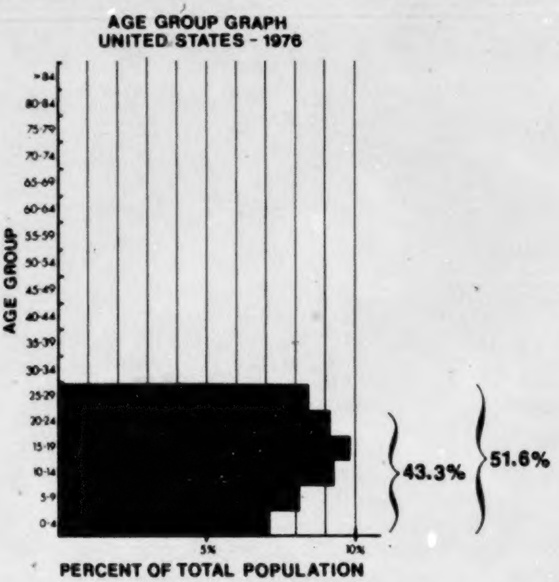
[†] For your information, here is a more precise definition of 'median'. If there are an odd number of measurements, the median equals the middle measurement when all the measurements are ordered. For example, for these five measurements 26, 26, 28, 34, 47, the median is 28. If there are an even number of measurements, the median is the average of the two middle measurements, when all the measurements are ordered. For example, for these six measurements, 26, 26, 28, 34, 47, 48, the median is $\frac{28 + 34}{2} = 31$.

P 6

T: How can we use the age-group graph to determine the median age of Americans in 1976?

S: One-half of the population is the same as 50% of the population. Starting at the 0-4 age group, add the percent of people in each age group until you reach 50%.

From the graph, estimate the percent in each age group and keep a running total. Your estimates may differ slightly from the estimates in this picture.



T: So 43.3% of Americans in 1976 were 24 years old or less, and 51.6% were 29 years old or less. What is a good estimate of the median age?

S: About 28 years old since 51.6 is closer to 50 than is 43.3.

Compare this answer to the students' estimates.

T: Do you think the median ages in other countries are higher, lower, or about the same?

Let the students express their opinions.

Exercise 3

Distribute Worksheets P14, P15, and P16.

T: These are the 1976 age-group graphs for the United States, Mexico, and Sweden.

Lead a discussion of the similarities and differences among the three graphs. Encourage comments about what each graph reveals about that country. Many points may be brought up, including,

- Mexico has the largest percent of young people; Sweden has the smallest.
- Sweden has the largest percent of old people; Mexico has the smallest.
- Sweden and United States both have a comparatively small percent in the 40-44 age group.
- Mexico may soon have many problems due to their rapidly increasing population.

T: Which country do you think has the fewest children in the 0-4 age group?

S: Sweden, because less than 6% of their people are in that group.

S: Sweden, because they have the smallest population.

T: Which country do you think has the most children in the 0-4 age group?

S: I think Mexico does since over 18% of their people are in that age group.

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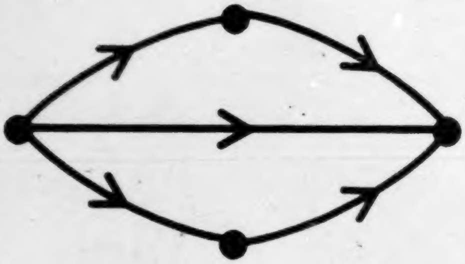
P6

S: I think the United States does since the United States' population is much larger than Mexico's.

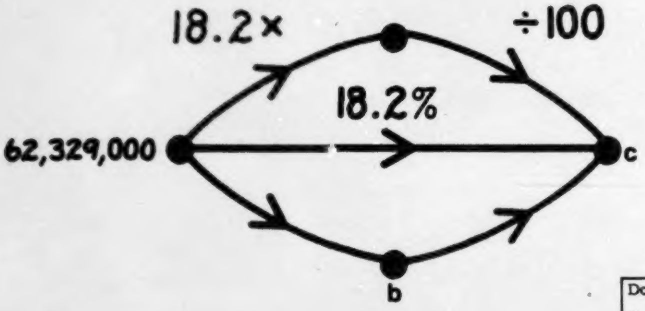
Draw this arrow picture on the board.



T: Let's calculate the number of children in the 0-4 age group in Mexico, United States, and Sweden. I'll do Mexico first. How should I label the arrow picture?

S: According to the graph, about 18.2% of all Mexicans are in the 0-4 age group. Label the green arrow 18.2%. Then the red arrow could be 18.2 × and the blue arrow ÷ 100.

S: Label the leftmost dot "62,329,000", the population of Mexico.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (pointing to b): What is 62,329,000 ÷ 100? Do not use your calculator.

S: 623,290.

Label b "623,290".

T: Therefore, how many Mexicans are 4 years old or less?

S: 18.2 x 623,290 = 11,343,878.

Write "Mexico: 11,343,878" on the board. In a similar manner, use the arrow picture to calculate that there are about 15,240,079 (7.1 x 2,146,490) Americans and about 557,124 (6.8 x 81,930) Swedes in the 0-4 age group. Note that there are more Americans than Mexicans in the 0-4 age group even though the percentage of Mexicans is larger.

Distribute Worksheets P17 and P18. Caution students to use the correct national population and/or the correct graph when answering each question. Allow time at the end of the period to check some of the answers and to discuss different problems the United States, Sweden, and Mexico may have. For example,

- the United States is closing schools, but Mexico needs to be building many schools;
- the United States and especially Sweden have a large percentage of people over 65 years old, who may require financial assistance;
- Mexico's population is rapidly increasing which could cause shortages of food, schools, and health facilities;
- New York City has problems due to a declining population, while Mexico City has problems due to a rapidly increasing population.

Name ____________ **P 17** *

1. What percent of Americans are less than 20 years old? 34.2%
   What percent of Mexicans are less than 20 years old? 57%
   What percent of Swedes are less than 20 years old? 27.1%
   Use this information to explain at least one problem that Mexico may face in the near future that Sweden and the United States probably will not face.
   Many answers are possible.

   How many Americans are less than 20 years old? 73,409,958
   How many Mexicans are less than 20 years old? 35,527,530
   How many Swedes are less than 20 years old? 2,220,303
2. What percent of Americans are 65 years old or older? 10.8%
   What percent of Mexicans are 65 years old or older? 3.5%
   What percent of Swedes are 65 years old or older? 15.1%
   Use this information to describe at least one possible difference between life in Mexico and life in Sweden and in the United States.
   Many answers are possible.

   How many Americans are 65 years old or older? 23,182,092
   How many Mexicans are 65 years old or older? 2,181,516
   How many Swedes are 65 years old or older? 1,237,143

Answers may vary due to rounding or estimating in reading the population pyramids.

Name ____________ **P 18** **

1. The median age of a population is the age that ½ or 50% of the people are younger than. In class you found that the median age in the United States is about 28 years. Calculate the medium age for:

   Sweden 35 years
   Mexico 16 or 17 years

2. About 3.5% of all Americans live in New York City. What is the population of New York City? 7,518,716
   About 18.2% of all Mexicans live in Mexico City. What is the population of Mexico City? 11,343,878

3. It is predicted that in the year 2000 the population of New York City will be 7,200,000 and that 2.7% of all Americans will live in New York City. If these predictions are true, what will the population of the United States be in the year 2000? 266,667,000

   It is predicted that in the year 2000 the population of Mexico City will be 32,000,000 and that 23% of all Mexicans will live in Mexico City. If these predictions are true, what will the population of Mexico be in the year 2000? 139,130,000

Answers may vary due to rounding.